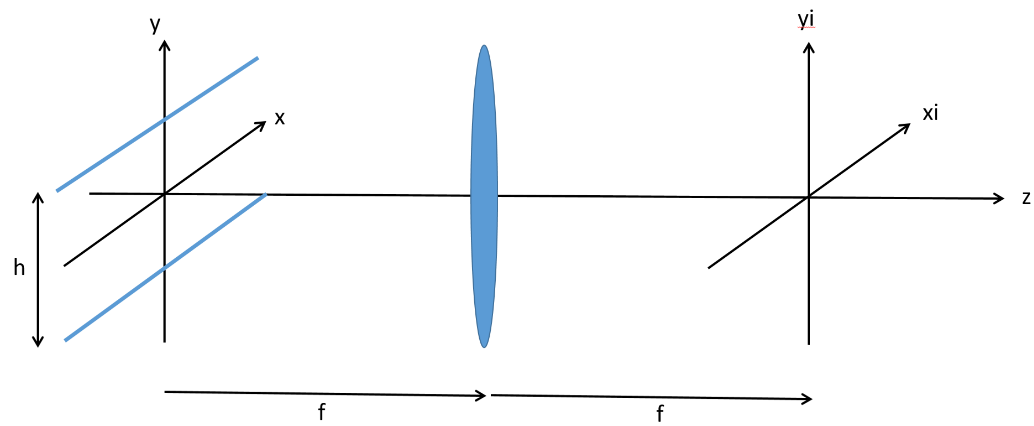
1. Young's double slit experiment illustrates that light is a wave and exhibits interference

a) A transmission mask with two narrow slits separated by a distance h is given by

$$t(x,y) = \delta\left(y - \frac{h}{2}\right) + \delta\left(y + \frac{h}{2}\right).$$

This mask is located at the front focal plane of a thin lens with focal length f. The mask is illuminated with a unit amplitude plane propagating along the z-axis. A screen is placed at the rear focal plane of the lens. Draw a sketch of the setup.



b) Calculate the irradiance pattern $I_i(x_i, y_i, 2f)$ on the screen. You can ignore the effects of the pupil of the lens, and be sure your answer is in terms of coordinates (x_i, y_i).

The field on the screen is given by

$$U(x_i, y_i, f+d) = \frac{exp\big(ik(f+d)\big)}{i\lambda f} exp\left[\frac{i\pi}{\lambda f}\left(1 - \frac{d}{f}\right)(x^2 + y^2)\right]$$
$$\times\ \mathcal{F}_{2D}\left\{U(x_o, y_o, 0)P\left(x_o + \frac{d}{f}x, y_o + \frac{d}{f}y\right)\right\};\quad \xi = \frac{x}{\lambda f}, \eta = \frac{y}{\lambda f}$$

with $d = f$, $P(\,) = 1$ *since the pupil effects are being ignored and* $U(x_o, y_o, 0) = t(x_o, y_o)$. *Given these requirements*

$$U(x_i, y_i, 2f) = \frac{exp(i2kf)}{i\lambda f}\ \mathcal{F}_{2D}\{t(x_o, y_o)\};\quad \xi = \frac{x_i}{\lambda f}, \eta = \frac{y_i}{\lambda f}$$

and

$$I(x_i, y_i, 2f) = \frac{1}{(\lambda f)^2} |\mathcal{F}_{2D}\{t(x_o, y_o)\}|^2; \quad \xi = \frac{x_i}{\lambda f}, \eta = \frac{y_i}{\lambda f}$$

The Fourier transform of the transmission mask is

$$\mathcal{F}_{2D}\{t(x_o, y_o)\} = 2\delta(\xi)cos\left[2\pi\frac{h}{2}\eta\right] = 2\lambda f\delta(x_i)cos\left[\pi\frac{hy_i}{\lambda f}\right]$$

Plugging into the irradiance expression

$$I(x_i, y_i, 2f) = 4cos^2\left[\pi\frac{hy_i}{\lambda f}\right]$$

c) What is the spacing between the peaks in the irradiance pattern?

