3.2.5 SHACK-HARTMANN SENSOR [BOOK 5.4.4]

START WITH SLIDES

$W_p(x_p, y_p)$ ... $W_L(x_1, y_1)$ ... sensor

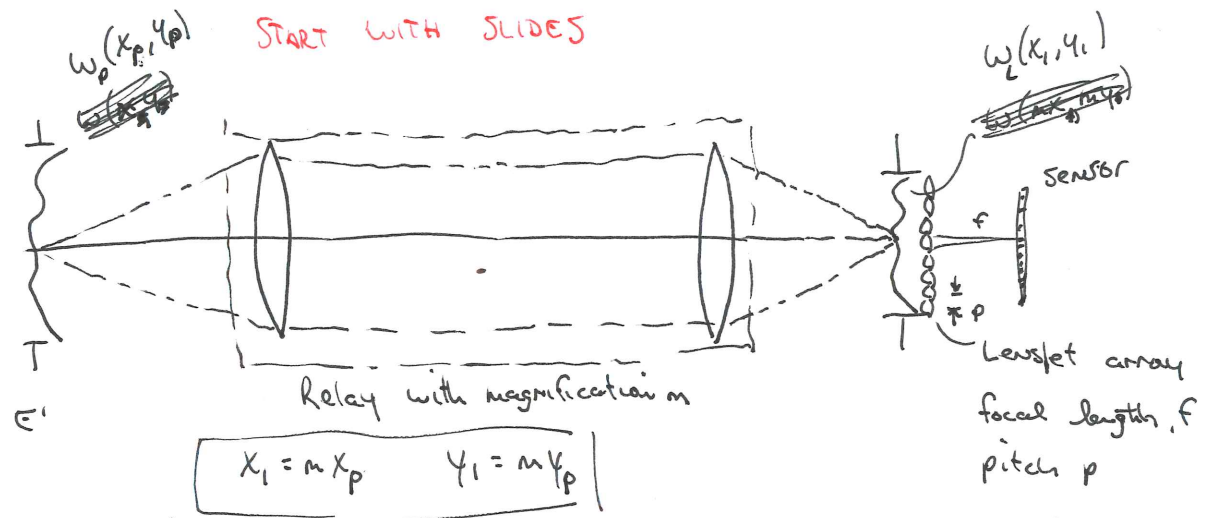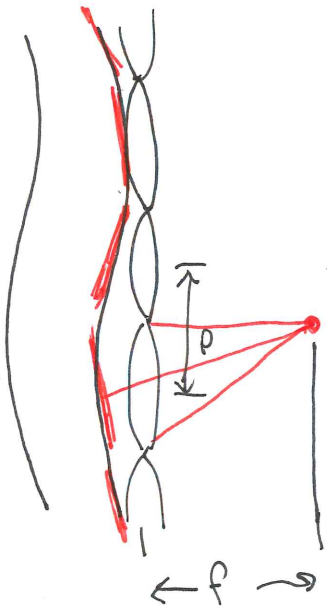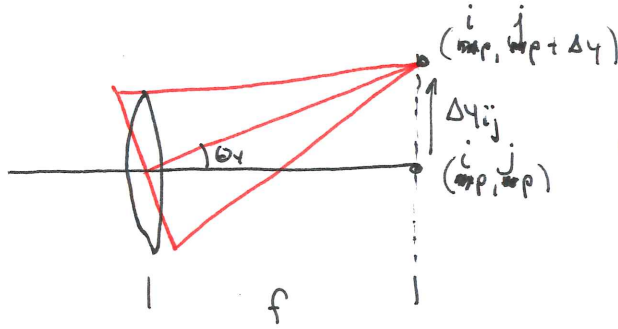Relay with magnification m

$$X_1 = m X_p \qquad Y_1 = m Y_p$$

Lenslet array
focal length, f
pitch p

Wavefront from exit pupil is relayed to lenslet array. If relay has magnification m, then a wavefront error $W_p(x_p, y_p)$ gets projected onto the lenslet array as $W_L(x_1, y_1)$. This means the wavefront is compressed in the transverse direction, but the phase remains the same.

