Harmonic Analysis

We typically describe 1D signals that vary in time in the time domain (i.e a function $f(t)$) and 2D signals that vary in space in the spatial domain (i.e. a function $f(x,y)$). Examples of time domain representations include voltage fluctuations in time coming from a microphone recording an audio signal. In the spatial domain, an example would be the irradiance pattern across the x,y plane of a camera sensor. Harmonic analysis is basically an alternative representation of these signals in the temporal-frequency and spatial-frequency domains. Essentially, the signals are broken up into a combination of sines and cosines. These sines and cosines can be described uniquely by their frequencies. The frequency domain describes how much of each sine and cosine is present (amplitude). In the spatial frequency domain, we also need to describe the orientation of the sinusoidal patterns. Harmonic Analysis describes how to convert function to the frequency domain.

Orthogonal Functions

An understanding of orthogonal functions is useful for developing the concepts of harmonic analysis. In 1D, a set of orthogonal functions $\{g_m(x)\}$ (also called basis functions) satisfies the following:

$$\int_{x_1}^{x_2} g_{m'}(x)\, g_m^*(x)\, dx = \begin{cases} C_m & \text{for } m = m' \\ 0 & \text{otherwise} \end{cases}$$

where $[x_1, x_2]$ is the domain of the orthogonality, C_m is a constant and m and m' are just indices to determine which member of the set is used.

Suppose we want to represent a function $f(x)$ as a linear combination of the orthogonal functions $\{g_m(x)\}$. We can write this as

$$f(x) = \sum_{m'} a_{m'} g_{m'}(x)$$

where $a_{m'}$ is a coefficient that describes how much of $g_{m'}(x)$ is present in $f(x)$. A method for finding these coefficients a_m is needed. Orthogonal functions makes this easy.

Multiply both sides by $g_m^*(x)$

$$f(x) g_m^*(x) = \sum_{m'} a_{m'} g_{m'}(x) g_m^*(x)$$

Integrate over the domain $[x_1, x_2]$

$$\int_{x_1}^{x_2} f(x) g_m^*(x)\, dx = \int_{x_1}^{x_2} \left[\sum_{m'} a_{m'} g_{m'}(x) g_m^*(x) \right] dx$$

The integral of a sum is the sum of the integrals

$$\int_{x_1}^{x_2} f(x) g_m^*(x)\, dx = \sum_{m'} a_{m'} \int_{x_1}^{x_2} g_{m'}(x) g_m^*(x)\, dx$$

All of the integrals on the right are zero except when $m = m'$

$$\int_{x_1}^{x_2} f(x) g_m^*(x)\, dx = a_m c_m$$

$$\boxed{a_m = \frac{1}{c_m} \int_{x_1}^{x_2} f(x) g_m^*(x)\, dx}$$

← projection of $f(x)$ onto basis functions $g_m(x)$

Example, consider the set of functions

$$\{ \ldots \sin 6\pi \xi_0 x ,\ \sin 4\pi \xi_0 x ,\ \sin 2\pi \xi_0 x ,\ 1 ,\ \cos 2\pi \xi_0 x ,\ \cos 4\pi \xi_0 x ,\ \cos 6\pi \xi_0 x \ldots \}$$

For short hand define $\Theta_m (2\pi \xi_0 x) = \begin{cases} \sin(2\pi |m| \xi_0 x) & m < 0 \\ \cos(2\pi m \xi_0 x) & m \geq 0 \end{cases}$

Define period $X = \dfrac{1}{\xi_0}$

Real functions so no need to worry about the complex conjugate

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \Theta_{m'} (2\pi \xi_0 x) \Theta_m (2\pi \xi_0 x)\, dx$$

Check to see if this set is orthogonal

CASE 1 $m' < 0 \quad m \geq 0$

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \sin(2\pi |m'| \xi_0 x) \cos(2\pi m \xi_0 x)\, dx = 0$$

since sin is odd and cos is even, their product in an odd function

CASE 2 $m' < 0$ $m < 0$ and CASE 3 $m' > 0$ $m > 0$ can be integrated and shown to be zero as long as $m' \neq m$.

What happens when $m = m'$

CASE 4 $m < 0 \quad m = m'$

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \sin^2 (2\pi |m| \xi_0 x) = \left[\frac{x}{2} - \frac{\sin(4\pi m \xi_0 x)}{8\pi m \xi_0} \right]_{-\frac{X}{2}}^{\frac{X}{2}} = \frac{X}{2} = \frac{1}{2\xi_0}$$

CASE 5 $m > 0 \quad m = m'$ Note $m \neq 0$

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \cos^2 (2\pi m \xi_0 x) = \left(\frac{x}{2} + \frac{\sin(4\pi m \xi_0 x)}{8\pi m \xi_0} \right]_{-\frac{X}{2}}^{\frac{X}{2}} = \frac{1}{2\xi_0}$$

Case 6 $m = m' = 0$

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} 1 \cdot dx = X = \frac{1}{\xi_0}$$

Kronecker delta function

$$\delta_{mm'} = \begin{cases} 1 & m = m' \\ 0 & \text{otherwise} \end{cases}$$

Summary of Results

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \Theta_{m'}(2\pi \xi_0 x)\, \Theta_m(2\pi \xi_0 x)\, dx = \begin{cases} \dfrac{1+\delta_{m0}}{2\xi_0} & m = m' \\ 0 & \text{otherwise} \end{cases}$$

ORTHOGONAL FOR $-\frac{X}{2} \le x \le \frac{X}{2}$

Turns out for this set $\Theta_m(2\pi\xi_0 x)$ is orthogonal for any lower limit x_1 as long as $x_2 = x_1 + NX$, N integer i.e. the domain is an integer number of periods.

