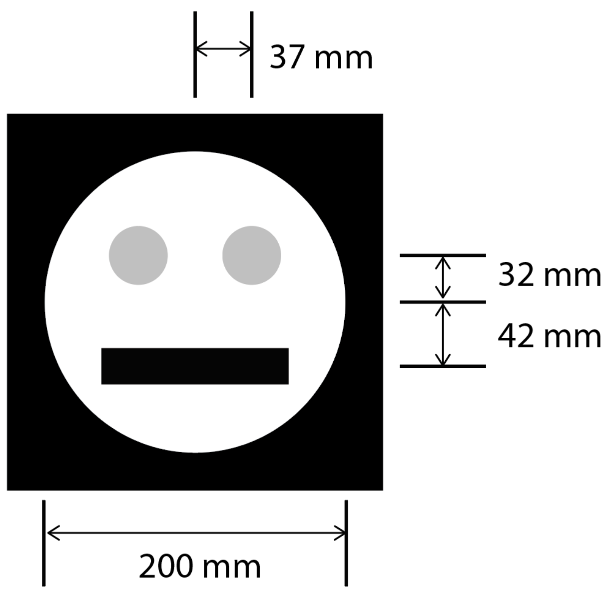
1. Suppose we want to make a transmission mask that looks like a face. The face is a circle with diameter 200 mm and it has 100% transmission, except in the region of the eyes and mouth. The eyes have a transmission of 25% and a diameter of 40 mm. The eyes are centered horizontally, 37 mm on either side of the midline of the face, and are centered vertically, 32 mm above the center of the face. The mouth is an opaque rectangle that has a width of 125 mm and a height of 25 mm. The center of the mouth on the midline, but shifted 42 mm below the center of the face.



a) Create a description of the transmission of this filter $t(x, y)$?

$$t(x,y) = cyl\left(\frac{r}{200}\right) - 0.75cyl\left(\frac{r}{40}\right) ** \big(\delta(x-37, y-32) + \delta(x+37, y-32)\big)$$

$$- rect\left(\frac{x}{125}, \frac{y}{25}\right) ** \delta(x, y+42)$$

where $r = \sqrt{x^2 + y^2}$. *An alternative, but much uglier would be*

$$t(x,y) = cyl\left(\frac{r}{200}\right) - 0.75cyl\left(\frac{\sqrt{(x-37)^2+(y-32)^2}}{40}\right)$$

$$- 0.75cyl\left(\frac{\sqrt{(x+37)^2+(y-32)^2}}{40}\right) - rect\left(\frac{x}{125}, \frac{y+42}{25}\right)$$

b) Calculate the Fourier transform $T(\xi,\eta)$ of this filter.

$$T(\xi,\eta) = 40000\frac{\pi}{4}somb(200\rho) - 1600 \times 0.75$$

$$\times \frac{\pi}{4} somb(40\rho)[exp(-i2\pi 37\xi)exp(-i2\pi 32\eta)$$

$$+ exp(i2\pi 37\xi)exp(-i2\pi 32\eta)]$$

$$- 3125 sinc(125\xi, 25\eta)exp(i2\pi 42\eta)$$

where $\rho = \sqrt{\xi^2+\eta^2}$. *Simplifying gives*

$$T(\xi,\eta) = 10000\pi somb(200\rho) - 600\pi somb(40\rho)exp(-i64\pi\eta)cos(74\pi\xi)$$

$$- 3125 sinc(125\xi, 25\eta)exp(i84\pi\eta)$$

2. An LSI system has an impulse response of $h(x) = Gaus(3x)$.

a) What is the transfer function $H(\xi)$ of the system?

$$H(\xi) = \frac{1}{3}Gaus\left(\frac{\xi}{3}\right)$$

b) For the input $f(x) = Gaus\left(\frac{x}{2}\right)$, what is the output $g(x)$ of the system?

$$F(\xi) = 2Gaus(2\xi)$$

$$G(\xi) = F(\xi)H(\xi) = \frac{2}{3}Gaus(2\xi)Gaus\left(\frac{\xi}{3}\right)$$

From the definition of the Gaus() function

$$G(\xi) = \frac{2}{3}exp[-\pi(2\xi)^2]exp\left[-\pi\left(\frac{\xi}{3}\right)^2\right] = \frac{2}{3}exp\left[-\pi\left(\frac{37}{9}\right)\xi^2\right]$$

$$G(\xi) = \frac{2}{3} Gaus\left(\frac{\sqrt{37}}{3}\xi\right)$$

$$g(x) = \frac{2}{\sqrt{37}} Gaus\left(\frac{3}{\sqrt{37}}x\right)$$

c) Write your result from part (b) as a Gaus() function.

