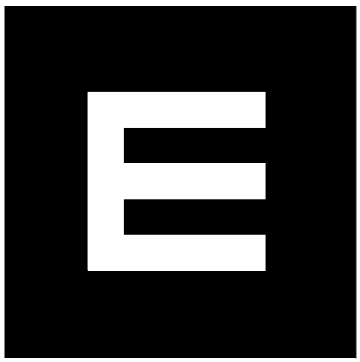
OPTI 512 Homework 5 Due: October 14, 2020

1. Let's redo problem 2 from homework 2, but now write the transmission of the mask below in terms of two *rect*() and two *delta* functions.



2. Write and expression for the 2D Fourier transform of the transmission mask in the preceding problem and provide a 2D plot of the *magnitude* of the result with the ranges $\xi \rightarrow [-0.75, .75]$ and $\eta \rightarrow [-0.75, .75]$. Also show cross-sectional plots along the ξ and η axes.

3. Let's redo problem 1 from homework 4, but now prove that $rect_1(x) * rect_2(x) = tri(x)$ using the properties of Fourier transforms and the table of common transforms in the notes.

4. A linear shift invariant system has an impulse response $h(x) = 7sinc(7x)$. Find the output $g(x)$ for the input $f(x) = cos(4\pi x)$. Use the Fourier domain approach to find the answer.

5. Find the 0^{th} order Hankel transform of $f(r) = cyl(2r) \bigstar\bigstar\, cyl(2r)$.

Formula Sheet

$f(x,y) ** h(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h(x-\alpha,y-\beta)d\alpha d\beta$	Convolution
$f(x,y) \bigstar\bigstar h(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h(\alpha-x,\beta-y)d\alpha d\beta$	Correlation
$\gamma_{fh}(x,y) = f(x,y) \bigstar\bigstar h^{*}(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h^{*}(\alpha-x,\beta-y)d\alpha d\beta$	Complex Correlation
$\mathcal{F}_{2D}\{f(x,y)\} = F(\xi,\eta) = \iint_{-\infty}^{\infty} f(x,y)exp[-i2\pi(\xi x+\eta y)]dxdy$	2D Fourier Transform
$\mathcal{F}_{2D}^{-1}\{f(\xi,\eta)\} = f(x,y) = \iint_{-\infty}^{\infty} F(\xi,\eta)exp[i2\pi(\xi x+\eta y)]d\xi d\eta$	Inverse Fourier Transform 2D
$\mathcal{H}_0\{f(r)\} = F(\rho) = 2\pi\int_0^{\infty} f(r)J_0(2\pi\rho r)rdr$	0th Order Hankel Transform
$cos(a) = \frac{1}{2}[exp(ia)+exp(-ia)]$ $sin(a) = \frac{1}{2i}[exp(ia)-exp(-ia)]$	Trig Identities
$rect(x) = \begin{cases} 0 & \lvert x\rvert > 1/2 \\ 1/2 & \lvert x\rvert = 1/2 \\ 1 & \lvert x\rvert < 1/2 \end{cases}$ $tri(x) = \begin{cases} 0 & \lvert x\rvert \geq 1 \\ 1-\lvert x\rvert & \lvert x\rvert < 1 \end{cases}$ $sinc(x) = \frac{sin(\pi x)}{\pi x}$ $Gaus(x) = exp[-\pi x^2]$ $\frac{1}{\lvert b\rvert} comb\left(\frac{x-x_o}{b}\right) = \sum_{n=-\infty}^{\infty} \delta(x-x_o-nb)$ $cyl(r) = \begin{cases} 0 & \lvert r\rvert > 1/2 \\ 1/2 & \lvert r\rvert = 1/2 \\ 1 & 0 \leq \lvert r\rvert < 1/2 \end{cases}$ $somb(r) = \frac{2J_1(\pi r)}{\pi r}$	Common Special Functions