Assume that the wavefront is slowly varying over the aperture of a given lenslet.

We can assume that the wavefront over a given lenslet is given by a tilted plane wave. The tilt is given by the local wavefront slope. A plane wave is focused to the rear focal plane of the lenslet. The focal spot, however, is displaced from the optical axis of the lenslet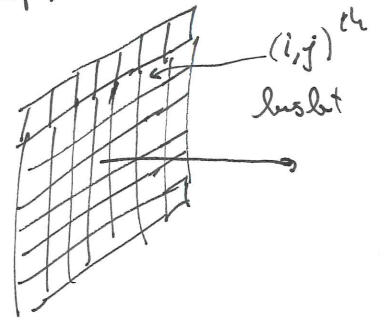 due to the tilt. By measuring the spot displacements, the gradient (i.e. x and y derivatives) of the wavefront error can be obtained. In turn, the gradient can be numerically integrated to recover the wavefront error.

$$\tan \Theta_y = \frac{\Delta y}{f}$$

Consider the $(i,j)^{th}$ luslet in the array. It is centered at a point (i_p, j_p)

but $\tan \Theta_y = - \dfrac{d\omega(i_p, j_p)}{dy}$

so

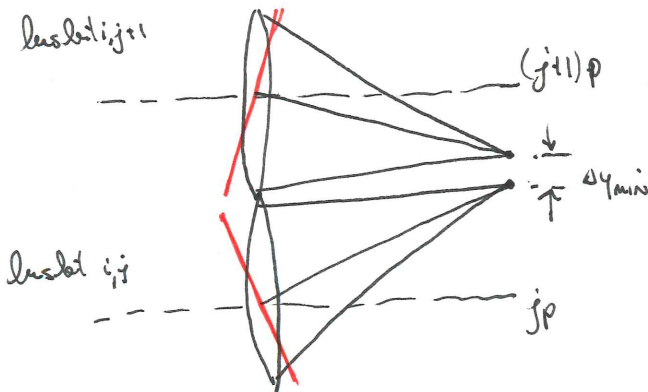$$\Delta y_{ij} = -f \frac{d\omega_c}{dy_1}(i_p, j_p)$$

Similarly

$$\Delta x_{ij} = -f \frac{d\omega_c}{dx_1}(i_p, j_p)$$

Practically, we don't get perfect point but instead some finite size to the focal spot. In this case, the spot centroid is used. There is a maximum change in slope that can occur between two adjacent lenslets.

Dynamic Range
MAXIMUM SLOPE CHANGE BETWEEN LENSLETS

Spots can merge or cross over if slope change is too big between lenslets

Wait

$$(j+1)p + \Delta y_{i,j+1} - \left[jp + \Delta y_{i,j} \right] \geq \Delta y_{min}$$

where Δy_{min} is some minimum separation between adjacent spots

So,

$$p + \Delta y_{i,j+1} - \Delta y_{i,j} \geq \Delta y_{min}$$

$$p - f \frac{d\omega_L}{dy}(i_p, (j+1)_p) + f \frac{d\omega_L}{dy}(i_p, j_p) \geq \Delta y_{min}$$

$$\boxed{\frac{d\omega_L}{dy_1}(i_p, (j+1)_p) - \frac{d\omega_L}{dy_1}(i_p, j_p) \leq \frac{p - \Delta y_{min}}{f}}$$

