Here, we will consider two types of system coherent and ~~and~~ incoherent. For coherent systems, the fields can interfere with one another, whereas incoherent systems have no interference. In actuality, there's a continual degree of coherence, but for this class, we'll just examine the extreme cases.

COHERENT SYSTEMS - linear in field

$$U_i(x_i, y_i, z') = \iint_{-\infty}^{\infty} \tilde{h}(x_i, y_i; x_o, y_o)\, U_o(x_o, y_o, z)\, dx_o\, dy_o$$

If shift invariant

impulse response

IRRADIANCE

$$\begin{tikzcd} \frac{1}{m} U_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z\right) \arrow[d, "\mathcal{F}\{\}"] & ** & \tilde{h}(x_i, y_i) \arrow[d, "\mathcal{F}\{\}"] & = & U_i(x_i, y_i, z') \arrow[r] & I_i(x_i, y_i, z') = |U_i|^2 \\ A_o(\xi, \eta, z) & \cdot & H(\xi, \eta) & = & A_i(\xi, \eta, z') \arrow[u, "\mathcal{F}^{-1}\{\}"'] & \end{tikzcd}$$

COHERENT TRANSFER FUNCTION (CTF)

From our previous results

$$\tilde{h}(x_i, y_i) = \frac{1}{\lambda^2 z'^2} \mathcal{F}_{2D}\{P(x_e, y_e)\} \quad \text{with } \xi = \frac{x_i}{\lambda z'},\ \eta = \frac{y_i}{\lambda z'}$$

$$CTF(\xi, \eta) = H(\xi, \eta) = \frac{1}{\lambda^2 z'^2} \mathcal{F}_{2D}\left\{\mathcal{F}_{2D}\{P(x_e, y_e)\}\right\}$$

$$\boxed{CTF(\xi, \eta) = P(-\lambda z'\xi, -\lambda z'\eta)}$$

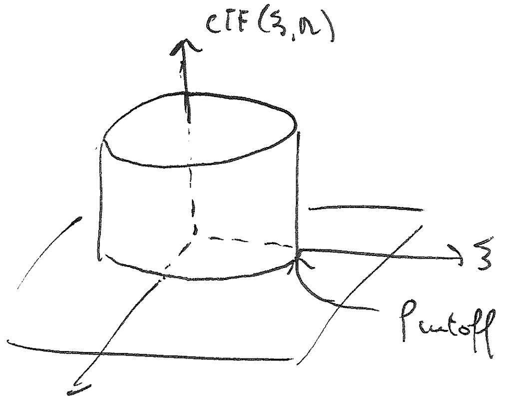
Scaled version of pupil function

For a coherent system with a circular pupil and no aberrations

$$P(x_\ell, y_\ell) = cyl\left(\frac{r_\ell}{d}\right) \qquad r_\ell^2 = x_\ell^2 + y_\ell^2$$

The Coherent Transfer Function is

$$CTF(\xi, \eta) = cyl\left(\frac{\lambda z' \rho}{d}\right) \qquad \text{where } \rho^2 = \xi^2 + \eta^2$$



Cutoff frequency

$$f_{cutoff} = \frac{d}{2\lambda z'}$$

In finite conjugate systems, $\frac{z'}{d}$ is called the working F-Number $F_w/\#$. For infinite conjugate system $z' \Rightarrow f$ and $F_w/\# \Rightarrow F/\#$

So $$\boxed{f_{cutoff} = \frac{1}{2\lambda F_w/\#}}$$

$$U_0(x_0, y_0, z) = \cos(2\pi \xi_0 x_0)$$

$$A_0(\xi, \eta, z) = \frac{1}{2}\left[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)\right]\delta(\eta)$$

$$CTF(\xi, \eta) = cyl\left(\frac{\lambda z' \rho}{d}\right)$$

$$A_i(\xi, \eta, z') = \begin{cases} \frac{1}{2}\left[\delta(\xi - \xi_0) + \delta(\xi + \xi_0)\right]\delta(\eta) & \xi_0 < f_{cutoff} \\ 0 & \text{otherwise} \end{cases}$$

$$U(x_i, y_i, z') = \begin{cases} \cos(2\pi \xi_0 x_i) & \xi < f_{cutoff} \\ 0 & \text{otherwise} \end{cases}$$