This previous example is just a Fourier series. A function $f(x)$ is

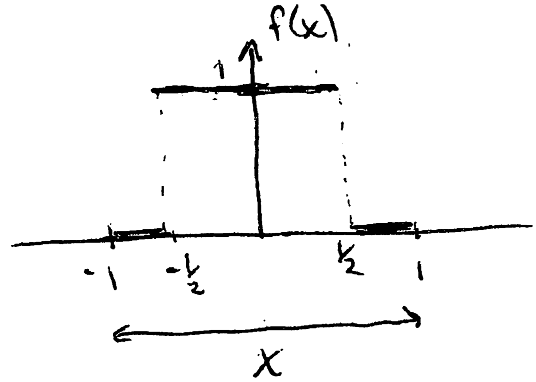
$$\boxed{\begin{aligned} f(x) &= \sum_m a_m \Theta_m(2\pi\xi_0 x) \\ \text{where} \quad a_m &= \frac{2\xi_0}{1+\delta_{m0}} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\, \Theta_m(2\pi\xi_0 x)\, dx \end{aligned}}$$

Note: A lot of math books will write this something along the lines of

$$f(x) = \frac{a_0}{2} + \sum_{N=1}^{\infty} a_N \cos Nx + \sum_{N=1}^{\infty} b_N \sin Nx$$

which is equivalent.

Example Fourier Series



$$f(x) = \begin{cases} 1 & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < |x| \le 1 \end{cases}$$

$$X = 2 \Rightarrow \xi_0 = \frac{1}{2}$$

So the Fourier series for $X = 2$ is determined from the expansion coefficients

$$a_m = \frac{2\left(\frac{1}{2}\right)}{1 + \delta_{m0}} \int_{-1}^{1} f(x)\, \Theta_m\!\left(2\pi\left(\tfrac{1}{2}\right)x\right) dx$$

$$a_0 = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx = \frac{1}{2}$$

Case $m < 0$

Note $f(x)$ is an even function and $\sin(2\pi m \xi_0 x)$ is and odd function, so the integral is always zero since their product is odd.

$$a_m = 0 \quad \text{for } m < 0$$

Case $m > 0$

Here, both $f(x)$ and $\cos(2\pi m \xi_0 x)$ are even, so their product is even

$$a_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos\left(2\pi m \tfrac{1}{2} x\right) dx = 2 \int_0^{\frac{1}{2}} \cos(m\pi x)\, dx = 2\, \frac{\sin(m\pi x)}{m\pi} \Bigg|_0^{\frac{1}{2}}$$

$$a_m = \frac{\sin\left(\frac{m\pi}{2}\right)}{\frac{m\pi}{2}} = \operatorname{sinc}\left(\frac{m}{2}\right) \quad \text{for } m > 0$$

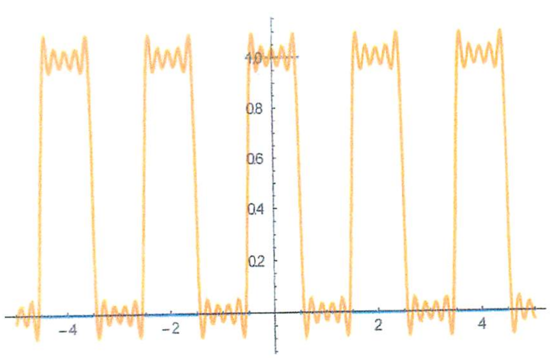
Continuing, recall when we defined the sinc() function that it is zero whenever its argument is a non-zero integer.

So when m is even, $a_m = 0$

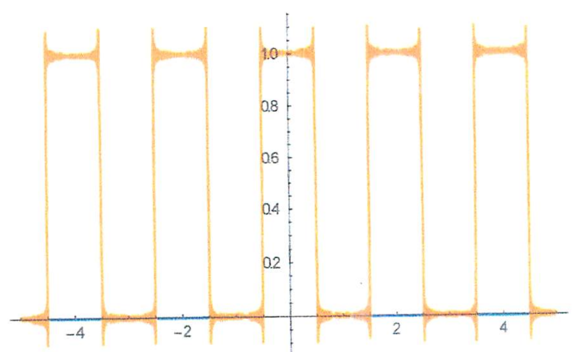
Putting this all together

$$a_m = \begin{cases} \frac{1}{2} & m = 0 \\ \mathrm{sinc}\left(\frac{m}{2}\right) & m = 1, 3, 5 \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \sum_m a_m \textcircled{c}_m\left(2\pi\left(\tfrac{1}{2}\right)x\right) = \frac{1}{2} + \mathrm{sinc}\left(\frac{1}{2}\right)\cos(\pi x) + \mathrm{sinc}\left(\frac{3}{2}\right)\cos(3\pi x) + \ldots$$



10 COSINE TERMS



100 COSINE TERMS

Fourier series creates a periodic reproduction of f(x) with a period X. Discontinuities in f(x) create a "ringing" effect (Gibb's Phenomenon) so an infinite number of terms are needed to fit these points.

Fourier Series are useful for representing <u>periodic</u> functions. They can also represent non-periodic functions, but only over the isolated region of width X.