The peaks of $cos^2()$ *occur when the argument is integer multiples of* π, *so the spacing between peaks is*

$$\Delta y_i = \frac{\lambda f}{h}$$

d) How does the irradiance pattern change when h is decreased?

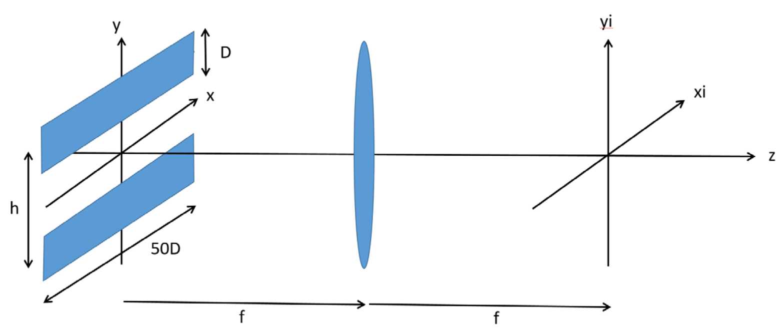
From part c, decreasing h *causes the spacing between the peaks to increase.*

e) Let's make the transmission mask have finite sized slits. The mask now looks like

$$t(x, y) = rect\left(\frac{x}{50D}, \frac{y}{D}\right) ** \left[\delta\left(y - \frac{h}{2}\right) + \delta\left(y + \frac{h}{2}\right)\right].$$

What is the constraint on D relative to h so that this is a physically realizable experiment?

The setup now looks like



From the diagram, it should be evident that $D/2 \leq h/2$, otherwise the openings start to overlap and give an impossible transmission value of 2 in the overlap regions. Thus

$$D \leq h.$$

f) Calculate the irradiance pattern $I_i(x_i, y_i, 2f)$ on the screen for the new transmission mask.

We know from part b that

$$I(x_i, y_i, 2f) = \frac{1}{(\lambda f)^2} |\mathcal{F}_{2D}\{t(x_o, y_o)\}|^2; \quad \xi = \frac{x_i}{\lambda f}, \eta = \frac{y_i}{\lambda f}$$

In this case,

$$t(x_o, y_o) = rect\left(\frac{x_o}{50D}, \frac{y_o - h/2}{D}\right) + rect\left(\frac{x_o}{50D}, \frac{y_o + h/2}{D}\right)$$

$$\mathcal{F}_{2D}\{t(x_o, y_o)\} = 50D^2 sinc(50D\xi, D\eta)[exp(-i\pi h\eta) + exp(i\pi h\eta)]$$

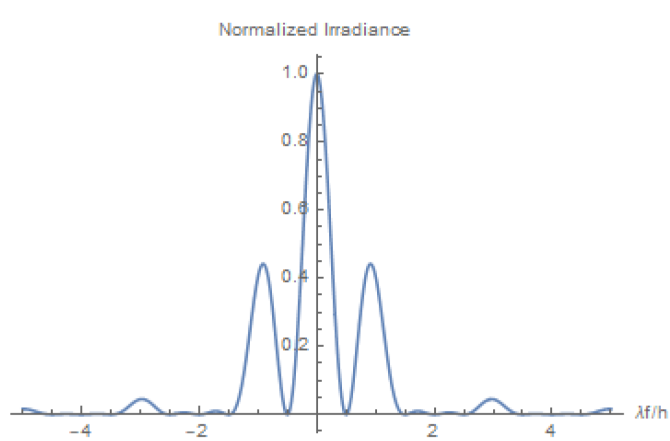
$$\mathcal{F}_{2D}\{t(x_o, y_o)\} = 100D^2 sinc(50D\xi, D\eta) cos[\pi h\eta]$$

