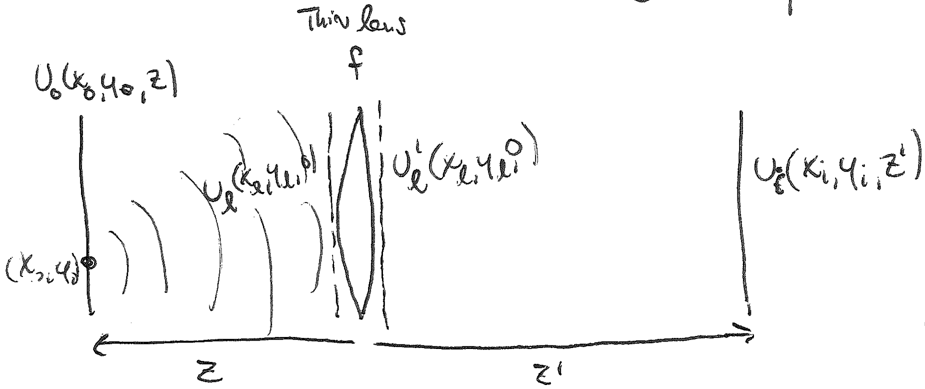
IMPULSE RESPONSE OF THIN LENS

The previous section used a lens to create the Fourier transform (to within a quadratic phase factor) of a object. The more conventional use for a lens, however, is imaging. Let's try to understand imaging with a thin lens from a wavefront perspective.

I'll try to maintain the sign convention from OPTI 502 to be consistent. We need to revisit linear systems for this analysis.



This assumes the origin is at the thin lens which makes $z<0$. For now z and z' are arbitrary.

From our linear system theory

$$U_i(x_i,y_i,z') = \iint_{-\infty}^{\infty} h(x_i,y_i;x_o,y_o)\,U_o(x_o,y_o,z)\,dx_o\,dy_o$$

OUTPUT — IMPULSE RESPONSE — INPUT

This says that the output is a superposition of the input times the impulse response. Note, this is the most general form and we need to show shift invariance to make this a convolution. We also need to figure of what h is. To find h, we put a delta function (point source in the input plane) and h is the output. This delta function creates a spherical wave the enters the thin lens

$$U_\ell(x_\ell,y_\ell,0) = -\frac{\exp(-ikz)}{i\lambda z}\exp\left[\frac{-i\pi}{\lambda z}\left[(x_\ell-x_o)^2+(y_\ell-y_o)^2\right]\right]$$

minus signs show up because $z<0$. Parabolic (Fresnel) approximation

Passing through the lens creates

$$U_\ell'(x_\ell, y_\ell, 0) = U_\ell(x_\ell, y_\ell, 0)\, P(x_\ell, y_\ell) \exp\left[\frac{-i\pi}{\lambda f}(x_\ell^2 + y_\ell^2)\right]$$

where again $P(x_\ell, y_\ell)$ is the pupil function of the lens.

Now propagate this field to the output plane.

since δ-function input

Note we used the form on page 135 for the Fresnel propagation and will expand momentarily

$$U_i(x_i, y_i, z') = h(x_i, y_i; x_0, y_0)$$

$$= \frac{\exp(ikz')}{i\lambda z'} \iint_{-\infty}^{\infty} U_\ell'(x_\ell, y_\ell, 0) \exp\left[\frac{i\pi}{\lambda z'}\left((x_i - x_\ell)^2 + (y_i - y_\ell)^2\right)\right] dx_\ell\, dy_\ell$$

$$h(x_i, y_i; x_0, y_0) = \frac{\exp(ik(z' - z))}{\lambda^2 z z'} \underbrace{\exp\left[\frac{i\pi}{\lambda z'}(x_i^2 + y_i^2)\right] \exp\left[\frac{-i\pi}{\lambda z}(x_0^2 + y_0^2)\right]}_{\text{Quadratic phase terms at input and output planes}}$$