Complex Fourier Series $\{g_m(x)\} = \{\exp(i2\pi m \xi_0 x)\}$

These are orthogonal

$$\int_{-\frac{X}{2}}^{\frac{X}{2}} \exp\left[i2\pi m' \xi_0 x\right] \exp\left[-i2\pi m \xi_0 x\right] dx = \begin{cases} X & m' = m \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\begin{aligned} f(x) &= \sum_m a_m e^{i2\pi m \xi_0 x} \\ a_m &= \frac{1}{X} \int_{-\frac{X}{2}}^{X/2} f(x) \exp\left[-i2\pi m \xi_0 x\right] dx \end{aligned}}$$

Complex Fourier series can be used to represent complex periodic functions (i.e. the real and imaginary parts are both periodic in X). In general, the coefficients a_m will be complex. We can still represent real periodic functions with a complex Fourier series.

Consider the previous example

$$a_m = \frac{1}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \exp\left[-i2\pi m \xi_0 x\right] dx \qquad f(x) = \begin{cases} 1 & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} < |x| \le 1 \end{cases}$$

$$X = 2 \rightarrow \xi_0 = \frac{1}{2}$$

$$a_m = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \exp\left(-i\pi m x\right] dx = \frac{1}{2} \operatorname{sinc}\left[\frac{m}{2}\right]$$

$$f(x) = \ldots + \frac{1}{2}\operatorname{sinc}\left(\frac{-3}{2}\right)\exp(-i3\pi x) + \frac{1}{2}\operatorname{sinc}\left(\frac{-1}{2}\right)\exp(-i\pi x) + \frac{1}{2} + \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}\right)\exp(i\pi x) + \frac{1}{2}\operatorname{sinc}\left(\frac{3}{2}\right)\exp(i3\pi x) + \ldots$$

$$f(x) = \frac{1}{2} + \operatorname{sinc}\left(\frac{1}{2}\right)\cos(\pi x) + \operatorname{sinc}\left(\frac{3}{2}\right)\cos(3\pi x) + \ldots$$ Just like previous

COMPARISON BETWEEN FOURIER SERIES AND COMPLEX FOURIER SERIES

FOURIER SERIES

$$f(x) = \sum_m a_m \Theta_m(2\pi \xi_0 x)$$

where
$$a_m = \frac{2\xi_0}{1+\delta_{m0}} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\,\Theta_m(2\pi\xi_0 x)\,dx$$

$$\Theta_m(2\pi\xi_0 x) = \begin{cases} \sin(2\pi |m| \xi_0 x) & m<0 \\ \cos(2\pi m \xi_0 x) & m \geq 0 \end{cases}$$

$$X = \text{period} = \frac{1}{\xi_0}$$

can represent real functions over interval $-\frac{X}{2} \leq x \leq \frac{X}{2}$ or periodic real functions with period X.

COMPLEX FOURIER SERIES

$$f(x) = \sum_m b_m \exp(i 2\pi m \xi_0 x)$$

where
$$b_m = \frac{1}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \exp(-i2\pi m \xi_0 x)\,dx$$

$$\exp(i2\pi m\xi_0 x) = \cos(2\pi m \xi_0 x) + i\sin(2\pi m \xi_0 x)$$

$$X = \text{period} = \frac{1}{\xi_0}$$

Can represent real or complex functions on $-\frac{X}{2} \leq x \leq \frac{X}{2}$ or periodic real or complex functions with period X.

m	-3	-2	-1	0	1	2
FOURIER	$\sin(6\pi\xi_0 x)$	$\sin(4\pi\xi_0 x)$	$\sin(2\pi\xi_0 x)$	1	$\cos(2\pi\xi_0 x)$	$\cos(4\pi\xi_0 x)$
COMPLEX FOURIER	$\exp(-i6\pi\xi_0 x)$	$\exp(-i4\pi\xi_0 x)$	$\exp(-i2\pi\xi_0 x)$	1	$\exp(i2\pi\xi_0 x)$	$\exp(i4\pi\xi_0 x)$

Suppose we represent $f(x) = \sum_m a_m \Theta_m (2\pi \xi_0 x)$ with

$$a_m = \frac{2\xi_0}{1+\delta_{m0}} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \Theta_m (2\pi \xi_0 x)\, dx$$

FOR $m = 0$

$$a_0 = \xi_0 \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \Theta_0 (2\pi \xi_0 x)\, dx = \frac{1}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\, dx$$

FOR $m > 0$

$$a_{m>0} = 2\xi_0 \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \Theta_{m>0} (2\pi \xi_0 x)\, dx = \frac{2}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \cos (2\pi m \xi_0 x)\, dx$$

FOR $m < 0$

$$a_{m<0} = 2\xi_0 \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \Theta_{m<0} (2\pi \xi_0 x)\, dx = \frac{2}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \sin (2\pi |m| \xi_0 x)\, dx$$

Suppose we want to represent $f(x)$ now as a complex Fourier series

$$f(x) = \sum_m b_m \exp (i 2\pi m \xi_0 x) \quad \text{with}$$

$$b_m = \frac{1}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x) \exp (i 2\pi m \xi_0 x)\, dx$$

FOR $m = 0$

$$b_0 = \frac{1}{X} \int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\, dx = a_0$$

Same coefficients for Fourier and complex Fourier. Represents average value of the function over interval