MAXIMUM SLOPE CHANGE
BETWEEN LENSLETS

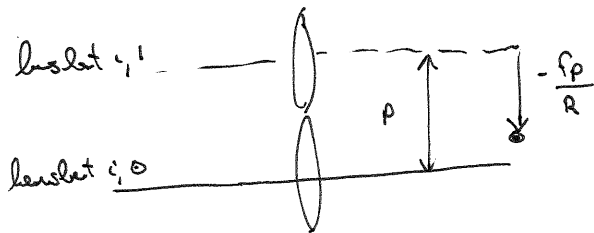$$\frac{\frac{d\omega_L}{dy_1}(i_p, (j+1)_p) - \frac{d\omega_L}{dy_1}(i_p, j_p)}{p} \simeq \boxed{\frac{d^2\omega_L}{dy_1^2}(i_p, j_p) \leq \frac{p - \Delta y_{min}}{pf}}$$

MAXIMUM WAVEFRONT CURVATURE

Simple Example in 1-D

$$\omega_L = \frac{y_1^2}{2R} \quad \Rightarrow \quad \frac{d\omega_L}{dy_1} = \frac{y_1}{R} \quad \Rightarrow \quad \frac{d^2\omega_L}{dy_1^2} = \frac{1}{R}$$

Suppose $\Delta y_{min} = 0$

$$\Delta y_{i,0} = 0$$

$$\Delta y_{i,1} = -\frac{fp}{R}$$

lenslet $i,1$

lenslet $i,0$

p

$-\frac{fp}{R}$

Maximum Slope Change Says

$$\frac{p}{R} - 0 \leq \frac{p}{f} \quad \Rightarrow \quad \frac{1}{R} \leq \frac{1}{f} \quad \Rightarrow \quad R \geq f$$
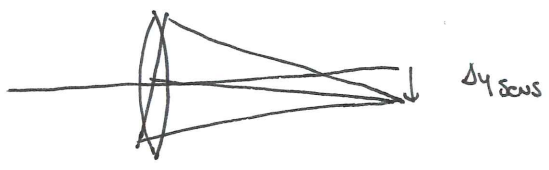
means $\Delta y_{i,1} \leq p$

MAXIMUM WAVEFRONT CURVATURE Says
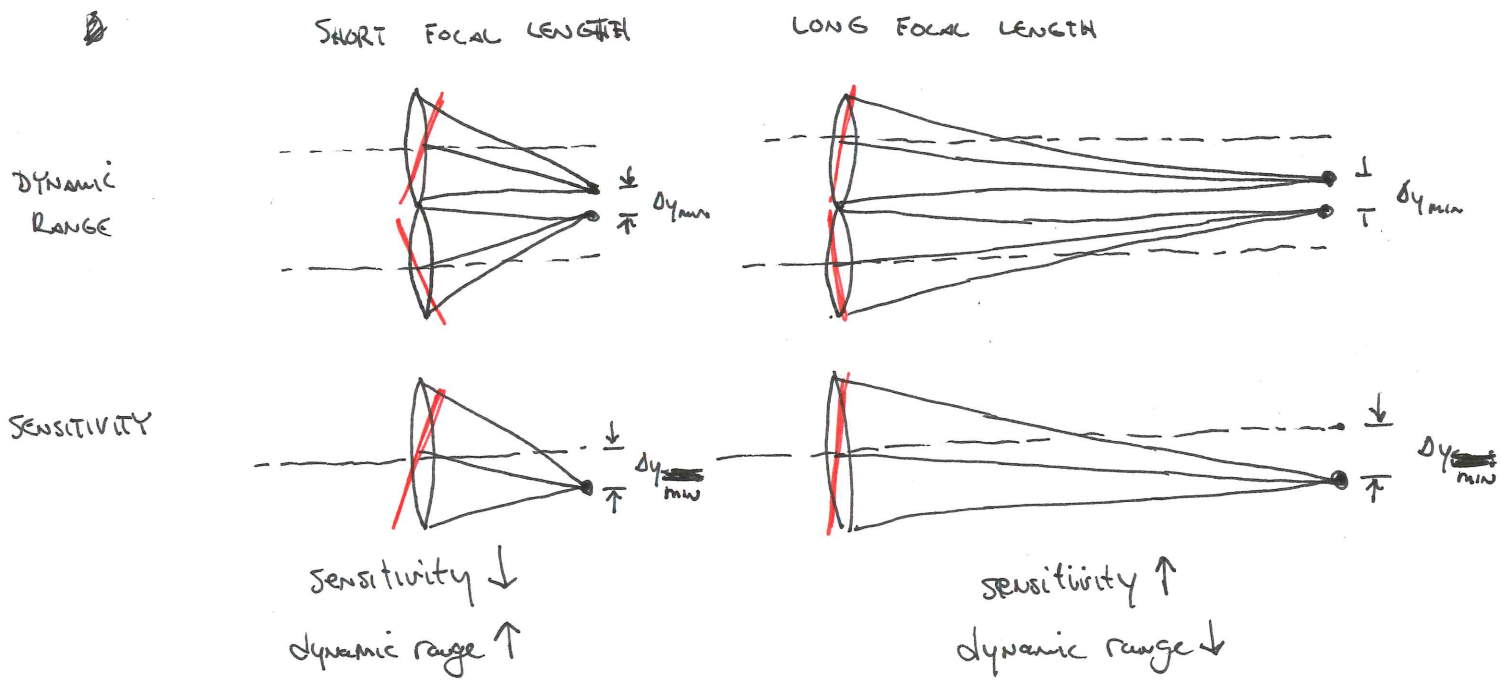
$$\frac{1}{R} \leq \frac{1}{f} \quad \text{same thing}$$

