

Do all four questions

1. Young's double slit experiment illustrates that light is a wave and exhibits interference

a) A transmission mask with two narrow slits separated by a distance  $h$  is given by

$$t(x, y) = \delta\left(y - \frac{h}{2}\right) + \delta\left(y + \frac{h}{2}\right).$$

This mask is located at the front focal plane of a thin lens with focal length  $f$ . The mask is illuminated with a unit amplitude plane propagating along the  $z$ -axis. A screen is placed at the rear focal plane of the lens. Draw a sketch of the setup.

b) Calculate the irradiance pattern  $I_i(x_i, y_i, 2f)$  on the screen. You can ignore the effects of the pupil of the lens, and be sure your answer is in terms of coordinates  $(x_i, y_i)$ .

c) What is the spacing between the peaks in the irradiance pattern?

d) How does the irradiance pattern change when  $h$  is decreased?

e) Let's make the transmission mask have finite sized slits. The mask now looks like

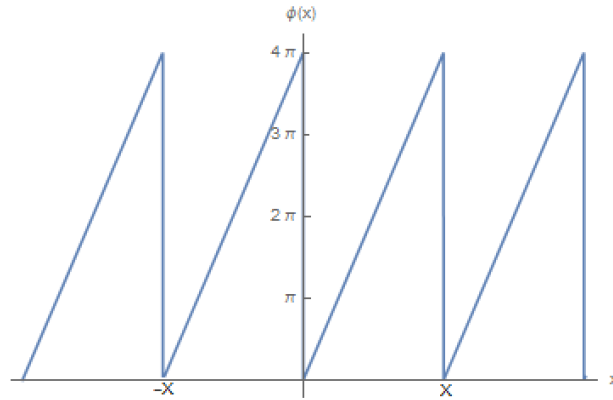
$$t(x, y) = \text{rect}\left(\frac{x}{50D}, \frac{y}{D}\right) ** \left[\delta\left(y - \frac{h}{2}\right) + \delta\left(y + \frac{h}{2}\right)\right].$$

What is the constraint on  $D$  relative to  $h$  so that this is a physically realizable experiment?

f) Calculate the irradiance pattern  $I_i(x_i, y_i, 2f)$  on the screen for the new transmission mask.

g) Sketch  $I_i(0, y_i, 2f)$  when  $D = h/2$ .

2. A variation on the kinoform lens we looked at in class is called a multi-order diffractive (MOD) lens. A plot of the periodic pattern  $\phi(x)$  is shown below. The peak height of the pattern is  $4\pi$ . The phase profile of the MOD lens is given by  $\phi(r^2)$ .



- a) Calculate the complex Fourier series coefficients  $a_m$  of the function  $f(x) = \exp(i\phi(x))$ . You can modify the integration range to 0 to  $X$  to simplify the calculation.
  - b) What is the diffraction efficiency  $\eta_m$  for the MOD lens?
  - c) What diffraction order has the maximum diffraction efficiency? What is the value of the diffraction efficiency for this order?
3. A coherent system with wavelength  $\lambda$  and image distance  $z'$  has a square pupil of width  $D$ .
- a) Calculate the impulse response  $\tilde{h}(x_i, y_i)$  of the system.
  - b) Calculate the Coherent Transfer Function  $CTF(\xi, \eta)$  of the system.
  - c) If incoherent light is now used with the system, calculate the point spread function  $PSF(x_i, y_i)$ .
  - d) Calculate the Optical Transfer Function  $OTF(\xi, \eta)$ .
  - e) The cutoff frequency in the  $\xi$ -direction is given by  $\xi_{cutoff}$  in this case where  $OTF(\xi, \eta) = 0$  for  $|\xi| > \xi_{cutoff}$ . Write an expression for  $\xi_{cutoff}$ .