$\delta(x-x_o)=0\ \ for\ x\neq x_o$ $\int_{x_1}^{x_2} f(\alpha)\delta(\alpha-x_o)d\alpha=f(x_o)\ \ for\ x_1<x_o<x_2$ $\delta\left(\frac{x-x_o}{b}\right)=\lvert b\rvert\delta(x-x_o)$ $f(x)\delta(x-x_o)=f(x_o)\delta(x-x_o)$ $\int_{-\infty}^{\infty} exp[-i2\pi(\xi-\xi_o)x]dx=\delta(\xi-\xi_o)$	Properties of Delta Functions
$f(x)=\sum_{m=-\infty}^{\infty} a_m exp[i2\pi m\xi_o x]\quad with\ \xi_o=\frac{1}{X}\ and\ X=period$ $where\ a_m=\frac{1}{X}\int_{-X/2}^{X/2} f(x)\ \ exp[-i2\pi m\xi_o x]dx$	Complex Fourier Series
$\mathcal{F}_{2D}\{rect(x,y)\}=sinc(\xi,\eta)$ $\mathcal{F}_{2D}\{tri(x,y)\}=sinc^2(\xi,\eta)$ $\mathcal{F}_{2D}\{Gaus(x,y)\}=Gaus(\xi,\eta)$ $\mathcal{F}_{2D}\{comb(x,y)\}=comb(\xi,\eta)$ $\mathcal{F}_{2D}\{\delta(x\pm x_o,y\pm y_o)\}=exp[\pm i2\pi x_o\xi]exp[\pm i2\pi y_o\eta]$ $\mathcal{F}_{2D}\{exp[\pm i2\pi\xi_o x]exp[\pm i2\pi\eta_o y]\}=\delta(\xi\mp\xi_o,\eta\mp\eta_o)$ $\mathcal{F}_{2D}\{cos(2\pi\xi_o x)\}=\frac{1}{2}[\delta(\xi-\xi_o)+\delta(\xi+\xi_o)]\delta(\eta)$ $\mathcal{F}_{2D}\{sin(2\pi\xi_o x)\}=\frac{1}{2i}[\delta(\xi-\xi_o)-\delta(\xi+\xi_o)]\delta(\eta)$ $\mathcal{H}_0\{cyl(r)\}=\frac{\pi}{4}somb(\rho)$	Common 2D Fourier Transforms

$\mathcal{F}_{2D}\{f(\pm x, \pm y)\} = F(\pm\xi, \pm\eta)$ $\mathcal{F}_{2D}\{f^*(\pm x, \pm y)\} = F^*(\mp\xi, \mp\eta)$ $\mathcal{F}_{2D}\{F(\pm x, \pm y)\} = f(\mp\xi, \mp\eta)$ $\mathcal{F}_{2D}\{F^*(\pm x, \pm y)\} = f^*(\pm\xi, \pm\eta)$ $\mathcal{F}_{2D}\{f_1(x)f_2(y)\} = \mathcal{F}_{1D}\{f_1(x)\}\mathcal{F}_{1D}\{f_2(y)\} = F_1(\xi)F_2(\eta)$ $\mathcal{F}_{2D}\{f_1(x)\} = F_1(\xi)\delta(\eta)$ $\mathcal{F}_{2D}\{f_2(y)\} = \delta(\xi)F_2(\eta)$ $\mathcal{F}_{2D}\left\{f\left(\frac{x}{b}, \frac{y}{d}\right)\right\} = \lvert bd\rvert F(b\xi, d\eta)$ $\mathcal{F}_{2D}\{f(x \pm x_o, y \pm y_o)\} = exp[\pm i2\pi x_o\xi]exp[\pm i2\pi y_o\eta]F(\xi, \eta)$ $\mathcal{F}_{2D}\left\{exp[\pm i2\pi\xi_o x]exp[\pm i2\pi\eta_o y]f\left(\frac{x \pm x_o}{b}, \frac{y \pm y_o}{b}\right)\right\}$ $= \lvert bd\rvert exp[\pm i2\pi x_o(\xi \mp \xi_o)]exp[\pm i2\pi y_o(\eta \mp \eta_o)]$ $\times F\big(b(\xi \mp \xi_o), d(\eta \mp \eta_o)\big)$ $\iint_{-\infty}^{\infty} f(x, y)dxdy = F(0,0) \; and \; \iint_{-\infty}^{\infty} F(\xi, \eta)d\xi d\eta = f(0,0)$ $\mathcal{H}_0\left\{f\left(\frac{r}{b}\right)\right\} = \lvert b\rvert^2 F(b\rho)$	General Properties of 2D Fourier & Hankel Transforms
$\mathcal{F}_{2D}\{f(x, y) ** h(x, y)\} = F(\xi, \eta)H(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y)h(x, y)\} = F(\xi, \eta) ** H(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y)\bigstar\bigstar h(x, y)\} = F(\xi, \eta)H(-\xi, -\eta)$ $\mathcal{F}_{2D}\{f(x, y)h(-x, -y)\} = F(\xi, \eta)\bigstar\bigstar H(\xi, \eta)$ $\mathcal{F}_{2D}\{\gamma_{fh}(x, y)\} = \mathcal{F}_{2D}\{f(x, y)\bigstar\bigstar h^*(x, y)\} = F(\xi, \eta)H^*(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y)h^*(x, y)\} = F(\xi, \eta)\bigstar\bigstar H^*(\xi, \eta) = \gamma_{FH}(\xi, \eta)$ $\mathcal{H}_0\{f(r) ** h(r)\} = \mathcal{H}_0\{f(r)\bigstar\bigstar h(r)\} = F(\rho)H(\rho)$ $\mathcal{H}_0\{f(r)h(r)\} = F(\rho) ** H(\rho) = F(\rho)\bigstar\bigstar H(\rho)$ $\mathcal{H}_0\{\gamma_{fh}(r)\} = \mathcal{H}_0\{f(r)\bigstar\bigstar h^*(r)\} = F(\rho)H^*(\rho)$	2D Transforms of Products, Correlations and Convolutions