FOR $m>0$

$$e^{\pm ix} = \cos x \pm i \sin x$$

$$b_{m>0} = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\exp(i2\pi m \xi_0 x)\,dx$$

$m=|m|$ when $m>0$

$$b_{m>0} = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\cos(2\pi|m|\xi_0 x)\,dx + i\frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\sin(2\pi|m|\xi_0 x)\,dx$$

$$b_{m>0} = \frac{1}{2\cancel{X}}a_{|m|} + i\frac{1}{2}a_{-|m|}$$

FOR $m<0$

$m=-|m|$ when $m<0$

$$b_{m<0} = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\exp(-i2\pi|m|\xi_0 x)\,dx$$

$$b_{m<0} = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\cos(2\pi|m|\xi_0 x)\,dx - i\frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\sin(2\pi|m|\xi_0 x)\,dx$$

$$b_{m<0} = \frac{1}{2}a_{|m|} - i\frac{1}{2}a_{-|m|}$$

SUMMARY

$a_{|m|}$ FOURIER COSINE COEFFICIENTS

COMPLEX FOURIER COEFFICIENTS $\rightarrow$

$$b_m = \begin{cases} a_0 & m=0 \\ \frac{1}{2}(a_{|m|} + ia_{-|m|}) & m>0 \\ \frac{1}{2}(a_{|m|} - ia_{-|m|}) & m<0 \end{cases}$$

$a_{-|m|}$ FOURIER SINE COEFFICIENTS

Fourier series with complex exponentials is preferred to using sines and cosines since math is easier (no trig identities required) and now both real and complex periodic functions can be easily represented.

FOURIER INTEGRAL

What if we want to represent a non-periodic function everywhere in space? Our answer comes from the complex Fourier series. Now we know for the Fourier series, the function $f(x)$ repeats itself at intervals of X. What happens if we make X really big? The function $f(x)$ never has a chance to repeat as $X \to \infty$. Also ξ_0 becomes infinitesimally small for $X \to \infty$, so this should be evoking thoughts of sums going to integrals in the limit.

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(\xi) e^{i2\pi\xi x} \, d\xi}$$

← $F(\xi)$ acting like our coefficients which describe how much $\sin(2\pi\xi x)$ and $\cos(2\pi\xi x)$ are present in $f(x)$

where

$$\boxed{F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi\xi x} \, dx}$$

← $F(\xi)$ is found by projecting $f(x)$ onto the basis function (complex conjugate)

These are called the Fourier integrals and are useful for represent both periodic and non-periodic functions across all space.

$F(\xi)$ is called the Fourier transform of $f(x)$ or frequency spectrum

$f(x)$ is also called the inverse Fourier transform of $F(\xi)$

Instead of discrete sines and cosines (complex exponentials) we now have a continuum of sines and cosines, so coefficients $F(\xi)$ continuous.

Fourier transform of a periodic function

A general periodic function with fundamental frequency ξ_0 is given by

$$f(x) = \sum_{m=-\infty}^{\infty} a_m e^{i2\pi m \xi_0 x}$$

The Fourier transform is

$$F(\xi) = \int_{-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} a_m e^{i2\pi m \xi_0 x} \right] e^{-i2\pi \xi x} \, dx$$

$$F(\xi) = \sum_{m=-\infty}^{\infty} a_m \int_{-\infty}^{\infty} e^{-i2\pi(\xi - m\xi_0)x} \, dx$$

From the orthogonality of the complex exponentials

$$\boxed{\int_{-\infty}^{\infty} e^{-i2\pi(\xi - m\xi_0)x} \, dx = \delta(\xi - m\xi_0)}$$

This relationship appears repeatedly

So

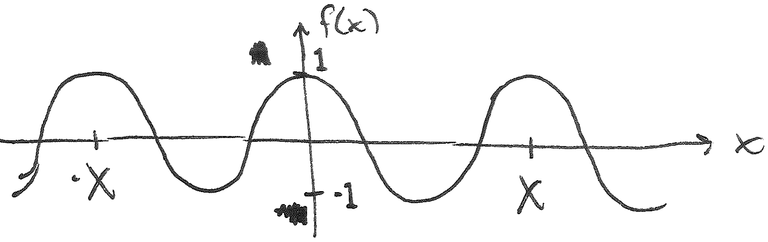
$$\boxed{F(\xi) = \sum_{m=-\infty}^{\infty} a_m \delta(\xi - m\xi_0)}$$

Fourier transform of a periodic function is a discrete set of delta functions with areas given by the coefficients.

In general, non-periodic functions will have a continuous Fourier transform and not discrete delta functions.

Simple periodic function example

$$f(x) = \text{[redacted]} \cos(2\pi \xi_0 x) \qquad \text{with} \quad \xi_0 = \frac{1}{X}$$



Periodic, so we can rewrite as a Fourier series. All terms are zero except for 2

$$f(x) = \frac{1\text{[redacted]}}{2}\left[\exp(i2\pi\xi_0 x) + \exp(-i2\pi\xi_0 x)\right]$$