If the two Gaus() functions were combined as above, then your answer is already a Gaus() function, so

$$g(x) = \frac{2}{\sqrt{37}} Gaus\left(\frac{3}{\sqrt{37}}x\right)$$

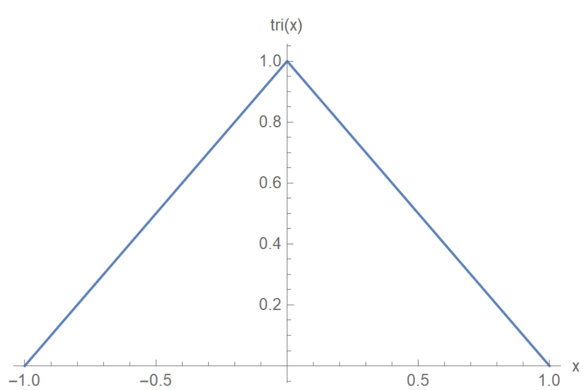
If the Gaus() functions were not combined as above, then you get something like

$$g(x) = Gaus\left(\frac{x}{2}\right) * Gaus(3x)$$

which is right where you started.

3. Compute the *complex* Fourier series for the function $f(x) = tri(x)$ defined over the range $-1 \le x < 1$, with period $X = 2$.

a) Sketch a plot of $f(x)$ over its range.



b) What is the fundamental frequency ξ_o of the series?

$$\xi_o = \frac{1}{X} = \frac{1}{2}$$

c) Calculate the coefficients a_m of the series.

$$a_m = \frac{1}{2}\int_{-1}^{1} tri(x)exp\left(-i2\pi m\left(\frac{1}{2}\right)x\right)dx$$

$$a_m = \frac{1}{2}\int_{-1}^{0}(x+1)exp(-i\pi mx)dx + \frac{1}{2}\int_{0}^{1}(1-x)exp(-i\pi mx)dx$$

$$a_m = \frac{1}{2}\int_{-1}^{0} xexp(-i\pi mx)dx + \frac{1}{2}\int_{-1}^{0} exp(-i\pi mx)dx + \frac{1}{2}\int_{0}^{1} exp(-i\pi mx)dx$$

$$-\frac{1}{2}\int_{0}^{1} xexp(-i\pi mx)dx$$

The second and third terms can be combined.

$$a_m = \frac{1}{2}\int_{-1}^{0} xexp(-i\pi mx)dx + \frac{1}{2}\int_{-1}^{1} exp(-i\pi mx)dx - \frac{1}{2}\int_{0}^{1} xexp(-i\pi mx)dx$$

For the integrals containing the x in front of the exponential, we can use the following relationship which can be found in most tables of integrals

$$\int xexp(\alpha x)dx = \left(\frac{x}{\alpha} - \frac{1}{\alpha^2}\right)exp(ax) \quad for\ \alpha \neq 0$$

We can apply this integral to the first and third terms above as long as $\alpha \neq 0$*, which means* $\alpha = -i\pi m$*, as long as* $m \neq 0$.

$$a_{m\neq 0} = \frac{1}{2}\left(\frac{x}{-i\pi m} - \frac{1}{(-i\pi m)^2}\right)exp(-i\pi mx)\Bigg|_{x=-1}^{x=0} + \frac{1}{2}\frac{exp(-i\pi mx)}{-i\pi m}\Bigg|_{x=-1}^{x=1}$$

$$-\frac{1}{2}\left(\frac{x}{-i\pi m} - \frac{1}{(-i\pi m)^2}\right)exp(-i\pi mx)\Bigg|_{x=0}^{x=1}$$

$$a_{m\neq 0} = \frac{1}{2}\left(\frac{1}{(\pi m)^2}\right) - \frac{1}{2}\left(\frac{1}{i\pi m} + \frac{1}{(\pi m)^2}\right) exp(i\pi m) + \frac{1}{2}\frac{exp(-i\pi m)}{-i\pi m} - \frac{1}{2}\frac{exp(i\pi m)}{-i\pi m}$$

$$-\frac{1}{2}\left(\frac{1}{-i\pi m} + \frac{1}{(\pi m)^2}\right) exp(-i\pi m) + \frac{1}{2}\left(\frac{1}{(\pi m)^2}\right)$$