Perfect transmission below cutoff frequency

INCOHERENT SYSTEMS - linear in irradiance

$$I_i(x_i, y_i, z') = \iint_{-\infty}^{\infty} \left|\tilde{h}(x_i - mx_o, y_i - my_o)\right|^2 I_o(x_o, y_o, z)\, dx_o\, dy_o$$

Point Spread Function

$$PSF(x_i, y_i) = \frac{1}{\lambda^2 z'^2} \left| \mathcal{F}_{2D}\{ P(x_e, y_e) \} \right|^2 \qquad \xi = \frac{x_i}{\lambda z'} \quad \eta = \frac{y_i}{\lambda z'}$$

$\frac{x_i}{m}, \frac{y_i}{m}$

$$I_o(x_o, y_o, z) \quad ** \quad PSF(x_i, y_i) \quad = \quad I_i(x_i, y_i, z')$$

$\mathcal{F}\{\}\downarrow$ $\qquad \downarrow \mathcal{F}\{\}$ $\qquad \uparrow \mathcal{F}^{-1}\{\}$

$$A_o(\xi, \eta, z) \quad \underset{\text{multiply}}{\cdot} \quad \mathcal{H}(\xi, \eta) \quad = \quad A_i(\xi, \eta, z')$$

OPTICAL TRANSFER FUNCTION (OTF)

$$OTF(\xi, \eta) = \mathcal{H}(\xi, \eta) = \frac{\mathcal{F}_{2D}\{PSF(x_i, y_i)\}}{\iint_{-\infty}^{\infty} PSF(x_i, y_i)\, dx_i\, dy_i}$$

In general, OTF is a complex function

← Normalization

Three properties

① $OTF(0,0) = 1$ Due to Normalization. Conservation of energy.
Put in constant object get out constant image with same energy.

② $OTF(-\xi, -\eta) = OTF^*(\xi, \eta)$ Symmetry about diagonal Hermitian
since PSF is real-positive function

③ $|OTF(\xi, \eta)| \leq |OTF(0,0)|$ Can't have higher OTF than 1.

$$OTF(\xi,\eta) = \mathcal{F}_{2D}\{PSF(x_i,y_i)\}\Big|_{norm} = \mathcal{F}_{2D}\{\tilde{h}(x_i,y_i)\,\tilde{h}^*(x_i,y_i)\}\Big|_{norm}$$

Recall that $\mathcal{F}_{2D}\{f(x,y)\,g^*(x,y)\} = F(\xi,\eta) \star\star\, G^*(\xi,\eta)$

where $F(\xi,\eta)$ and $G(\xi,\eta)$ are the Fourier transforms of $f(x,y)$ and $g(x,y)$.

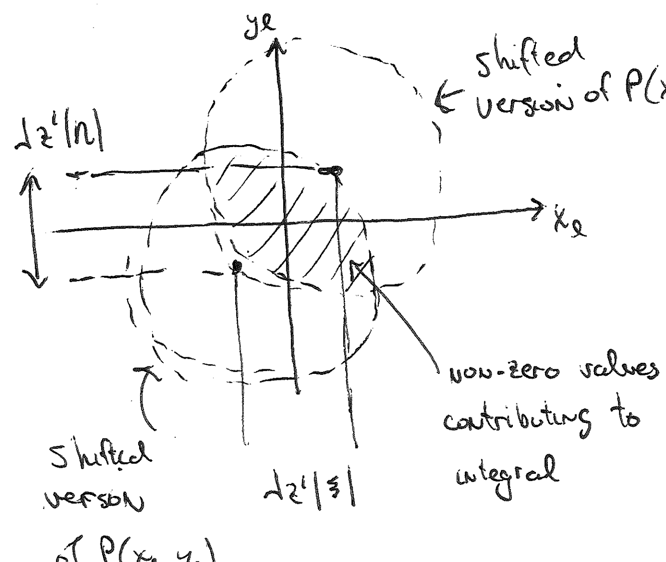
Since $\mathcal{F}_{2D}\{\tilde{h}(x_i,y_i)\} = P(-\lambda z'\xi, -\lambda z'\eta)$, the OTF is just a normalized autocorrelation of the pupil function.