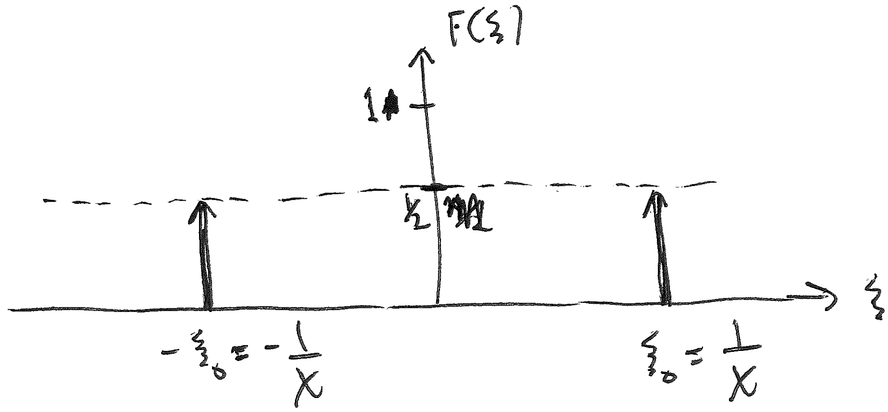
The Fourier transform is

$$F(\xi) = \frac{1\text{[redacted]}}{2}\int_{-\infty}^{\infty} \exp(i2\pi\xi_0 x)\exp(-i2\pi\xi x)\,dx$$

$$+ \frac{1\text{[redacted]}}{2}\int_{-\infty}^{\infty} \exp(-i2\pi\xi_0 x)\exp(-i2\pi\xi x)\,dx$$

$$F(\xi) = \frac{1\text{[redacted]}}{2}\int_{-\infty}^{\infty} \exp(-i2\pi(\xi-\xi_0)x)\,dx + \frac{1\text{[redacted]}}{2}\int_{-\infty}^{\infty}\exp(-i2\pi(\xi+\xi_0)x)\,dx$$

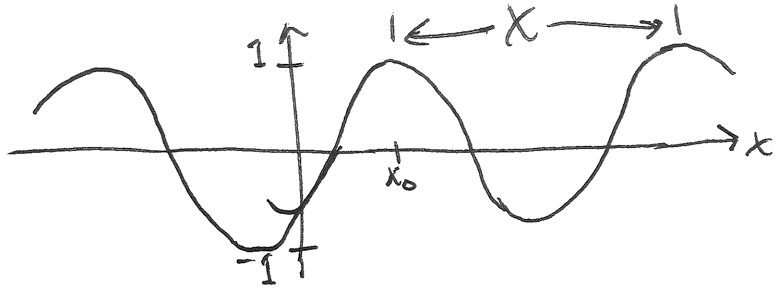
$$F(\xi) = \frac{1\text{[redacted]}}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right] = \frac{1\text{[redacted]}}{2\xi_0}\delta\delta\left(\frac{\xi}{\xi_0}\right)$$



Fourier transform of cosine is a real function consisting of two delta functions at $\pm\xi_0$

Slightly more complicated periodic function example

$$f(x) = \cos(2\pi \xi_0 (x - x_0)) \quad \text{with } \xi_0 = \frac{1}{X}$$



Periodic, so rewrite as Fourier series

$$f(x) = \frac{1}{2}\left[\exp(i2\pi\xi_0(x-x_0)) + \exp(-i2\pi\xi_0(x-x_0))\right]$$

$$f(x) = \frac{1}{2}\left[\exp(-i2\pi\xi_0 x_0)\exp(i2\pi\xi_0 x) + \exp(i2\pi\xi_0 x_0)\exp(-i2\pi\xi_0 x)\right]$$

The Fourier transform is

$$F(\xi) = \frac{1}{2}\exp(-i2\pi\xi_0 x_0)\int_{-\infty}^{\infty}\exp(-i2\pi(\xi-\xi_0)x)\,dx$$
$$+ \frac{1}{2}\exp(i2\pi\xi_0 x_0)\int_{-\infty}^{\infty}\exp(-i2\pi(\xi+\xi_0)x)\,dx$$

$$F(\xi) = \frac{1}{2}\left[\exp(-i2\pi\xi_0 x_0)\,\delta(\xi-\xi_0) + \exp(i2\pi\xi_0 x_0)\,\delta(\xi+\xi_0)\right]$$

$$F(\xi) = \frac{1}{2}\left[\exp(-i2\pi\xi x_0)\,\delta(\xi-\xi_0) + \exp(-i2\pi\xi x_0)\,\delta(\xi+\xi_0)\right]$$

This step uses properties of delta functions from 19

$$\delta(-\alpha+\alpha_0) = \delta(\alpha-\alpha_0)$$

$$f(\alpha)\,\delta(\alpha-\alpha_0) = f(\alpha_0)\,\delta(\alpha-\alpha_0)$$

Finally

$$F(\xi) = \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]\exp\left(-i2\pi\xi x_0\right)$$

Ok so let's write the Fourier transform as

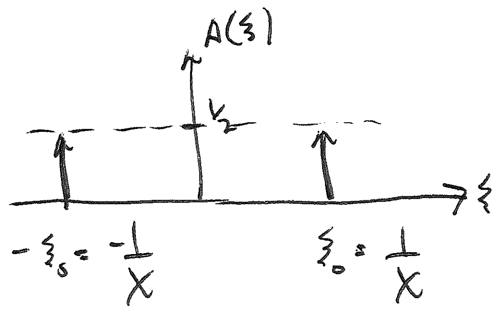
$$F(\xi) = A(\xi)\exp\left(-i\Phi(\xi)\right)$$

where $A(\xi)$ is called the <u>Amplitude Spectrum</u>

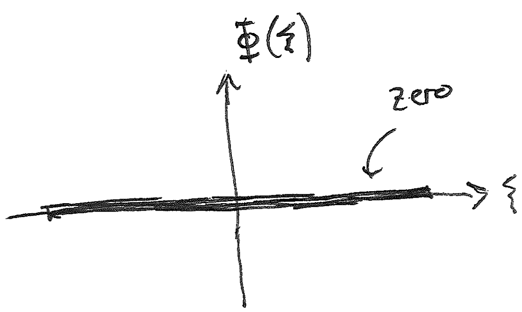
and $\Phi(\xi)$ is called the <u>Phase Spectrum</u>

