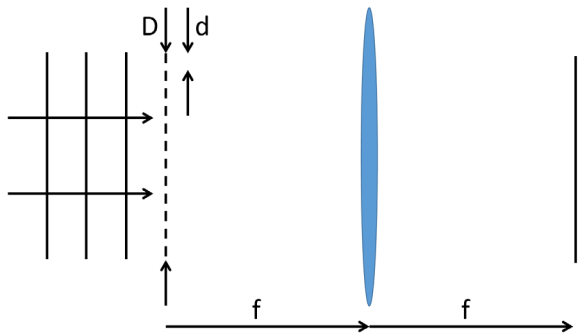
OPTI 512 Review Problem 2

A Ronchi ruling is a glass plate with alternating transparent and opaque strips. The transmission of the ruling is given by

$$t(x,y) = rect\left(\frac{x}{D},\frac{y}{D}\right)\left[rect\left(\frac{2y}{d}\right) ** \frac{1}{d}comb\left(\frac{y}{d}\right)\right],$$

Where D is the width and height of the ruling and d is the period of the ruling pattern.



If the ruling is illuminated with a unit amplitude plane wave and the ruling is located in the front focal plane of a thin lens (you can ignore the effects of the pupil function of the lens), do the following:

(a) Show that the irradiance pattern in the rear focal plane of the lens is

$$I(x_f,y_f,2f) = \left|U(x_f,y_f,2f)\right|^2 = \frac{D^4}{4\lambda^2 f^2}sinc^2\left(\frac{m}{2}\right)\left[\sum_m sinc\left(\frac{Dx_f}{\lambda f}, D\left(\frac{y_f}{\lambda f}-\frac{m}{d}\right)\right)\right]^2$$

For an object located in front of the lens, the field on the focal plane is given by

$$U(x_f,y_f,2f) = \frac{exp(ik2f)}{i\lambda f}\mathcal{F}\{U(x_o,y_o,0)\},$$

with $\xi = x_f/\lambda f$ *and* $\eta = y_f/\lambda f$. *Taking the Ronchi ruling as the origin and noting the unit amplitude plane wave illumination,* $U(x_o,y_o,0) = t(x_o,y_o)$.

Fourier transforming, we get

$$U(x_f,y_f,2f) = \frac{exp(ik2f)}{i\lambda f}D^2 sinc(D\xi,D\eta) ** \left[\delta(\xi)\frac{d}{2}sinc\left(\frac{d\eta}{2}\right)comb(d\eta)\right].$$

From the definition of the comb function

$$U(x_f,y_f,2f) = \frac{exp(ik2f)}{i2\lambda f}D^2 sinc(D\xi,D\eta) ** \left[sinc\left(\frac{d\eta}{2}\right)\sum_m \delta\left(\xi,\eta-\frac{m}{d}\right)\right].$$

From the multiplication properties of delta functions

$$U(x_f, y_f, 2f) = \frac{exp(ik2f)}{i2\lambda f} D^2 sinc(D\xi, D\eta) ** \sum_m sinc\left(\frac{m}{2}\right) \delta\left(\xi, \eta - \frac{m}{d}\right).$$

Now, perform the convolution

$$U(x_f, y_f, 2f) = \frac{exp(ik2f)}{i2\lambda f} D^2 \sum_m sinc\left(\frac{m}{2}\right) sinc\left(D\xi, D\left(\eta - \frac{m}{d}\right)\right).$$

Substitute $\xi = x_f/\lambda f$ *and* $\eta = y_f/\lambda f$*, so that*

$$U(x_f, y_f, 2f) = \frac{exp(ik2f)}{i2\lambda f} D^2 \sum_m sinc\left(\frac{m}{2}\right) sinc\left(\frac{Dx_f}{\lambda f}, D\left(\frac{Dy_f}{\lambda f} - \frac{m}{d}\right)\right).$$

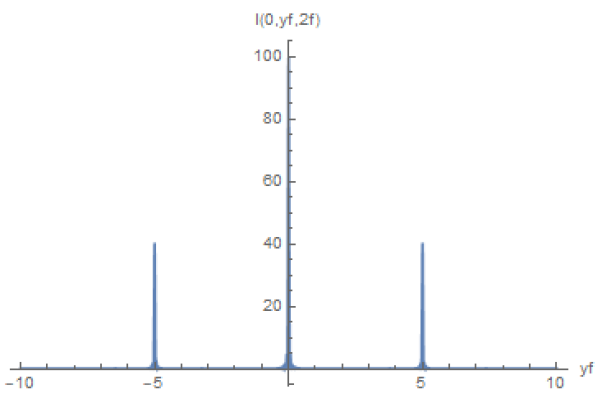
Finally, take the squared magnitude to get the irradiance pattern

$$I(x_f, y_f, 2f) = \left|U(x_f, y_f, 2f)\right|^2 = \frac{D^4}{4\lambda^2 f^2}\left[\sum_m sinc\left(\frac{m}{2}\right) sinc\left(\frac{Dx_f}{\lambda f}, D\left(\frac{y_f}{\lambda f} - \frac{m}{d}\right)\right)\right]^2$$

(b) The irradiance pattern is often approximated by

$$I(x_f, y_f, 2f) \cong \frac{D^4}{4\lambda^2 f^2} \sum_m sinc^2\left(\frac{m}{2}\right) sinc^2\left(\frac{Dx_f}{\lambda f}, D\left(\frac{y_f}{\lambda f} - \frac{m}{d}\right)\right),$$

Which basically says that the overlap between adjacent *sinc* functions is negligible. Plot the approximation of $I(0, y_f, 2f)$ for $-10\ mm \le y_f \le 10\ mm$, $D = 1\ mm$, $f = 100\ mm$, $\lambda = 0.5\ \mu m$, and $d = 10\ \mu m$ to convince yourself that this is a valid approximation.



(c) Find a general expression for the location y_f of the peak in the case $m = 1$.
The peaks occur when the argument of the sinc function is zero or when

$$\left(\frac{y_f}{\lambda f} - \frac{1}{d}\right) = 0$$

$$y_f = \frac{\lambda f}{d}$$

(d) This setup can be used as a spectrometer to determine the wavelength distribution or spectrum of the incident light. The input now is a series of plane waves, each with a different wavelength

and amplitude corresponding to its spectral content. The $m = \pm 1$ peaks will disperse into a weighted version of the incident spectrum. If a sensor with $5\ \mu m$ pixels is placed at the rear focal plane of the lens, what is the minimum wavelength difference $\Delta\lambda$ that can be resolved with the setup? Use $f = 100\ mm$ and $d = 10\ \mu m$. Do the peaks move inwards or outwards when going from longer to shorter wavelengths?

Consider two wavelengths λ_1 and λ_2. Based on part (c) the separation between the peaks is

$$y_{f2} - y_{f1} = \frac{(\lambda_2 - \lambda_1)f}{d}$$

At the minimum separation of $5\ \mu m$, we have

$$\Delta\lambda = \frac{(0.005\ mm)0.01\ mm}{100\ mm} = 0.5\ nm.$$

From part (c), y_f is proportional to λ, so shorter wavelengths means that the peaks move inward.

The equation sheet will have the formulas for the three cases of object in front of lens, object against lens, and object behind lens. Know how to use these formulas, especially how to convert the spatial frequency variables back into spatial coordinates.