$$\cdot \iint_{-\infty}^{\infty} P(x_\ell, y_\ell) \underbrace{\exp\left[\frac{i\pi}{\lambda}\left(-\frac{1}{z} + \frac{1}{z'} - \frac{1}{f}\right)(x_\ell^2 + y_\ell^2)\right]}_{\text{Quadratic Phase terms}} \underbrace{\exp\left[-i2\pi\left[\left(-\frac{x_0}{\lambda z} + \frac{x_i}{\lambda z'}\right)x_\ell + \left(-\frac{y_0}{\lambda z} + \frac{y_i}{\lambda z'}\right)y_\ell\right]\right]}_{\text{FOURIER TRANSFORM KERNEL}} dx_\ell\, dy_\ell$$

Let's figure out under what conditions (approximations) we can get rid of the quadratic phase factors.

Inside the integral we have

$$\exp\left[\frac{i\pi}{\lambda}\left(-\frac{1}{z} + \frac{1}{z'} - \frac{1}{f}\right)(x_\ell^2 + y_\ell^2)\right]$$ This is equal to 1 when

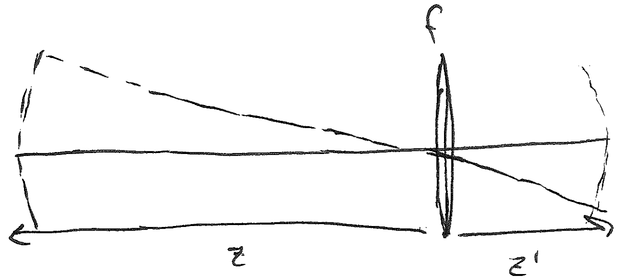
$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f} \quad \text{or if you prefer} \quad \frac{1}{z'} = \frac{1}{f} + \frac{1}{z}$$

Gaussian Imaging equation! So if the input plane is the object plane and the output plane is the image plane (i.e. they are conjugate) then this term disappears.

Let's now look at $\exp\left[\frac{i\pi}{\lambda z'}(x_i^2+y_i^2)\right]$ and $\exp\left[\frac{-i\pi}{\lambda z}(x_o^2+y_o^2)\right]$ in front

The first one disappears if the image plane was instead an image sphere with radius $-z'$ centered on the middle of the thin lens.

The second term disappears if the object plane was instead an object sphere with radius $-z$ centered on the middle of the thin lens.



Lenses really want to map a curved object onto a curved image.

We typically are imaging planes to planes so these quadratic phase terms are real effects that contribute to an aberration known as Field Curvature.

Often lens designers try to design real lens to minimize this effect and under these conditions, the two quadratic terms can be ignored.

We can also ignore these terms if we are only interested in the irradiance at the image plane $I(x_i, y_i, z') = |U(x_i, y_i, z')|^2$ since the $|\,|^2$ eliminates them.

If $-z \gg 0$ and the field of view is small, then $\exp\left[\frac{-i\pi}{\lambda z}(x_o^2+y_o^2)\right]$ changes very little over the object plane and can be ignored.

If $z' \gg 0$ and the field of view is small, then $\exp\left[\frac{i\pi}{\lambda z'}(x_i^2+y_i^2)\right]$ changes very little over the image plane and can be ignored.

Assuming that we have satisfied the Gaussian Imaging equation and are in a situation where the quadratic phase factors can be ignored, then

$$h(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 z z'} \iint_{-\infty}^{\infty} P(x_\ell,y_\ell)\exp\left[-i2\pi\left[\left(-\frac{x_o}{\lambda z}+\frac{x_i}{\lambda z'}\right)x_\ell + \left(-\frac{y_o}{\lambda z}+\frac{y_i}{\lambda z'}\right)y_\ell\right]\right]dx_\ell\, dy_\ell$$