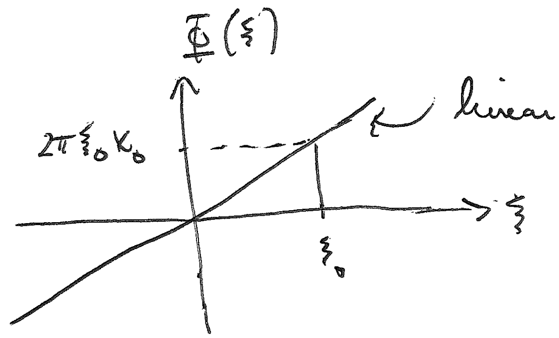
$\Big\}$ Both real functions

<u>$\cos(2\pi\xi_0 x)$</u>	<u>$\cos(2\pi\xi_0(x-x_0))$</u>
$A(\xi) = \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]$	$A(\xi) = \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]$
$\Phi(\xi) = 0$	$\Phi(\xi) = 2\pi\xi x_0$



$\Rightarrow$ same





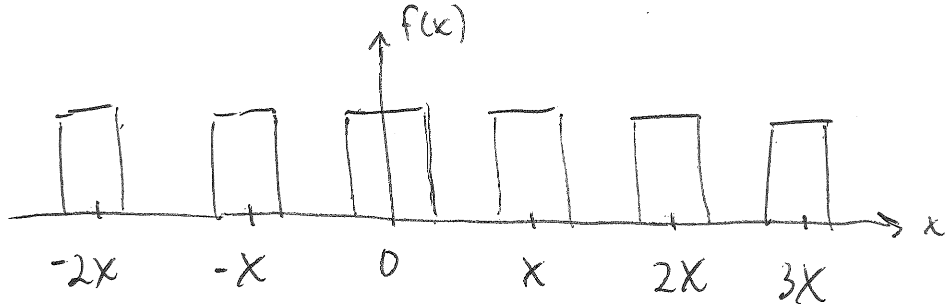
The amplitude spectrum describes the period of the cosine.

The phase spectrum describes the shift of the cosine.

Even more complicated periodic function example

$$f(x) = \sum_{m=-\infty}^{\infty} \text{rect}\left(\frac{x - mX}{X/2}\right)$$

(rectangular) square wave pattern



Periodic, so rewrite as Fourier Series

$$f(x) = \sum_m a_m \exp(i2\pi m \xi_0 x) \qquad \xi_0 = \frac{1}{X}$$

where

$$a_m = \frac{1}{X}\int_{-\frac{X}{2}}^{X/2} f(x) \exp(-i2\pi m \xi_0 x)\, dx$$

$$a_m = \xi_0 \int_{-\frac{X}{4}}^{\frac{X}{4}} \exp(-i2\pi m \xi_0 x)\, dx \qquad \text{note } \frac{X}{4} = \frac{1}{4\xi_0}$$

$$a_m = \cancel{\xi_0} \left. \frac{\exp(-i2\pi m \xi_0 x)}{-i2\pi m \cancel{\xi_0}} \right|_{-\frac{1}{4\xi_0}}^{\frac{1}{4\xi_0}}$$

$$a_m = \frac{1}{-i2\pi m}\left[\exp\left(-\frac{i\pi m}{2}\right) - \exp\left(\frac{i\pi m}{2}\right)\right]$$

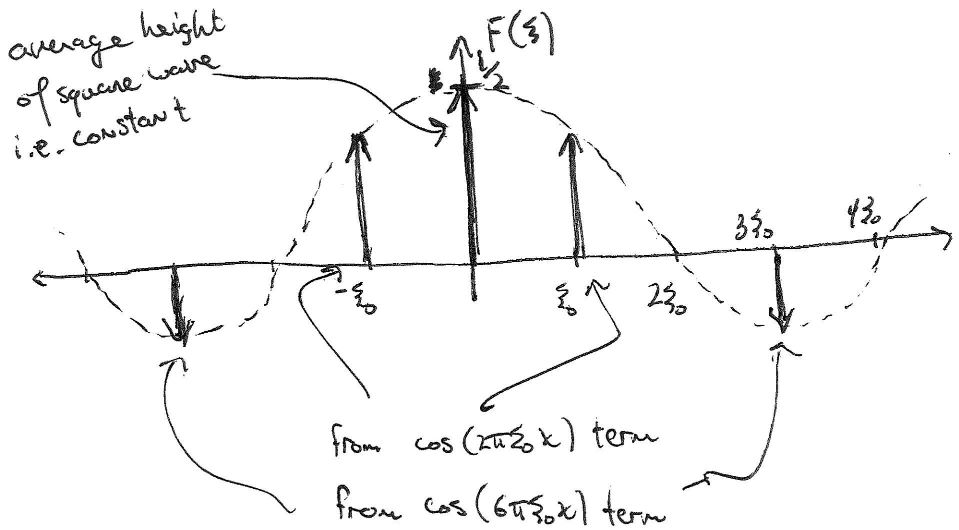
$$a_m = \frac{1}{2}\,\text{sinc}\left(\frac{m}{2}\right)$$