Finally,

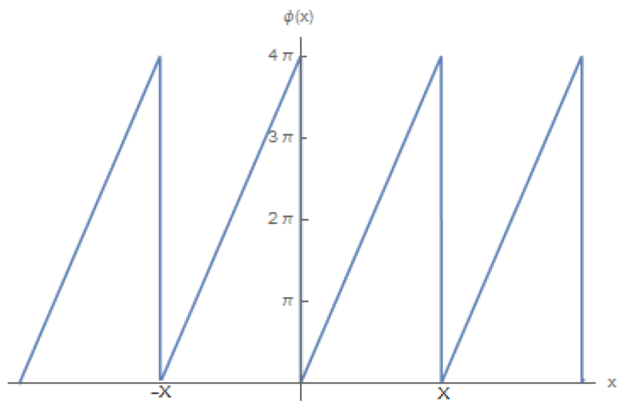
$$I(x_i, y_i, 2f) = \left[\frac{100D^2}{\lambda f}\right]^2 sinc^2\left(\frac{50Dx_i}{\lambda f}, \frac{Dy_i}{\lambda f}\right) cos^2\left[\frac{\pi h y_i}{\lambda f}\right]$$

g) Sketch $I_i(0, y_i, 2f)$ when $D = h/2$.

$$I(0, y_i, 2f) = \left[\frac{25h^2}{\lambda f}\right]^2 sinc^2\left(0, \frac{hy_i}{2\lambda f}\right) cos^2\left[\frac{\pi h y_i}{\lambda f}\right]$$



2. A variation on the kinoform lens we looked at in class is called a multi-order diffractive (MOD) lens. A plot of the periodic pattern $\phi(x)$ is shown below. The peak height of the pattern is 4π. The phase profile of the MOD lens is given by $\phi(r^2)$.



a) Calculate the complex Fourier series coefficients a_m of the function $f(x) = exp(i\phi(x))$. You can modify the integration range to 0 to X to simplify the calculation.

For $0 \leq x \leq X$*, the phase* $\phi(x) = 4\pi x/X$ *and* $f(x) = exp(i\,4\pi x/X)$*. The coefficients of the Fourier series are given by*

$$a_m = \frac{1}{X}\int_0^X exp\left(\frac{i4\pi x}{X}\right)\, exp\left[-\frac{i2\pi m x}{X}\right] dx$$

since $\xi_o = \frac{1}{X}$.

$$a_m = \frac{1}{X}\int_0^X exp\left[-\frac{i2\pi(m-2)x}{X}\right] dx$$

$$a_m = \frac{1}{X}\left[\frac{-X}{i2\pi(m-2)}\right](exp[-i2\pi(m-2)] - 1)$$

$$a_m = \left[\frac{1}{\pi(m-2)}\right] exp[-i\pi(m-2)](sin[\pi(m-2)])$$

$$a_m = exp[-i\pi(m-2)]sinc(m-2)$$

b) What is the diffraction efficiency η_m for the MOD lens?

$$\eta_m = |a_m|^2 = sinc^2(m-2)$$

c) What diffraction order has the maximum diffraction efficiency? What is the value of the diffraction efficiency for this order?

All of the coefficients are zero, except $a_2 = 1$ *since the sinc function is zero whenever the argument is a non-zero integer. The diffraction efficiency in this case is 100%.*

3. A coherent system with wavelength λ and image distance z' has a square pupil of width D.

a) Calculate the impulse response $\tilde{h}(x_i, y_i)$ of the system.

$$\tilde{h}(x_i, y_i) = \frac{1}{\lambda^2 z'^2} \mathcal{F}_{2D}\left\{rect\left(\frac{x_l}{D}, \frac{y_l}{D}\right)\right\}; \quad \xi = \frac{x_i}{\lambda z'}, \eta = \frac{y_i}{\lambda z'}$$

$$\tilde{h}(x_i, y_i) = \frac{D^2}{\lambda^2 z'^2} sinc(D\xi, D\eta) = \frac{D^2}{\lambda^2 z'^2} sinc\left(\frac{Dx_i}{\lambda z'}, \frac{Dy_i}{\lambda z'}\right)$$

b) Calculate the Coherent Transfer Function $CTF(\xi, \eta)$ of the system.