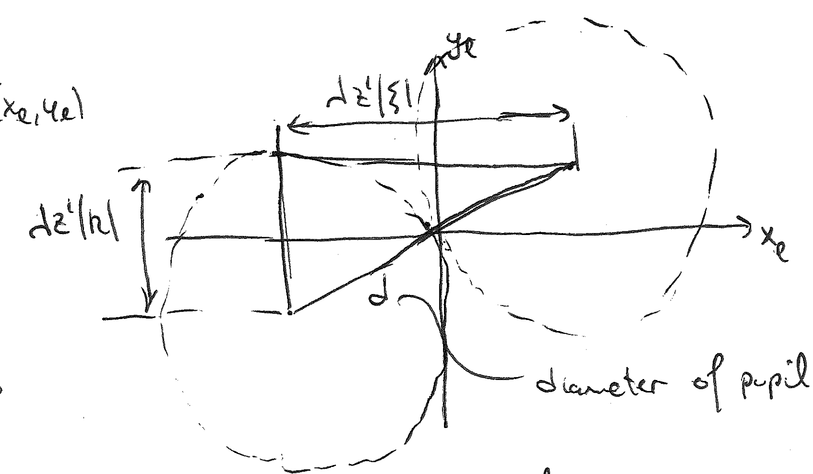
autocorrelation

$$OTF(\xi,\eta) = \frac{\displaystyle\iint_{-\infty}^{\infty} P\left(x_\ell + \frac{\lambda z'\xi}{2}, y_\ell + \frac{\lambda z'\eta}{2}\right) P^*\left(x_\ell - \frac{\lambda z'\xi}{2}, y_\ell - \frac{\lambda z'\eta}{2}\right) dx_\ell\, dy_\ell}{\displaystyle\iint_{-\infty}^{\infty} |P(x_\ell,y_\ell)|^2\, dx_\ell\, dy_\ell}$$

$\leftarrow$ area of Pupil assures $OTF(0,0)=1$

CIRCULAR PUPIL - NO ABERRATIONS





No overlap, so integral is zero

$$\lambda^2 z'^2 \xi^2 + \lambda^2 z'^2 \eta^2 = d^2$$ Condition where shifted pupils are just touching

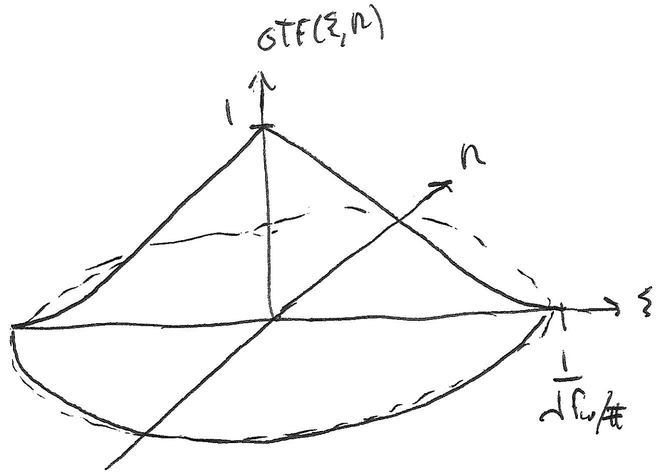
$$\rho^2_{cutoff} = \xi^2 + \eta^2 = \frac{d^2}{(\lambda z')^2}$$

$$\boxed{\rho_{cutoff} = \frac{d}{\lambda z'} = \frac{1}{\lambda F_w/\#}}$$ cutoff frequency is twice that of coherent case.

The circular pupil with no aberrations is about the ~~case~~ only case where the OTF can be calculated in closed form

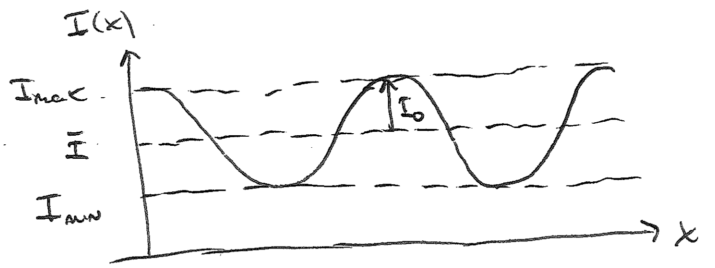
$$
OTF(\xi,\eta) = \begin{cases} \frac{2}{\pi}\left[\cos^{-1}\left(\lambda f_{w/\#}\,\rho\right) - \rho\,\lambda f_{w/\#}\sqrt{1-\rho^2(\lambda f_{w/\#})^2}\right] & ;\ \rho \le \frac{1}{\lambda f_{w/\#}} \\ 0 & \text{otherwise} \end{cases}
$$

where $\rho^2 = \xi^2 + \eta^2$



Pattern is roughly linear, but slight roll-off near cutoff frequency. This is the best OTF you can get with a system. Real systems are usually measured against this "diffraction-limited" OTF

