OPTI 512 Homework 3 Due: September 30, 2020

1. This homework is going to analyze the properties of the complex Fourier series and the Fourier transform. For the questions below, the following function will be used

$$f(x) = \frac{1}{4} rect(x) + \frac{3}{4} rect(2x).$$

Plot this function.

2. First, let's figure out the complex Fourier series for this function assuming a period $X_1 = 3/2$.

(a) What is the frequency ξ_o associated with this period?

(b) Determine an expression for the expansion coefficients a_m for the complex Fourier series in terms of a pair of *sinc*() functions.

(c) Armed with the expansion coefficients, a function $g_1(x)$ describing the complex Fourier series of $f(x)$ can be obtained with

$$g_1(x) = \sum_{m=-\infty}^{\infty} a_m exp[i2\pi m \xi_o x].$$

The function $g_1(x)$ perfectly matches the function $f(x)$ over the region $-3/4 \leq x \leq 3/4$ (i.e. over the period X_1). The side effect of representing $f(x)$ in terms of its Fourier series is that outside of this range, the function is repeated at intervals of X_1. The sum is therefore a periodic function and can be expressed as

$$g_1(x) = \sum_{m=-\infty}^{\infty} f(x - mX_1).$$

Plot $g_1(x)$ using this last expression over the range $-5 \leq x \leq 5$.

(d) In class, we showed that the Fourier transform of a periodic function is just a series of delta functions weighted by the Fourier expansion coefficients. In this case, the Fourier transform $G_1(\xi)$ is equal to

$$G_1(\xi) = \sum_{m=-\infty}^{\infty} a_m \delta(\xi - m\xi_o).$$

Plot $X_1 G_1(\xi)$ over the range $-6 \leq \xi \leq 6$.

3. Repeat question 2 now assuming a period $X_2 = 3$.

(a) What is the frequency ξ_o associated with this period?

(b) Determine an expression for the expansion coefficients a_m for the complex Fourier series in terms of a pair of *sinc*() functions.

(c) Armed with the expansion coefficients, a function $g_2(x)$ describing the complex Fourier series of $f(x)$ can be obtained with

$$g_2(x) = \sum_{m=-\infty}^{\infty} a_m exp[i2\pi m\xi_o x].$$

The function $g_2(x)$ perfectly matches the function $f(x)$ over the region $-3/2 \leq x \leq 3/2$ (i.e. over the period X_2). The side effect of representing $f(x)$ in terms of its Fourier series is that outside of this range, the function is repeated at intervals of X_2. The sum is therefore a periodic function and can be expressed as

$$g_2(x) = \sum_{m=-\infty}^{\infty} f(x - mX_2).$$

Plot $g_2(x)$ using this last expression over the range $-5 \leq x \leq 5$.

(d) In this case, the Fourier transform $G_2(\xi)$ is now equal to

$$G_2(\xi) = \sum_{m=-\infty}^{\infty} a_m \delta(\xi - m\xi_o).$$

Plot $X_2 G_2(\xi)$ over the range $-6 \leq \xi \leq 6$.

4. Now let's look at the Fourier transform of $f(x)$.

   (a) Use the Fourier integral to calculate $F(\xi)$ in terms of a pair of *sinc()* functions.

   (b) Plot $F(\xi)$ over the range $-6 \leq \xi \leq 6$.

   (c) Based on the results of problems 2(c) and 3(c), what happens to the plot of the complex Fourier series $g(x)$ as the period X becomes large?

   (d) Based on the results of problems 2(d), 3(d) and 4(b), describe what happens to the plot $XG(\xi)$ as the period X becomes large?