The Fourier transform is

$$F(\xi) = \frac{1}{2}\int_{-\infty}^{\infty}\left[\sum_m \mathrm{sinc}\left(\frac{m}{2}\right)\exp\left(i2\pi m\xi_0 x\right)\right]\exp\left(-i2\pi\xi x\right)dx$$

$$F(\xi) = \frac{1}{2}\sum_m \mathrm{sinc}\left(\frac{m}{2}\right)\int_{-\infty}^{\infty}\exp\left(-i2\pi\left(\xi - m\xi_0\right)x\right)dx$$

$$F(\xi) = \frac{1}{2}\sum_m \mathrm{sinc}\left(\frac{m}{2}\right)\delta\left(\xi - m\xi_0\right)$$



Delta functions modulated by a sinc() function

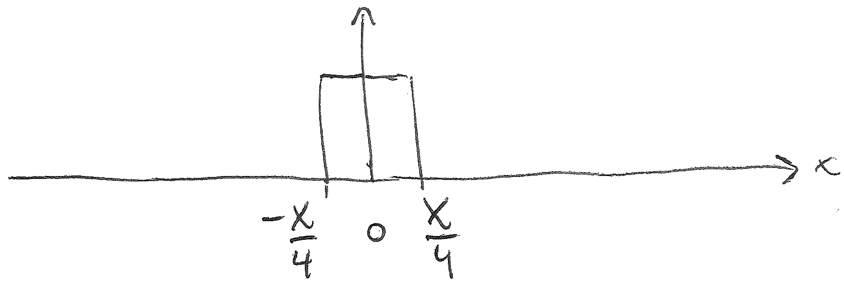
The areas of the delta functions correspond to the coefficients of the Fourier series.

$F(\xi)$ here is a real function, so $A(\xi)$ amplitude spectrum equals $F(\xi)$ and the phase spectrum $\Phi(\xi) = 0$.

As with the previous examples, if the square wave is shifted left or right, the phase spectrum becomes linear, but $A(\xi)$ is unchanged

Final example with a NON-periodic function

$$f(x) = \text{rect}\left(\frac{x}{X/2}\right)$$ single square



No Fourier series since f(x) is not periodic.

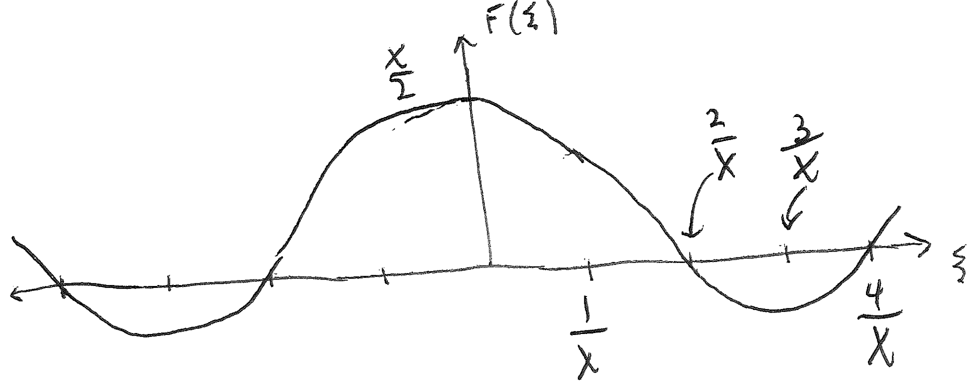
The Fourier transform is

$$F(\xi) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{x}{X/2}\right) \exp(-i2\pi\xi x)\, dx$$

$$F(\xi) = \int_{-\frac{X}{4}}^{X/4} \exp(-i2\pi\xi x)\, dx = \left.\frac{\exp(-i2\pi\xi x)}{-i2\pi\xi}\right|_{-\frac{X}{4}}^{\frac{X}{4}}$$

$$F(\xi) = \frac{1}{-i2\pi\xi}\left[\exp\left(-i2\pi\xi\frac{X}{4}\right) - \exp\left(i2\pi\xi\frac{X}{4}\right)\right]$$

$$F(\xi) = \frac{\frac{X}{2}}{\frac{X}{2}} \frac{\sin\left(\pi\xi\frac{X}{2}\right)}{\pi\xi} = \frac{X}{2}\,\text{sinc}\left(\frac{X\xi}{2}\right)$$



F(ξ) continuous since f(x) not periodic

2D Fourier Integral

So far, we have only looked at the Fourier transform in 1D. In the 1D case, the function $f(x)$ is transformed to a function $F(\xi)$. The variables here are arbitrary. We could have easily used t and ν in place of x and ξ to represent a time signal and its corresponding temporal frequency. In 2D, we are typically concerned with spatial distributions, so Cartesian coordinates (x, y) are often used. Their respective spatial frequencies are given by the variables (ξ, η). The 2D Fourier integrals are defined as

$$f(x,y) = \iint_{-\infty}^{\infty} F(\xi,\eta) \exp\left(i2\pi(\xi x + \eta y)\right) d\xi\, d\eta$$

$$F(\xi,\eta) = \iint_{-\infty}^{\infty} f(x,y) \exp\left(-i2\pi(x\xi + y\eta)\right) dx\, dy$$