Practically, Δy_{min} will be limited by the pixel size on the detector or the size of the focal spot (like Rayleigh criterion how close can these spots get and still be resolved?)

Sensitivity - what's the smallest absolute slope that can be measured?

$$\Delta y_{sens} = -f \frac{d\omega_{min}}{dy_1}$$

$$\left| \frac{\cancel{\frac{\partial \omega_L(ip,jp)}{\partial \text{weight}}}}{dy} \right|_{min} = \frac{\Delta y_{sens}}{f}$$

SHORT FOCAL LENGTH	LONG FOCAL LENGTH

DYNAMIC RANGE

SENSITIVITY

sensitivity ↓   sensitivity ↑
dynamic range ↑   dynamic range ↓

Effect of Relay System

$$\omega_p(x_p, y_p) = \omega_L(x_1, y_1) \quad \text{where} \quad x_1 = m x_p \; ; \; y_1 = m y_p \quad m = \text{relay magnification}$$

$$\frac{dx_1}{dx_p} = m \; ; \; \frac{dy_1}{dy_p} = m$$

chain Rule

$$\frac{d\omega_p}{dx_p} = \frac{d\omega}{dx_1} \frac{dx_1}{dx_p} \quad \Rightarrow \quad \boxed{\begin{aligned} \frac{d\omega_L}{dx_1} &= \frac{1}{m} \frac{d\omega_p}{dx_p} \\[4pt] \frac{d\omega_L}{dy_1} &= \frac{1}{m} \frac{d\omega_p}{dy_p} \end{aligned}}$$
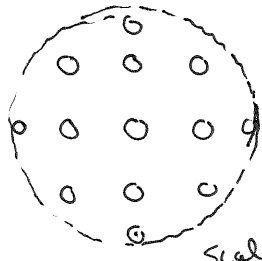
slope at lenslet →   ← slope at E'

For $m < 1$, slope is steeper at lenslet array than at E'

Dynamic Range ↓

SHACK HARTMANN SPOT PATTERN

$$\text{cyl}\left(\frac{r_1}{m D_{E_1}}\right) \sum_{i,j}\left[\delta\left(x_1 - ip + f\frac{dW_L}{dx_1}(ip,jp),\ y_1 - jp + f\frac{dW_L}{dy_1}(ip,jp)\right) * B(x_1,y_1)\right]$$