Direction Cosines

$$\alpha = cos\theta_x;\ \beta = cos\theta_y; \gamma = cos\theta_z$$

Plane Wave with amplitude A and direction cosines (α, β, γ)

$$U(x, y, z) = Aexp\left[i\frac{2\pi}{\lambda}(\alpha x + \beta y + \gamma z)\right]$$

Spherical Wave with amplitude A centered on origin

$$U(x,y,z) = \frac{A}{r_{01}} exp(\pm ikr_{01});\ \ r_{01} = \sqrt{x^2+y^2+z^2};\ \ +diverging/-converging$$

Parabolic Approximation to Spherical Wave with amplitude A centered on (0,0,0)

$$U(x,y,z) = \frac{A}{z} exp(\pm ikz) exp\left[\pm \frac{i\pi}{\lambda z}(x^2+y^2)\right];\ \ +diverging/-converging$$

Gaussian Beam of amplitude A

$$U(x,y,z) = A\frac{w_o}{w(z)} exp\left(-\frac{x^2+y^2}{w^2(z)}\right) exp\left[i\left(kz + k\frac{x^2+y^2}{2R(z)} - \Phi(z)\right)\right]$$

$$z_o = \frac{kw_o^2}{2};\ \ w(z) = w_o\left(1+\frac{z^2}{z_o^2}\right)^{1/2};\ R(z) = z\left(1+\frac{z_o^2}{z^2}\right);\ \Phi(z) = tan^{-1}\left(\frac{z}{z_o}\right);\ \theta = \frac{\lambda}{\pi w_o}$$

Angular Spectrum Propagation

$$A(\xi,\eta;z) = A(\xi,\eta;0) exp\left(i2\pi z\sqrt{\frac{1}{\lambda^2} - \xi^2 - \eta^2}\right)$$

Fresnel Approximation to Angular Spectrum Propagation

$$A(\xi,\eta;z) = A(\xi,\eta;0) exp(ikz) exp[-i\pi\lambda z(\xi^2+\eta^2)]$$

Fraunhofer Diffraction

$$U(x,y,z) = \frac{exp(ikz)}{i\lambda z} exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right] \mathcal{F}_{2D}\{U(x_o,y_o,0)\};\quad \xi = \frac{x}{\lambda z}, \eta = \frac{y}{\lambda z}$$