$F(\xi,\eta)$ is the <u>2D Fourier Transform</u> or <u>Fourier spectrum</u> of $f(x,y)$

Let's see what happens to cosines in 2D. Find the 2D Fourier transform $H(\xi,\eta)$ of the function $h(x,y)$ where

$$h(x,y) = A + B\cos(2\pi\xi_0 x)$$

Note: This is a 2D function, but independent of y

$$H(\xi,\eta) = \iint_{-\infty}^{\infty} \left(A + B\cos(2\pi\xi_0 x)\right) \exp\left(-i2\pi(x\xi + y\eta)\right) dx\, dy$$

$$H(\xi,\eta) = A\iint_{-\infty}^{\infty} \exp\left(-i2\pi(x\xi + y\eta)\right) dx\, dy + B\iint_{-\infty}^{\infty} \cos(2\pi\xi_0 x) \exp\left(-i2\pi(x\xi + y\eta)\right) dx\, dy$$

$$H(\xi,\eta) = A\int_{-\infty}^{\infty} \exp(-i2\pi x\xi)\,dx \int_{-\infty}^{\infty} \exp(-i2\pi y\eta)\,dy$$

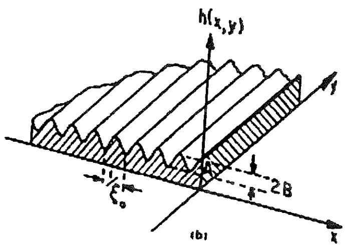
$$+ \frac{B}{2}\left[\int_{-\infty}^{\infty} \exp(-i2\pi(x(\xi-\xi_0)))\,dx + \int_{-\infty}^{\infty}\exp(-i2\pi x(\xi+\xi_0))\,dx\right]\int_{-\infty}^{\infty} \exp(-i2\pi y\eta)\,dy$$

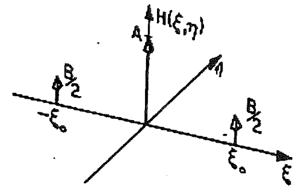
All of these integrals are delta functions

$$H(\xi,\eta) = A\,\delta(\xi)\,\delta(\eta) + \frac{B}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]\delta(\eta)$$

Can also write as

$$H(\xi,\eta) = A\,\delta(\xi,\eta) + \frac{B}{2}\left[\delta(\xi-\xi_0,\eta) + \delta(\xi+\xi_0,\eta)\right]$$





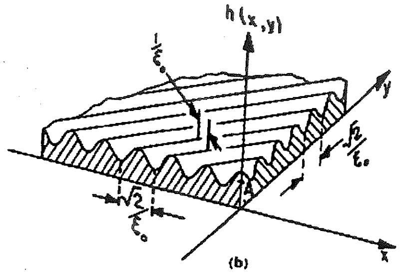
$h(x,y)$ only varies in x-direction and is constant in y-direction. The average height of $h(x,y)$ is A and the amplitude of the cosine is B (i.e. the cosine ranges in height from ($A-B$ to $A+B$). The spatial frequency ξ_0 is related to the period of the cosine along the X-axis as $1/\xi_0$. The Fourier transform $H(\xi,\eta)$ has one delta function at the origin associated with the constant A. It also has a pair of delta functions along the ξ-axis associated with the cosine term much like what we saw with the cosine in 1D.

Now let's rotate the cosine pattern 45°. This can be described as

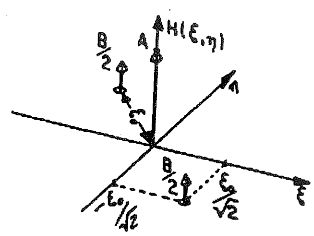
$$h(x,y) = A + B\cos\left[2\pi\left(\frac{\xi_0}{\sqrt{2}}\right)(x-y)\right]$$

$$H(\xi,\eta) = \iint_{-\infty}^{\infty} \left[A + \frac{B}{2}\exp\left(i2\pi\left(\frac{\xi_0}{\sqrt{2}}\right)x\right)\exp\left(-i2\pi\left(\frac{\xi_0}{\sqrt{2}}\right)y\right) + \frac{B}{2}\exp\left(-i2\pi\left(\frac{\xi_0}{\sqrt{2}}\right)x\right)\exp\left(i2\pi\left(\frac{\xi_0}{\sqrt{2}}\right)y\right)\right]\exp\left(-i2\pi(\xi x + \eta y)\right)\,dx\,dy$$

$$H(\xi,\eta) = A\,\delta(\xi,\eta) + \frac{B}{2}\left[\delta\left(\xi - \frac{\xi_0}{\sqrt{2}}, \eta + \frac{\xi_0}{\sqrt{2}}\right) + \delta\left(\xi + \frac{\xi_0}{\sqrt{2}}, \eta - \frac{\xi_0}{\sqrt{2}}\right)\right]$$



(b)



The spectrum $H(\xi,\eta)$ simply rotates 45° degrees as well.

The basic 1D and 2D Fourier integrals have been introduced. We'll expand on Fourier transforming other functions soon, but first need a few more tools. Also, the integrals here are tedious. We will show how to generalize these calculations so that once you know Fourier transform pairs, you can quickly write down the Fourier transform of scaled, shifted and combinations of functions. Here, we learned our first two Fourier transform pairs.

1. a constant becomes a delta function at the origin
2. cosines become pairs of delta functions a distance $\pm\xi_0$ from the origin.