4. The field at the plane  $z = 0$  is periodic in the x-direction and can be written as a complex Fourier series given by

$$U(x, y, 0) = \sum_m a_m \exp(i2\pi m \xi_0 x)$$

- a) Calculate the angular spectrum  $A(\xi, \eta; 0)$  of this field.
- b) Use the Fresnel approximation transfer function to calculate  $A(\xi, \eta; z)$ , the angular spectrum a distance  $z$  away.
- c) Calculate the field  $U(x, y, z)$  at this new plane.
- d) At what distance  $z$  does (there are multiple possible distances, but use the most basic one)

$$U(x, y, z) = \exp(ikz) U(x, y, 0)?$$

- e) When this condition is met, how do the irradiance patterns at the input plane and the output plane compare?

## Formula Sheet

$f(x, y) ** h(x, y) = \iint_{-\infty}^{\infty} f(\alpha, \beta) h(x - \alpha, y - \beta) d\alpha d\beta$	Convolution
$f(x, y) \star \star h(x, y) = \iint_{-\infty}^{\infty} f(\alpha, \beta) h(\alpha - x, \beta - y) d\alpha d\beta$	Correlation
$\begin{aligned} \gamma_{fh}(x, y) &= f(x, y) \star \star h^*(x, y) \\ &= \iint_{-\infty}^{\infty} f(\alpha, \beta) h^*(\alpha - x, \beta - y) d\alpha d\beta \end{aligned}$	Complex Correlation
$\mathcal{F}_{2D}\{f(x, y)\} = F(\xi, \eta) = \iint_{-\infty}^{\infty} f(x, y) \exp[-i2\pi(\xi x + \eta y)] dx dy$	2D Fourier Transform
$\mathcal{F}_{2D}^{-1}\{f(\xi, \eta)\} = f(x, y) = \iint_{-\infty}^{\infty} F(\xi, \eta) \exp[i2\pi(\xi x + \eta y)] d\xi d\eta$	Inverse Fourier Transform 2D
$\mathcal{H}_0\{f(r)\} = F(\rho) = 2\pi \int_0^{\infty} f(r) J_0(2\pi r \rho) r dr$	0 <sup>th</sup> Order Hankel Transform
$\begin{aligned} \cos(a) &= \frac{1}{2} [\exp(ia) + \exp(-ia)] \\ \sin(a) &= \frac{1}{2i} [\exp(ia) - \exp(-ia)] \end{aligned}$	Trig Identities
$\begin{aligned} \text{rect}(x) &= \begin{cases} 0 &  x  > 1/2 \\ 1/2 &  x  = 1/2 \\ 1 &  x  < 1/2 \end{cases} \\ \text{tri}(x) &= \begin{cases} 0 &  x  \geq 1 \\ 1 -  x  &  x  < 1 \end{cases} \\ \text{sinc}(x) &= \frac{\sin(\pi x)}{\pi x} \\ \text{Gaus}(x) &= \exp[-\pi x^2] \\ \frac{1}{ b } \text{comb}\left(\frac{x - x_0}{b}\right) &= \sum_{n=-\infty}^{\infty} \delta(x - x_0 - nb) \\ \text{cyl}(r) &= \begin{cases} 0 &  r  > 1/2 \\ 1/2 &  r  = 1/2 \\ 1 & 0 \leq  r  < 1/2 \end{cases} \\ \text{somb}(r) &= \frac{2J_1(\pi r)}{\pi r} \end{aligned}$	Common Special Functions