Recall that the magnification $m = \frac{z'}{z}$

$$h(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 z z'} \iint_{-\infty}^{\infty} P(x_\ell,y_\ell)\exp\left[\frac{-i2\pi}{\lambda z'}\left[\left(x_i - \frac{x_o z'}{z}\right)x_\ell + \left(y_i - \frac{y_o z'}{z}\right)y_\ell\right]\right]dx_\ell\, dy_\ell$$

$$h(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 z z'}\, \mathcal{F}_{2D}\left\{P(x_\ell,y_\ell)\exp\left[i2\pi\left[\frac{m x_o}{\lambda z'}\right]x_\ell\right]\exp\left[i2\pi\left[\frac{m y_o}{\lambda z'}\right]y_\ell\right]\right\}$$

where $\xi = \frac{x_i}{\lambda z'}$ and $\eta = \frac{y_i}{\lambda z'}$

Let's let $\tilde{P}(\xi,\eta) = \mathcal{F}_{2D}\{P(x_\ell,y_\ell)\}$

So our Fourier transform becomes

$$\tilde{P}(\xi,\eta) ** \delta\left(\xi - \frac{m x_o}{\lambda z'}\right)\delta\left(\eta - \frac{m y_o}{\lambda z'}\right)$$

or $\tilde{P}\left(\xi - \frac{m x_o}{\lambda z'}, \eta - \frac{m y_o}{\lambda z'}\right)$

or $\tilde{P}\left(\frac{x_i - m x_o}{\lambda z'}, \frac{y_i - m y_o}{\lambda z'}\right)$

The image point conjugate to an object point located at (x_o,y_o) is just $(m x_o, m y_o)$ as predicted by geometrical objects, so this expression is just the Fourier transform of the pupil function centered on the ideal image point predicted from geometrical optics.

So the impulse response under all our assumptions is

$$h(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 zz'}\,\tilde{P}\left(\frac{x_i - mx_o}{\lambda z'},\ \frac{y_i - my_o}{\lambda z'}\right)$$

Aside from a complex constant, the impulse response is just the Fourier transform of the pupil function center of the geometric image point.

We see a strong connection to geometrical optics theory with the Gaussian imaging equation showing up and the Fourier transform of the pupil function centered on the Gaussian image point. This shouldn't be too surprising since the Fresnel approximation basically require us to be near the axis (i.e. paraxial region). We can see further this connection by rewinding a couple steps. We saw

$$h(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 zz'}\,\mathcal{F}_{2D}\left\{P(x_\ell,y_\ell)\exp\left[i2\pi\left[\frac{mx_o}{\lambda z'}\right]x_\ell\right]\exp\left[i2\pi\left[\frac{my_o}{\lambda z'}\right]y_\ell\right]\right\}$$

~~The reasons for these minus signs will become obvious shortly~~

Another assumption of geometrical optics is that the lens aperture can be infinitely large. Let's let $P(x_\ell,y_\ell) = 1$ to explore the impulse response when we're in the geometrical optics regime.

$$h_g(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\lambda^2 zz'}\,\mathcal{F}_{2D}\left\{\exp\left[i2\pi\left[\frac{mx_o}{\lambda z'}\right]x_\ell\right]\exp\left[i2\pi\left[\frac{my_o}{\lambda z'}\right]y_\ell\right]\right\}$$

geometrical optics

$$\xi = \frac{x_i}{\lambda z'} \qquad \eta = \frac{y_i}{\lambda z'}$$

$$h_g(x_i,y_i;x_o,y_o) = \frac{\exp(\ )}{\lambda^2 zz'}\,\delta\left(\xi - \frac{mx_o}{\lambda z'},\ \eta - \frac{my_o}{\lambda z'}\right)$$

$$h_g(x_i,y_i;x_o,y_o) = \frac{\exp(\ )}{\lambda^2 zz'}\,\delta\left(\frac{x_i - mx_o}{\lambda z'},\ \frac{y_i - my_o}{\lambda z'}\right)$$