Contrast or Visibility of a sinusoidal pattern



$$
\text{Contrast} = \frac{I_{max} - I_{MIN}}{I_{max} + I_{MIN}} = \frac{I_0}{\bar{I}}
$$

Contrast = 1 when $I_{MIN} = 0 \rightarrow \bar{I} = I_0$ minimum is black

Contrast = 0 when $I_{max} = I_{MIN} \rightarrow I_0 = 0$ pattern is constant

What happens when we put a sinusoidal pattern with contrast = 1 into our incoherent system

$$I_o(x_o, y_o, z) = \frac{1}{2} + \frac{1}{2}\cos(2\pi \xi_o x_o)$$

Remember this needs to be strictly positive since irradiance

$$A_o(\xi, \eta, z) = \frac{1}{2}\delta(\xi,\eta) + \frac{1}{4}\left(\delta(\xi-\xi_o) + \delta(\xi+\xi_o)\right)\delta(\eta)$$

$OTF(\xi, \eta)$

$$A_i(\xi, \eta, z') = \frac{1}{2}\delta(\xi,\eta)\, OTF(\xi,\eta) + \frac{1}{4}\left(\delta(\xi-\xi_o) + \delta(\xi+\xi_o)\right)\delta(\eta)\, OTF(\xi,\eta)$$

$$A_i(\xi, \eta, z') = \frac{1}{2}\delta(\xi,\eta)\, OTF(0,0) + \frac{1}{4} OTF(\xi_o,0)\,\delta(\xi-\xi_o)\,\delta(\eta) + \frac{1}{4} OTF(-\xi_o,0)\,\delta(\xi+\xi_o)\,\delta(\eta)$$

For this case of circular aberration free pupil, OTF is real. Hermitian property from page 170 says

$$OTF(-\xi, -\eta) = OTF^*(\xi, \eta)$$

which means $OTF(-\xi_o, 0) = OTF(\xi_o, 0)$ in this case.

Also from page 170, normalization means $OTF(0,0) = 1$.

$$A_i(\xi, \eta, z') = \frac{1}{2}\delta(\xi,\eta) + OTF(\xi_o,0)\,\frac{1}{4}\left(\delta(\xi-\xi_o) + \delta(\xi+\xi_o)\right)\delta(\eta)$$

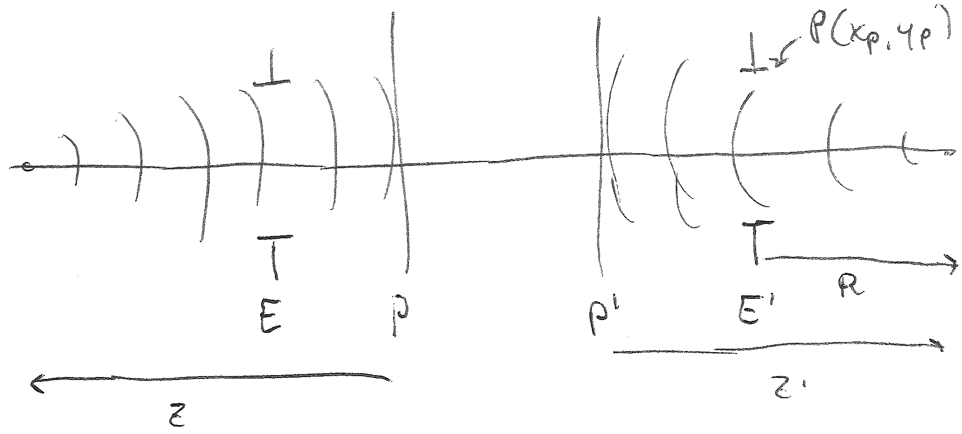
$$U_i(x_i, y_i, z') = \frac{1}{2} + \frac{OTF(\xi_o,0)}{2}\cos(2\pi \xi_o x_i)$$

$$\text{Contrast} = \frac{\frac{1}{2} + \frac{OTF(\xi_o,0)}{2} - \left(\frac{1}{2} - \frac{OTF(\xi_o,0)}{2}\right)}{\frac{1}{2} + \frac{OTF(\xi_o,0)}{2} + \left(\frac{1}{2} - \frac{OTF(\xi_o,0)}{2}\right)}$$

$$\boxed{\text{Contrast} = OTF(\xi_o, 0)}$$

OTF in this case describes the reduction in contrast of a sinusoidal pattern..