$$a_{m\neq 0} = \left(\frac{1}{(\pi m)^2}\right) + \frac{sin(\pi m)}{\pi m} - \frac{1}{2}\left(\frac{1}{i\pi m}\right)[exp(i\pi m) - exp(-i\pi m)]$$

$$-\frac{1}{2}\left(\frac{1}{(\pi m)^2}\right)[exp(i\pi m) + exp(-i\pi m)]$$

$$a_{m\neq 0} = \left(\frac{1}{(\pi m)^2}\right) + sinc(m) - \left(\frac{1}{\pi m}\right) sin(m\pi) - \left(\frac{1}{(\pi m)^2}\right) cos(m\pi)$$

$$a_{m\neq 0} = \frac{1}{(\pi m)^2}(1 - (-1)^m)$$

In the special case when $m = 0$*, the integrals reduce to*

$$a_0 = \frac{1}{2}\int_{-1}^{0} xdx + \frac{1}{2}\int_{-1}^{1} dx - \frac{1}{2}\int_{0}^{1} xdx.$$

Integrating gives

$$a_0 = \frac{1}{4}x^2\Big|_{x=-1}^{x=0} + \frac{1}{2}x\Big|_{x=-1}^{x=1} - \frac{1}{4}x^2\Big|_{x=0}^{x=1}.$$

$$a_0 = -\frac{1}{4} + \left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right) - \frac{1}{4} = \frac{1}{2}$$

Alternative Method

Using Fourier transforms

$$a_m = \frac{1}{2}\int_{-1}^{1} tri(x) exp\left(-i2\pi m\left(\frac{1}{2}\right)x\right) dx$$

The function tri() is zero outside of the range $-1 \leq x < 1$*, so the equivalent integral is*

$$a_m = \frac{1}{2}\int_{-\infty}^{\infty} tri(x)exp\left(-i2\pi m\left(\frac{1}{2}\right)x\right)dx$$

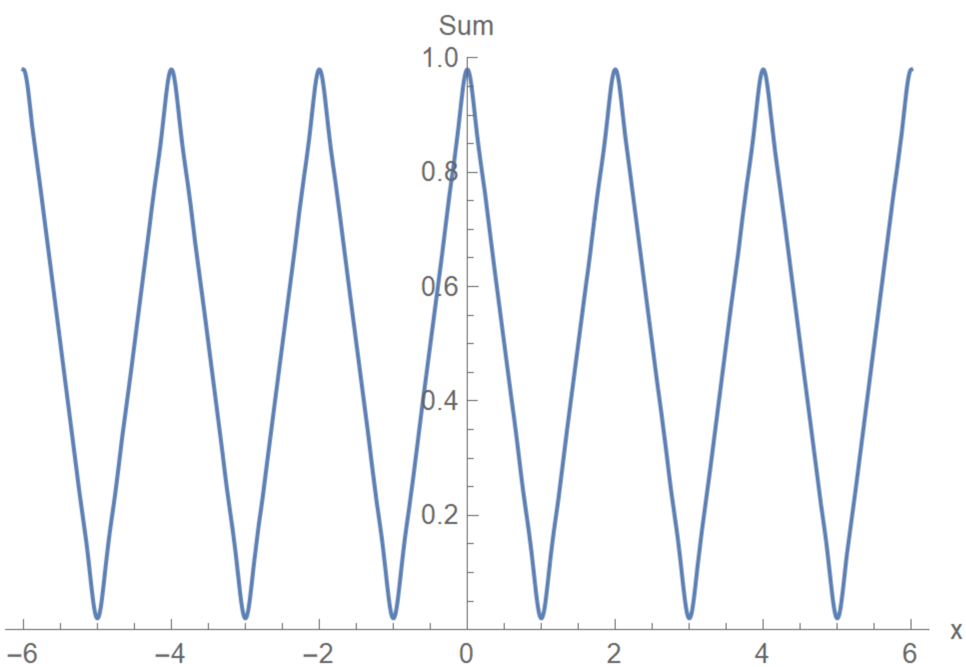
Substitute $\xi = m/2$ *and we have*

$$a_m = \frac{1}{2}\int_{-\infty}^{\infty} tri(x)exp(-i2\pi\xi x)dx = \frac{1}{2}\mathcal{F}\{tri(x)\}$$

$$a_m = \frac{1}{2}sinc^2\left(\frac{m}{2}\right)$$

d) Plot over the range $-6 \le x < 6$, the *real part* of the truncated Fourier series

$$\sum_{m=-10}^{10} a_m exp(i2\pi m\xi_o x)$$



4. A transmission mask is described by

$$t(x,y) = cyl(r)\left[rect(x)rect(y) - rect\left(2x - \frac{1}{2}\right)rect(y)\right]$$

where $r = \sqrt{x^2 + y^2}$.

a) Make a surface plot of this transmission function.