$\delta(x - x_o) = 0 \text{ for } x \neq x_o$ $\int_{x_1}^{x_2} f(\alpha)\delta(\alpha - x_o)d\alpha = f(x_o) \text{ for } x_1 < x_o < x_2$ $\delta\left(\frac{x - x_o}{b}\right) =  b \delta(x - x_o)$ $f(x)\delta(x - x_o) = f(x_o)\delta(x - x_o)$ $\int_{-\infty}^{\infty} \exp[-i2\pi(\xi - \xi_o)x]dx = \delta(\xi - \xi_o)$	Properties of Delta Functions
$f(x) = \sum_{m=-\infty}^{\infty} a_m \exp[i2\pi m \xi_o x] \text{ with } \xi_o = \frac{1}{X} \text{ and } X = \text{period}$ $\text{where } a_m = \frac{1}{X} \int_{-X/2}^{X/2} f(x) \exp[-i2\pi m \xi_o x] dx$	Complex Fourier Series
$\mathcal{F}_{2D}\{\text{rect}(x, y)\} = \text{sinc}(\xi, \eta)$ $\mathcal{F}_{2D}\{\text{tri}(x, y)\} = \text{sinc}^2(\xi, \eta)$ $\mathcal{F}_{2D}\{\text{Gaus}(x, y)\} = \text{Gaus}(\xi, \eta)$ $\mathcal{F}_{2D}\{\text{comb}(x, y)\} = \text{comb}(\xi, \eta)$ $\mathcal{F}_{2D}\{\delta(x \pm x_o, y \pm y_o)\} = \exp[\pm i2\pi x_o \xi] \exp[\pm i2\pi y_o \eta]$ $\mathcal{F}_{2D}\{\exp[\pm i2\pi \xi_o x] \exp[\pm i2\pi \eta_o y]\} = \delta(\xi \mp \xi_o, \eta \mp \eta_o)$ $\mathcal{F}_{2D}\{\cos(2\pi \xi_o x)\} = \frac{1}{2} [\delta(\xi - \xi_o) + \delta(\xi + \xi_o)] \delta(\eta)$ $\mathcal{F}_{2D}\{\sin(2\pi \xi_o x)\} = \frac{1}{2i} [\delta(\xi - \xi_o) - \delta(\xi + \xi_o)] \delta(\eta)$ $\mathcal{H}_0\{\text{cyl}(r)\} = \frac{\pi}{4} \text{somb}(\rho)$	Common 2D Fourier Transforms

$\mathcal{F}_{2D}\{f(\pm x, \pm y)\} = F(\pm\xi, \pm\eta)$ $\mathcal{F}_{2D}\{f^*(\pm x, \pm y)\} = F^*(\mp\xi, \mp\eta)$ $\mathcal{F}_{2D}\{F(\pm x, \pm y)\} = f(\mp\xi, \mp\eta)$ $\mathcal{F}_{2D}\{F^*(\pm x, \pm y)\} = f^*(\pm\xi, \pm\eta)$ $\mathcal{F}_{2D}\{f_1(x)f_2(y)\} = \mathcal{F}_{1D}\{f_1(x)\}\mathcal{F}_{1D}\{f_2(y)\} = F_1(\xi)F_2(\eta)$ $\mathcal{F}_{2D}\{f_1(x)\} = F_1(\xi)\delta(\eta)$ $\mathcal{F}_{2D}\{f_2(y)\} = \delta(\xi)F_2(\eta)$ $\mathcal{F}_{2D}\left\{f\left(\frac{x}{b}, \frac{y}{d}\right)\right\} =  bd F(b\xi, d\eta)$ $\mathcal{F}_{2D}\{f(x \pm x_o, y \pm y_o)\} = \exp[\pm i2\pi x_o\xi]\exp[\pm i2\pi y_o\eta]F(\xi, \eta)$ $\mathcal{F}_{2D}\left\{\exp[\pm i2\pi\xi_o x]\exp[\pm i2\pi\eta_o y]f\left(\frac{x \pm x_o}{b}, \frac{y \pm y_o}{b}\right)\right\}$ $=  bd \exp[\pm i2\pi x_o(\xi \mp \xi_o)]\exp[\pm i2\pi y_o(\eta \mp \eta_o)]$ $\times F(b(\xi \mp \xi_o), d(\eta \mp \eta_o))$ $\iint_{-\infty}^{\infty} f(x, y)dx dy = F(0,0) \text{ and } \iint_{-\infty}^{\infty} F(\xi, \eta)d\xi d\eta = f(0,0)$ $\mathcal{H}_0\left\{f\left(\frac{r}{b}\right)\right\} =  b ^2 F(b\rho)$	<p>General Properties of 2D Fourier &amp; Hankel Transforms</p>
$\mathcal{F}_{2D}\{f(x, y) ** h(x, y)\} = F(\xi, \eta)H(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y)h(x, y)\} = F(\xi, \eta) ** H(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y) \star \star h(x, y)\} = F(\xi, \eta)H(-\xi, -\eta)$ $\mathcal{F}_{2D}\{f(x, y)h(-x, -y)\} = F(\xi, \eta) \star \star H(\xi, \eta)$ $\mathcal{F}_{2D}\{\gamma_{fh}(x, y)\} = \mathcal{F}_{2D}\{f(x, y) \star \star h^*(x, y)\} = F(\xi, \eta)H^*(\xi, \eta)$ $\mathcal{F}_{2D}\{f(x, y)h^*(x, y)\} = F(\xi, \eta) \star \star H^*(\xi, \eta) = \gamma_{FH}(\xi, \eta)$ $\mathcal{H}_0\{f(r) ** h(r)\} = \mathcal{H}_0\{f(r) \star \star h(r)\} = F(\rho)H(\rho)$ $\mathcal{H}_0\{f(r)h(r)\} = F(\rho) ** H(\rho) = F(\rho) \star \star H(\rho)$ $\mathcal{H}_0\{\gamma_{fh}(r)\} = \mathcal{H}_0\{f(r) \star \star h^*(r)\} = F(\rho)H^*(\rho)$	<p>2D Transforms of Products, Correlations and Convolutions</p>