↑ scaled exit pupil   ↑ Spot shifts due to aberration   ↑ blur due to source size and diffraction

$$r_1^2 = x_1^2 + y_1^2$$

$$\text{cyl}\left(\frac{r}{m D_{E_1}}\right)\sum_{i,j} B\left(x_1 - ip + f\frac{dW_L}{dx_1}(ip,jp),\ y_1 - jp + f\frac{dW_L}{dy_1}(ip,jp)\right)$$

Centroid of $B(x,y_1)$ is at (d_x, d_y) where

$$d_x = \frac{\int x_1 B(x_1,y_1)\,dx_1 dy_1}{\int B(x_1,y_1)\,dx_1 dy_1}, \quad d_y = \frac{\int y_1 B(x_1,y_1)\,dx_1 dy_1}{\int B(x_1,y_1)\,dx_1 dy_1}$$

Centroids of the spots are given by

$$\text{cyl}\left(\frac{r_1}{m D_{E_1}}\right)\sum_{i,j}\delta\left(x_1 - ip + f\frac{dW_L}{dx_1}(ip,jp) - d_x,\ y_1 - jp + f\frac{dW_L}{dy_1}(ip,jp) - d_y\right)$$

Centroid of $B(x,y)$ shifts entire pattern by constant amount.
The preceding is only true for ~~interior~~ lenslets interior to the cyl() function
Spots that are clipped by the cyl() function will have displaced centroids
and induce slight errors.

Example $W_p(x_p, y_p) = W_{020}(x_p^2 + y_p^2) \implies$ Defocus in Exit Pupil

$$W_L(x_1,y_1) = \frac{W_{020}}{m^2}(x_1^2 + y_1^2)$$

$$\frac{dW_L}{dx_1} = \frac{dW_{020}}{m^2} x_1 \qquad \frac{dW_L}{dy_1} = \frac{dW_{020}}{m^2} y_1$$

SPOT PATTERN

$$\text{cyl}\left(\frac{r_i}{mD_{\varepsilon'}}\right) \sum_{ij} \delta\left(x_1 - \left(1 - \frac{dw_{020}}{m^2}f\right)ip - \Delta x, \; y_i - \left(1 - \frac{dw_{020}}{m^2}f\right)jp - \Delta y\right)$$

$$\left[1 - \frac{dw_{020}}{m^2}f\right]p \quad \text{uniformly spaced grid}$$

when $w_{020} = 0$ spots full separation $= p$

when $w_{020} > 0$ spots uniform but compress

when $w_{020} < 0$ spots uniform but expand

3.2.5.1   Fitting Shack-Hartmann Data to Zernike Polynomials   [Book 5.4.5]

STEPS TO MEASURING A WAVEFRONT WITH A SHACK HARTMANN SYSTEM

① Calibrate system with a perfect plane wave

    This gives a set of spots for the case where $W(x_1, y_1) = 0$

    Only need to do this once

② Measure the test system

    This gives a set of spots that are displaced by the aberrations of the system

③ Measure the distance between the ideal and aberrated spots to get

$$\{\Delta x_{ij}, \; \Delta y_{ij}\}_{ij}$$

④ Convert this set to $\left\{\dfrac{dw}{dx_1}^{(ip, jp)}, \; \dfrac{dw}{dy}^{(ip, jp)}\right\}_{ij}$ with eqs. from pg. [129]

Want to integrate these slopes to get $W(x_1, y_1)$ back

One way to do this is to fit to Zernike polynomials

$$\text{Before we did} \quad W(\bar{x}, \bar{y}) = \sum a_{nm} Z_n^m(\bar{x}, \bar{y})$$

$$\frac{dW(\bar{x},\bar{y})}{d\bar{x}} = \sum a_{nm} \frac{dZ_n^m(\bar{x},\bar{y})}{d\bar{x}} \qquad \frac{dW(\bar{x},\bar{y})}{d\bar{y}} = \sum a_{nm} \frac{dZ_n^m(\bar{x},\bar{y})}{d\bar{y}}$$