It is easier to see what is going on if the rect() is put into standard form

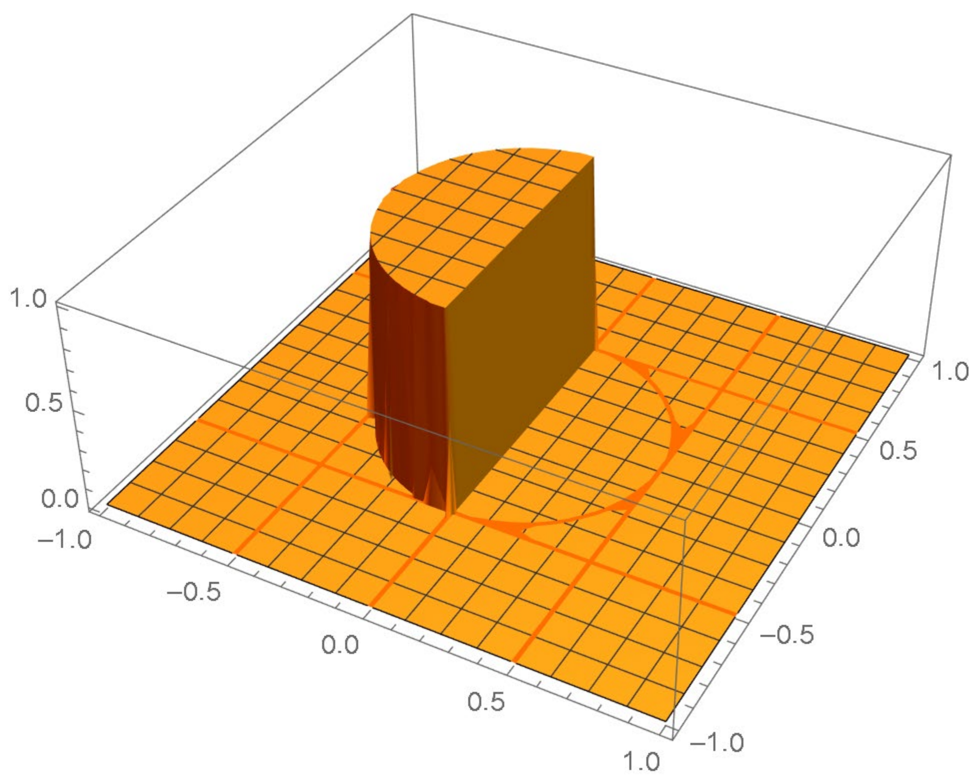
$$t(x,y) = cyl(r)\left[rect(x)rect(y) - rect\left(2\left(x-\frac{1}{4}\right)\right)rect(y)\right]$$

$$t(x,y) = cyl(r)\left[rect(x)rect(y) - rect\left(\frac{x-{}^{1}/_{4}}{{}^{1}/_{2}}\right)rect(y)\right]$$

The first set of rects multiple the whole cyl function by 1. The second set of rects subtracts off a rect for $x \geq 0$*, so only the negative portion of the cyl is left.*

Equivalently, this can be rewritten as

$$t(x,y) = cyl(r)\left[rect\left(\frac{x+{}^{1}/_{4}}{{}^{1}/_{2}}\right)rect(y)\right]$$



b) What is the Fourier transform $T(\xi,\eta)$ of this mask? You can leave your final answer in the form of a convolution.

$$T(\xi,\eta) = \frac{\pi}{4} somb(\rho) ** \left[sinc(\xi)sinc(\eta) - \frac{1}{2} sinc\left(\frac{\xi}{2}\right) exp\left[-i2\pi\left(\frac{1}{4}\right)\xi\right] sinc(\eta)\right]$$

or equivalently

$$T(\xi,\eta) = \frac{\pi}{4} somb(\rho) ** \left[\frac{1}{2} sinc\left(\frac{\xi}{2}\right) exp\left[i2\pi\left(\frac{1}{4}\right)\xi\right] sinc(\eta)\right]$$