Direction Cosines

$$\alpha = \cos\theta_x; \beta = \cos\theta_y; \gamma = \cos\theta_z$$

Plane Wave with amplitude  $A$  and direction cosines  $(\alpha, \beta, \gamma)$

$$U(x, y, z) = A \exp\left[i \frac{2\pi}{\lambda} (\alpha x + \beta y + \gamma z)\right]$$

Spherical Wave with amplitude  $A$  centered on origin

$$U(x, y, z) = \frac{A}{r_{01}} \exp(\pm ikr_{01}); \quad r_{01} = \sqrt{x^2 + y^2 + z^2}; \quad +diverging/-converging$$

Parabolic Approximation to Spherical Wave with amplitude  $A$  centered on  $(0,0,z)$

$$U(x, y, z) = \frac{A}{z} \exp(\pm ikz) \exp\left[\pm \frac{i\pi}{\lambda z} (x^2 + y^2)\right]; \quad +diverging/-converging$$

Gaussian Beam of amplitude  $A$

$$U(x, y, z) = A \frac{w_0}{w(z)} \exp\left(-\frac{x^2 + y^2}{w^2(z)}\right) \exp\left[i\left(kz + k\frac{x^2 + y^2}{2R(z)} - \Phi(z)\right)\right]$$

$$z_0 = \frac{kw_0^2}{2}; \quad w(z) = w_0 \left(1 + \frac{z^2}{z_0^2}\right)^{1/2}; \quad R(z) = z \left(1 + \frac{z_0^2}{z^2}\right); \quad \Phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

Angular Spectrum Propagation

$$A(\xi, \eta; z) = A(\xi, \eta; 0) \exp\left(i2\pi z \sqrt{\frac{1}{\lambda^2} - \xi^2 - \eta^2}\right)$$

Fresnel Approximation to Angular Spectrum Propagation

$$A(\xi, \eta; z) = A(\xi, \eta; 0) \exp(ikz) \exp[-i\pi\lambda z(\xi^2 + \eta^2)]$$

Fraunhofer Diffraction

$$U(x, y, z) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z} (x^2 + y^2)\right] \mathcal{F}_{2D}\{U(x_0, y_0, 0)\}; \quad \xi = \frac{x}{\lambda z}, \eta = \frac{y}{\lambda z}$$