GENERALIZATION TO REAL SYSTEM WITH ABERRATIONS



With real systems, we usually reference everything to the exit pupil. When aberrations are present, the pupil function is generalized to the complex pupil function

$$P(x_p, y_p) = \underbrace{P_0(x_p, y_p)}_{\text{Transmission of pupil}} \exp\left[i \frac{2\pi}{\lambda} \overbrace{\underbrace{W(x_p, y_p)}_{\text{wavefront error}}}^{\text{aberrations}}\right]$$

wavefront error: Deviation of actual wavefront from ideal reference sphere radius R

Pretty much everything we have done regarding PSF and OTF still holds. The main difference is that the OTF will usually be a complex function. To further analyze what happens define the following

$$MTF(\xi, \eta) = |OTF(\xi, \eta)| \quad \text{MODULATION TRANSFER FUNCTION}$$

$$PTF(\xi, \eta) = ARG(OTF(\xi, \eta)) \quad \text{PHASE TRANSFER FUNCTION}$$

So we can write

$$OTF(\xi, \eta) = MTF(\xi, \eta) \exp[i \, PTF(\xi, \eta)]$$

When we put a unit contrast sinusoidal pattern into our real, aberrated system

$$I_o(x_o, y_o, z) = \frac{1}{2} + \frac{1}{2}\cos(2\pi \xi_o x_o)$$

$\Downarrow$ Image spectrum as before on page 173

$$A_i(\xi, \eta, z') = \frac{1}{2}\delta(\xi, \eta) + \frac{1}{4}OTF(\xi_o, 0)\,\delta(\xi - \xi_o)\,\delta(\eta) + \frac{1}{4}OTF(-\xi_o, 0)\,\delta(\xi + \xi_o)\,\delta(\eta)$$

but now when we use the Hermitian property from page 170

$$OTF(-\xi, -\eta) = OTF^*(\xi, \eta)$$

$$OTF(-\xi_o, 0) = MTF(-\xi_o, 0)\exp(i\,PTF(-\xi_o, 0))$$

$$= MTF(\xi_o, 0)\exp(-i\,PTF(\xi_o, 0))$$

The image spectrum now becomes

$$A_i(\xi, \eta, z') = \frac{1}{2}\delta(\xi, \eta) + \frac{1}{4}MTF(\xi_o, 0)\exp(i\,PTF(\xi_o, 0))\,\delta(\xi - \xi_o)\,\delta(\eta)\;\cancel{\tfrac{1}{4}MTF(\xi_o, 0)\exp(-i}$$

$$+ \frac{1}{4}MTF(\xi_o, 0)\exp(-i\,PTF(\xi_o, 0))\,\delta(\xi + \xi_o)\,\delta(\eta)$$

Inverse Fourier transforming gives

$$I_i(x_i, y_i, z') = \frac{1}{2} + \frac{MTF(\xi_o, 0)}{2}\cos(2\pi \xi_o x_i + PTF(\xi_o, 0))$$

$$\text{Contrast} = MTF(\xi_o, 0)$$

We still have cosine pattern, but it is phase shifted by $PTF(\xi_o, 0)$

LINE SPREAD FUNCTION

Suppose instead of putting a point into our incoherent system to get the point spread function, we put an infinitely thin line as the object. The image in this case is called the Line Spread Function (LSF)

$$I_o(x_o, y_o, z) = \delta(x_o) \cdot 1$$

We know the image is

$$I_i(x_i, y_i, z') = I_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z'\right) ** \, PSF(x_i, y_i) = LSF(x_i, y_i)$$

FOURIER TRANSFORMING THIS GIVES

$$\mathcal{F}_{2D}\{LSF(x_i, y_i)\} = 1 \cdot \delta(\eta) \cdot OTF(\xi, \eta) = OTF(\xi, 0)\, \delta(\eta)$$

The Fourier transform of the LSF gives a cross-sectional slice through the OTF. This gives a means of measuring the OTF of a system. All you need to do is capture an image of an infinitely narrow slit an Fourier transform it. The problem is finding an infinitely narrow slit.