Fresnel Diffraction Free space Propagation

$$U(x,y,z) = \frac{exp(ikz)}{i\lambda z} exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right] \mathcal{F}_{2D}\left\{U(x_o,y_o,0) exp\left[\frac{i\pi}{\lambda z}(x_o^2+y_o^2)\right]\right\};$$
$$\xi = \frac{x}{\lambda z}, \eta = \frac{y}{\lambda z}$$

Fresnel Diffraction Object Against a Lens

$$U(x,y,f) = \frac{exp(ikf)}{i\lambda f} exp\left[\frac{i\pi}{\lambda f}(x^2+y^2)\right] \mathcal{F}_{2D}\{U(x_o,y_o,0)P(x_o,y_o)\};$$
$$\xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}$$

Fresnel Diffraction Object In Front of Lens

$$U(x,y,f+d) = \frac{exp\big(ik(f+d)\big)}{i\lambda f} exp\left[\frac{i\pi}{\lambda f}\left(1-\frac{d}{f}\right)(x^2+y^2)\right]$$
$$\times\ \mathcal{F}_{2D}\left\{U(x_o,y_o,0)P\left(x_o+\frac{d}{f}x, y_o+\frac{d}{f}y\right)\right\}; \quad \xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}$$

Fresnel Diffraction Object Behind a Lens

$$U(x,y,f) = \frac{1}{i\lambda d}\frac{Af}{d} exp\left[\frac{i\pi}{\lambda d}(x^2+y^2)\right] \mathcal{F}_{2D}\left\{U(x_o,y_o,0)P\left(x_o\frac{f}{d}, y_o\frac{f}{d}\right)\right\};$$
$$\xi = \frac{x}{\lambda d}, \eta = \frac{y}{\lambda d}$$

Geometrical Image through an optical system with magnification m

$$U_g(x_i,y_i,z') = -\frac{exp\big(ik(z'-z)\big)}{|m|} U_o\left(\frac{x_i}{m},\frac{y_i}{m},z\right);$$
$$I_g(x_i,y_i,z') = \frac{1}{|m|^2}\left|U_o\left(\frac{x_i}{m},\frac{y_i}{m},z\right)\right|^2$$

Impulse Response of Diffraction Limited System

$$\tilde{h}(x_i,y_i) = \frac{1}{\lambda^2 z'^2}\mathcal{F}_{2D}\{P(x_l,y_l)\}; \quad \xi = \frac{x_i}{\lambda z'}, \eta = \frac{y_i}{\lambda z'}$$

Definitions for Coherent and Incoherent Systems

$$CTF(\xi,\eta) = \mathcal{F}_{2D}\{\tilde{h}(x_i,y_i)\}$$
$$PSF(x_i,y_i) = \left|\tilde{h}(x_i,y_i)\right|^2$$
$$OTF(\xi,\eta) = \mathcal{F}_{2D}\{PSF(x_i,y_i)\}; Normalize\ so\ OTF(0,0) = 1$$
$$MTF(\xi,\eta) = |OTF(\xi,\eta)|$$
$$PTF(\xi,\eta) = Arg\big(OTF(\xi,\eta)\big)$$

Coherent Imaging System

$$U(x_i, y_i, z') = \tilde{h}(x_i, y_i) ** U_g(x_i, y_i, z')$$

$$A(\xi, \eta; z') = CTF(\xi, \eta) A_g(\xi, \eta; z'),$$

where the A's are the Fourier transforms of the respective fields.

Incoherent Imaging System

$$I(x_i, y_i, z') = PSF(x_i, y_i) ** I_g(x_i, y_i, z')$$

$$A(\xi, \eta; z') = OTF(\xi, \eta) A_g(\xi, \eta; z'),$$

where the A's are the Fourier transforms of the respective irradiances.

Incoherent Diffraction Limited OTF for Circular Pupil

$$OTF(\beta) = \frac{2}{\pi}\left[cos^{-1}(\beta) - \beta\sqrt{1-\beta^2}\right] \; where\ \beta = (\lambda F_w/\#)\rho \; and\ \rho \leq 1/\lambda F_w/\#$$

Otherwise $OTF(\beta) = 0$.