Fresnel Diffraction Free space Propagation

$$U(x, y, z) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z} (x^2 + y^2)\right] \mathcal{F}_{2D}\left\{U(x_0, y_0, 0) \exp\left[\frac{i\pi}{\lambda z} (x_0^2 + y_0^2)\right]\right\};$$

$$\xi = \frac{x}{\lambda z}, \eta = \frac{y}{\lambda z}$$

## Fresnel Diffraction Object Against a Lens

$$U(x, y, f) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x^2 + y^2)\right] \mathcal{F}_{2D}\{U(x_o, y_o, 0)P(x_o, y_o)\};$$

$$\xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}$$

## Fresnel Diffraction Object In Front of Lens

$$U(x, y, f + d) = \frac{\exp(ik(f + d))}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}\left(1 - \frac{d}{f}\right)(x^2 + y^2)\right]$$

$$\times \mathcal{F}_{2D}\left\{U(x_o, y_o, 0)P\left(x_o + \frac{d}{f}x, y_o + \frac{d}{f}y\right)\right\}; \quad \xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}$$

## Fresnel Diffraction Object Behind a Lens

$$U(x, y, f) = \frac{1}{i\lambda d} \frac{Af}{d} \exp\left[\frac{i\pi}{\lambda d}(x^2 + y^2)\right] \mathcal{F}_{2D}\left\{U(x_o, y_o, 0)P\left(x_o \frac{f}{d}, y_o \frac{f}{d}\right)\right\};$$

$$\xi = \frac{x}{\lambda d}, \eta = \frac{y}{\lambda d}$$

Geometrical Image through an optical system with magnification  $m$ 

$$U_g(x_i, y_i, z') = -\frac{\exp(ik(z' - z))}{|m|} U_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z\right);$$

$$I_g(x_i, y_i, z') = \frac{1}{|m|^2} \left|U_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z\right)\right|^2$$

## Impulse Response of Diffraction Limited System

$$\tilde{h}(x_i, y_i) = \frac{1}{\lambda^2 z'^2} \mathcal{F}_{2D}\{P(x_l, y_l)\}; \quad \xi = \frac{x_i}{\lambda z'}, \eta = \frac{y_i}{\lambda z'}$$

## Definitions for Coherent and Incoherent Systems

$$CTF(\xi, \eta) = \mathcal{F}_{2D}\{\tilde{h}(x_i, y_i)\}$$

$$PSF(x_i, y_i) = |\tilde{h}(x_i, y_i)|^2$$

$$OTF(\xi, \eta) = \mathcal{F}_{2D}\{PSF(x_i, y_i)\}; \text{ Normalize so } OTF(0,0) = 1$$

$$MTF(\xi, \eta) = |OTF(\xi, \eta)|$$

$$PTF(\xi, \eta) = \text{Arg}(OTF(\xi, \eta))$$



## Coherent Imaging System

$$U(x_i, y_i, z') = \tilde{h}(x_i, y_i) ** U_g(x_i, y_i, z')$$

$$A(\xi, \eta; z') = CTF(\xi, \eta)A_g(\xi, \eta; z'),$$

where the A's are the Fourier transforms of the respective fields.

## Incoherent Imaging System

$$I(x_i, y_i, z') = PSF(x_i, y_i) ** I_g(x_i, y_i, z')$$

$$A(\xi, \eta; z') = OTF(\xi, \eta)A_g(\xi, \eta; z'),$$

where the A's are the Fourier transforms of the respective irradiances.

## Incoherent Diffraction Limited OTF for Circular Pupil

$$OTF(\beta) = \frac{2}{\pi} \left[ \cos^{-1}(\beta) - \beta \sqrt{1 - \beta^2} \right] \text{ where } \beta = (\lambda F_w / \#) \rho \text{ and } \rho \leq 1 / \lambda F_w / \#$$

Otherwise  $OTF(\beta) = 0$ .