$$CTF(\xi, \eta) = \mathcal{F}_{2D}\{\tilde{h}(x_i, y_i)\} = rect\left(\frac{\lambda z' \xi}{D}, \frac{\lambda z' \eta}{D}\right)$$

c) If incoherent light is now used with the system, calculate the point spread function $PSF(x_i, y_i)$.

$$PSF(x_i, y_i) = \left|\tilde{h}(x_i, y_i)\right|^2 = \left[\frac{D^2}{\lambda^2 z'^2}\right]^2 sinc^2\left(\frac{Dx_i}{\lambda z'}, \frac{Dy_i}{\lambda z'}\right)$$

d) Calculate the Optical Transfer Function $OTF(\xi, \eta)$.

$$OTF(\xi, \eta) = \mathcal{F}_{2D}\{PSF(x_i, y_i)\}; Normalize\ so\ OTF(0,0) = 1$$

$$OTF(\xi, \eta) = tri\left(\frac{\lambda z' \xi}{D}, \frac{\lambda z' \eta}{D}\right)$$

e) The cutoff frequency in the ξ-direction is given by ξ_{cutoff} in this case where $OTF(\xi,\eta) = 0$ for $|\xi| > \xi_{cutoff}$. Write an expression for ξ_{cutoff}.

The tri() function goes to zero when its argument is greater than or equal to one, so

$$\frac{\lambda z' \xi_{cutoff}}{D} = 1 \Rightarrow \xi_{cutoff} = \frac{D}{\lambda z'}$$

4. The field at the plane $z = 0$ is periodic in the x-direction and can be written as a complex Fourier series given by

$$U(x,y,0) = \sum_m a_m exp(i2\pi m\xi_o x)$$

a) Calculate the angular spectrum $A(\xi,\eta;0)$ of this field.

$$A(\xi,\eta;0) = \sum_m a_m \delta(\xi - m\xi_o, \eta)$$

b) Use the Fresnel approximation transfer function to calculate $A(\xi,\eta;z)$, the angular spectrum a distance z away.

$$A(\xi,\eta;z) = \sum_m a_m \delta(\xi - m\xi_o, \eta)\, exp(ikz) exp[-i\pi\lambda z(\xi^2 + \eta^2)]$$

Using the multiplication properties of delta functions

$$A(\xi,\eta;z) = exp(ikz) \sum_m a_m \delta(\xi - m\xi_o, \eta)\, exp[-i\pi\lambda z(m^2\xi_o^2)]$$

c) Calculate the field $U(x,y,z)$ at this new plane.

$$U(x,y,z) = exp(ikz) \sum_m a_m exp(i2\pi m\xi_o x)\, exp[-i\pi\lambda z(m^2\xi_o^2)]$$

d) At what distance z does (there are multiple possible distances, but use the most basic one) $U(x,y,z) = exp(ikz)\, U(x,y,0)$?

We are seeking z values where

$$exp(ikz)\sum_m a_m exp(i2\pi m\xi_o x)\, exp[-i\pi\lambda z(m^2\xi_o^2)] = exp(ikz)\sum_m a_m exp(i2\pi m\xi_o x)$$

Comparing both sides, we want

$$exp[-i\pi\lambda z(m^2\xi_o^2)] = 1,$$

so the argument of the exponential should be integer multiples of $i2\pi$, *which leads to the basic separation of*

$$\pi\lambda z(m^2\xi_o^2) = 2\pi$$

$$z = \frac{2}{\lambda\xi_o^2} = \frac{2X^2}{\lambda}$$

since if this holds for $m = 1$, *then it will hold for higher values of m as well.*

e) When this condition is met, how do the irradiance patterns at the input plane and the output plane compare?

The irradiance patterns are identical.

$$I(x,y,z) = |exp(ikz)\, U(x,y,0)|^2 = |U(x,y,0)|^2 = I(x,y,0)$$