Let's look at the delta function for a moment

$$\delta\left(\frac{x_i - m x_o}{\lambda z'}, \frac{y_i - m y_o}{\lambda z'}\right) \qquad \text{multiply by } \frac{-1}{m}$$

$$= \delta\left(\frac{x_o - \frac{x_i}{m}}{\frac{-\lambda z'}{m}}, \frac{y_o - \frac{y_i}{m}}{\frac{-\lambda z'}{m}}\right) \qquad \text{Now use scaling property}$$

$$= \frac{\lambda^2 z'^2}{m^2}\,\delta\left(x_o - \frac{x_i}{m}, y_o - \frac{y_i}{m}\right)$$

Back to geometrical optics impulse response

$$h_g(x_i,y_i;x_o,y_o) = \frac{\exp(ik(z'-z))}{\cancel{\lambda^2} z \cancel{z'}}\,\frac{\cancel{\lambda^2} z'^{\cancel{2}}}{m^2}\,\delta\left(x_o - \frac{x_i}{m}, y_o - \frac{y_i}{m}\right)$$

$$\boxed{h_g(x_i,y_i;x_o,y_o) = \frac{|m|}{m}\,\frac{\exp(ik(z'-z))}{|m|}\,\delta\left(x_o - \frac{x_i}{m}, y_o - \frac{y_i}{m}\right)}$$

Using this in the superposition integral, we can find the output of our geometrical optics system

output of geo optics

$$U_g(x_i,y_i,z') = \frac{-\exp(ik(z'-z))}{|m|}\iint_{-\infty}^{\infty}\delta\left(x_o - \frac{x_i}{m}, y_o - \frac{y_i}{m}\right)U_o(x_o,y_o,z)\,dx_o\,dy_o$$

From sifting

$$\boxed{U_g(x_i,y_i,z') = \frac{-\exp(ik(z'-z))}{|m|}\,U_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z\right)}$$

OUTPUT IS JUST A SCALED VERSION OF THE INPUT WITH THE MAGNIFICATION AS THE SCALE FACTOR.

Again let's rewind to the form of the impulse response on page 163

$$h(x_i,y_i;x_o,y_o) = \frac{\exp\left(ik(z'-z)\right)}{\lambda^2 z z'} \cancel{\exp} \iint_{-\infty}^{\infty} P(x_\ell,y_\ell)\exp\left[\frac{-i2\pi}{\lambda z'}\left[\left(x_i - \frac{x_o z'}{z}\right)x_\ell + \left(y_i - \frac{y_o z'}{z}\right)y_\ell\right]\right]dx_\ell\, dy_\ell$$

define $\tilde{x}_o = m x_o$ and $\tilde{y}_o = m y_o$

$$h(x_i-\tilde{x}_o, y_i-\tilde{y}_o) = \frac{\exp\left(ik(z'-z)\right)}{\lambda^2 z z'} \iint_{-\infty}^{\infty} P(x_\ell,y_\ell)\exp\left(\frac{-i2\pi}{\lambda z'}\left[(x_i-\tilde{x}_o)x_\ell + (y_i-\tilde{y}_o)y_\ell\right]\right)dx_\ell\, dy_\ell$$

shift invariant

Our superposition integral becomes

$$U(x_i,y_i,z') = \frac{1}{m^2}\iint_{-\infty}^{\infty} h(x_i-\tilde{x}_o, y_i-\tilde{y}_o)\, U_o\left(\frac{\tilde{x}_o}{m}, \frac{\tilde{y}_o}{m}, z\right) d\tilde{x}_o\, d\tilde{y}_o$$

which is now in the form of a convolution

$$U(x_i,y_i,z') = \frac{1}{m^2} h(x_i,y_i) ** U_o\left(\frac{x_i}{m}, \frac{y_i}{m}, z\right)$$

BUT THIS IS RELATED TO $U_g(x_i,y_i,z')$ geometrical image

$$U(x_i,y_i,z') = \frac{|m|}{-m^2 \exp\left(ik(z'-z)\right)} h(x_i,y_i) ** U_g(x_i,y_i,z')$$

$$\boxed{U(x_i,y_i,z') = \tilde{h}(x_i,y_i) ** U_g(x_i,y_i,z')}$$

Convolution of geometrical image with scaled version of impulse response

where $\tilde{h}(x_i,y_i) = \frac{1}{\lambda^2 z'^2}\iint_{-\infty}^{\infty} P(x_\ell,y_\ell)\exp\left[\frac{-i2\pi}{\lambda z'}(x_i x_\ell + y_i y_\ell)\right]dx_\ell\, dy_\ell$

define $\tilde{x}_\ell = \frac{x_\ell}{\lambda z'}$ $\quad \tilde{y}_\ell = \frac{y_\ell}{\lambda z'}$