Set up a matrix system

$$\overbrace{}^{\tilde{\tilde{A}}}$$

(hatched)	$\dfrac{d\bar{Z}_1^{-1}(\bar{x}_1,\bar{q}_1)}{d\bar{x}}$	$\cdots$	$\dfrac{d\bar{Z}_N^{n}(\bar{x}_1,\bar{q}_1)}{d\bar{x}}$
$\vdots$	$\vdots$		$\vdots$
(hatched)	$\dfrac{d\bar{Z}_1^{-1}(\bar{x}_N,\bar{q}_N)}{d\bar{x}}$	$\cdots$	$\dfrac{d\bar{Z}_N^{n}(\bar{x}_N,\bar{q}_N)}{d\bar{x}}$
(hatched)	$\dfrac{d\bar{Z}_1^{-1}(\bar{x}_1,\bar{q}_1)}{d\bar{q}}$	$\cdots$	$\dfrac{d\bar{Z}_N^{n}(\bar{x}_1,\bar{q}_1)}{d\bar{q}}$
$\vdots$	$\vdots$		$\vdots$
(hatched)	$\dfrac{d\bar{Z}_1^{-1}(\bar{x}_N,\bar{q}_N)}{d\bar{q}}$	$\cdots$	$\dfrac{d\bar{Z}_N^{n}(\bar{x}_N,\bar{q}_N)}{d\bar{q}}$

$$\overbrace{}^{\bar{x}}$$

$$\begin{bmatrix} \text{(hatched)} \\ a_{1,-1} \\ a_{1,1} \\ \vdots \\ a_{N,N} \end{bmatrix}$$

$=$

$$\overbrace{}^{\bar{b}}$$

$$\begin{bmatrix} \dfrac{dw(\bar{x}_1,\bar{q}_1)}{d\bar{x}} \\[8pt] \dfrac{dw(\bar{x}_2,\bar{q}_2)}{d\bar{x}} \\[8pt] \vdots \\ \dfrac{dw(\bar{x}_N,\bar{q}_N)}{d\bar{x}} \\[8pt] \dfrac{dw(\bar{x}_1,\bar{q}_1)}{d\bar{q}} \\[8pt] \vdots \\ \dfrac{dw(\bar{x}_N,\bar{q}_N)}{d\bar{q}} \end{bmatrix}$$

dN rows

(red note) DON'T USE $\dfrac{dZ_0^0}{dx}$ or $\dfrac{dZ_0^0}{dy}$ because it gives a column of zeros.

$$\tilde{\tilde{D}}\,\tilde{\tilde{A}} = \tilde{w} \qquad \tilde{\tilde{A}}\,\bar{x} = \bar{b} \qquad \bar{x},\bar{q} \text{ Normalized coordinates}$$

~~(crossed out) DNS~~ $\quad \bar{x} = \left[\tilde{\tilde{A}}^{\mathsf{T}}\tilde{\tilde{A}}\right]^{-1}\tilde{\tilde{A}}^{\mathsf{T}}\bar{b}$

NOTE: The coordinates $\bar{x},\bar{q}$ are normalized coordinates

$$\bar{x} = \frac{x_1}{\left[\frac{m D_{E'}}{2}\right]} \qquad\qquad \bar{y} = \frac{y_1}{\left[\frac{m D_{E'}}{2}\right]}$$

$$\frac{dZ_n^m}{d\bar{x}} = \frac{dZ_n^m}{dx}\frac{dx}{d\bar{x}} = \frac{m D_{E'}}{2}\frac{dZ_n^m}{dx}$$

Similar for $\dfrac{dZ_n^m}{d\bar{q}}$ and $\dfrac{dw}{d\bar{x}}\ \dfrac{dw}{d\bar{q}}$

NOTE IN BOOK SECTION 5.4.5 unnormalized coordinates are used instead. Just be consistent between both sides of matrix equation