In reality, we use a narrow slit of width d.

$$I_o(x_o, y_o, z) = rect\left(\frac{x_o}{d}\right) \cdot 1$$

$$I_i(x_i, y_i, z') = rect\left(\frac{x_i}{d}\right) ** \, PSF(x_i, y_i)$$

$$\mathcal{F}_{2D}\{I_i(x_i, y_i, z')\} = d\,sinc(d\xi)\overset{\delta(\eta)}{} \, OTF(\xi, \eta)$$

$$OTF(\xi, 0) = \frac{\mathcal{F}_{2D}\{I_i(x_i, y_i, z')\}}{d\,sinc(d\xi)}$$

Works great until numerator get small.

EDGE SPREAD FUNCTION

Revisiting the line spread function

$$LSF(x_i, y_i) = \delta(x_i) ** PSF(x_i, y_i)$$

$$= \iint_{-\infty}^{\infty} PSF(\alpha, \beta)\, \delta(x_i - \alpha)\, d\alpha\, d\beta$$

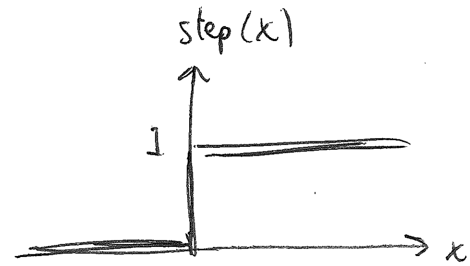
Remember convolution commutes

$$= \iint_{-\infty}^{\infty} PSF(x_i, \beta)\, \delta(x_i - \alpha)\, d\alpha\, d\beta$$

$$LSF(x_i, y_i) = \int_{-\infty}^{\infty} PSF(x_i, \beta)\, d\beta$$

Line Spread Function is just the integration of the PSF along one direction.

Define the Edge Spread Function $ESF(x_i, y_i)$ as



$$ESF(x_i, y_i) = PSF(x_i, y_i) ** step(x_i)$$

$$= \iint_{-\infty}^{\infty} PSF(\alpha, \beta)\, step(x_i - \alpha)\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} PSF(\alpha, \beta)\, d\beta \right] step(x_i - \alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} LSF(\alpha, y_i)\, step(x_i - \alpha)\, d\alpha$$

$$ESF(x_i, y_i) = \int_{-\infty}^{x_i} LSF(\alpha, y_i)\, d\alpha$$ This is where $step(x_i - \alpha) = 1$.

$$\frac{d}{dx_i} ESF(x_i, y_i) = \frac{d}{dx_i} \int_{-\infty}^{x} LSF(\alpha, y_i)\, d\alpha = LSF(x_i, y_i)$$

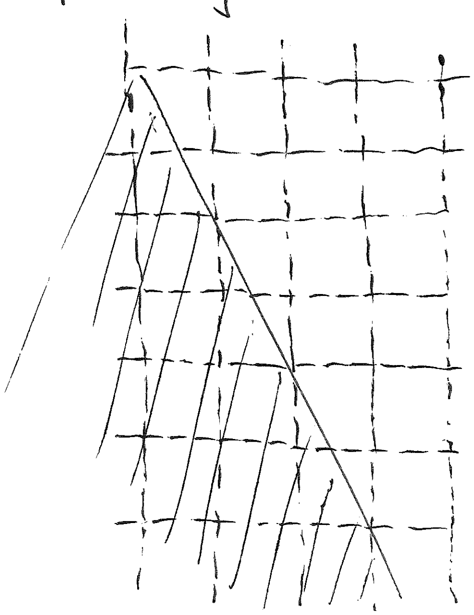
2nd FUNDAMENTAL THEOREM OF CALCULUS

$$\boxed{\frac{d}{dx_i} ESF(x_i, y_i) = LSF(x_i, y_i)}$$

This gives us another method to measure OTF. First, use an object that is opaque on one side and transparent on the other. Second, record the image formed by this object. This is $ESF(x_i, y_i)$. Next, differentiate $ESF(x_i, y_i)$ to get $LSF(x_i, y_i)$. Finally, Fourier transform $LSF(x_i, y_i)$ to get a slice through the OTF.

For both LSF and ESF, ~~to~~ we only looked in one orientation. We can always rotate the coordinate system so that other orientations are considered.

For digital imaging systems, the ESF technique is often used but with the edge tilted with respect to the pixel grid. This gives sampled versions of the ESF, but different locations give different sampling offsets which can be merged into a single super-sampled ESF prior to further processing.



This is a favorite question of people interviewing optical engineering candidates.