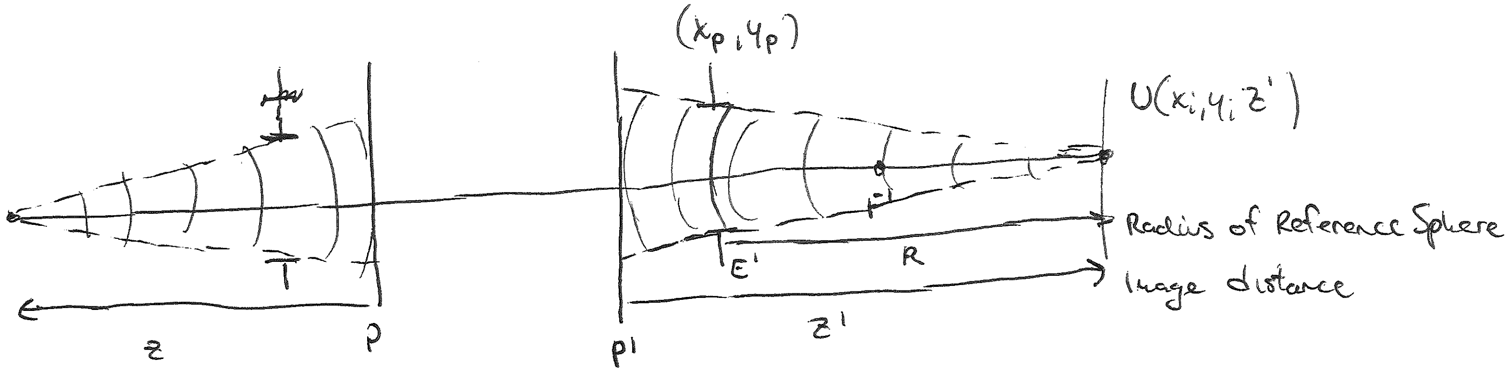
$$\boxed{\tilde{h}(x_i,y_i) = \mathcal{F}_{2D}\left\{P(\lambda z'\tilde{x}_\ell, \lambda z'\tilde{y}_\ell)\right\}}$$

where $\xi = x_i$ and $\eta = y_i$

Scale version of impulse response is just Fourier transform of scaled version of pupil function

Real Optical Systems

Thick lenses and multi-element lenses obviously don't meet the definition of a thin lens, but we can use a slightly modified version of the preceding results. Recall, that any optical system can be represented by its cardinal points and pupils.



This sort of acts like a thin lens, but the input to the lens is at the front principal plane P and the output of the lens is magically mapped to P'. The exit pupil E' is like the "object placed behind the lens" case we did on page 158. The pupil function is now defined by the shape and size of the exit pupil. The pupil function can also be made complex and its phase term encodes the aberrations of the optical system. The field on the image plane now is

$$U(x_i, y_i; z') = \tilde{h}(x_i, y_i) ** U_g(x_i, y_i; z') \qquad \tilde{x}_p = \frac{x_p}{\lambda R} \quad \tilde{y}_p = \frac{y_p}{\lambda R}$$

$$\text{where } \tilde{h}(x_i, y_i) = \mathcal{F}_{2D}\left\{ P(\lambda R \tilde{x}_p, \lambda R \tilde{y}_p) \right\}$$

$$\text{and } \xi = x_i \text{ and } \eta = y_i$$

R is the distance from the exit pupil E' to the image plane. This is known as the reference sphere radius. (x_p, y_p) are coordinates in the exit pupil plane.