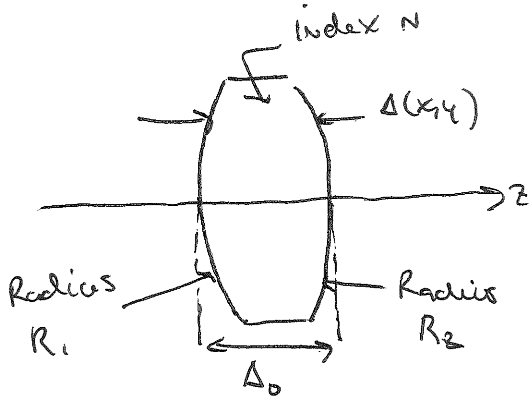
Effects of THIN LENSES ON WAVE PROPAGATION

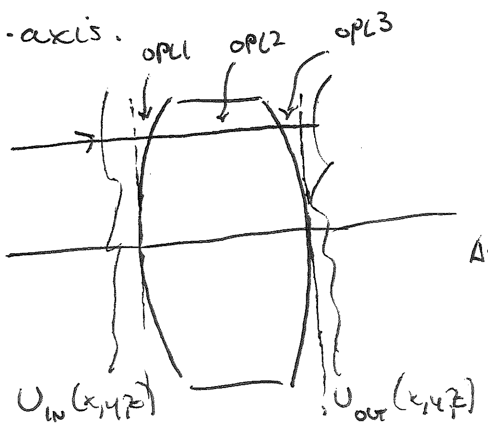
The thin lens is an idealized construct that is useful for understanding imaging properties of systems. In geometrical optics we are usually concerned with how rays are deviated by lenses. Here, we will instead look at how thin lenses deform wavefronts.



A spherical lens consists of two spherical surfaces with radii R_1 and R_2. Our sign convention will be that $R_1 > 0$ since its center of curvature is to the right of the lens and $R_2 < 0$ since its center of curvature is to the left of the lens shown here.

The center thickness of the is Δ_0 and the general thickness of the lens at any point (x,y) is $\Delta(x,y)$.

The basic approximation for a thin lens is that a ray entering the left side of the lens at a point (x,y) emerges from the right side of the lens at a point (x,y) but heading in a new direction. This means the displacement of the ray is negligible within the body of the lens. Under this approximation we can just look at the Optical Path Length ~~Difference~~ (OPL) introduced along a direction parallel to the z-axis.



$$\text{TOTAL OPL} = N\,\Delta(x,y) + (\Delta_0 - \Delta(x,y))$$

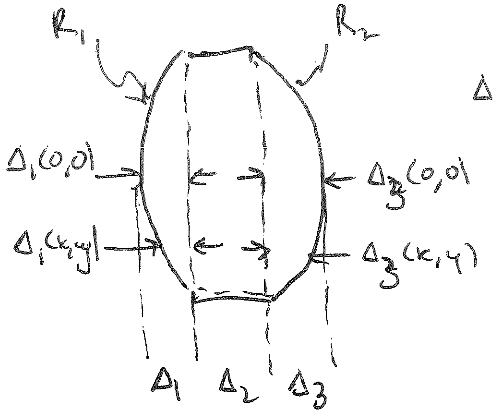
$$\text{Phase} = \frac{2\pi}{\lambda}(\text{OPL}) = kN\Delta(x,y) + k(\Delta_0 - \Delta(x,y))$$

As a phase filter

$$t(x,y) = \exp\left[i\left(kN\Delta(x,y) + k(\Delta_0 - \Delta(x,y))\right)\right]$$

$$U_{OUT}(x,y,z) = \exp(ik\Delta_0)\exp\left(ik(N-1)\Delta(x,y)\right)U_{IN}(x,y,z)$$

Let's further break up $\Delta(x,y) = \Delta_1(x,y) + \Delta_2(x,y) + \Delta_3(x,y)$



$$\Delta_1(x,y) = \Delta_1(0,0) - \left(R_1 - \sqrt{R_1^2 - (x^2+y^2)}\right)$$

$$\Delta_2(x,y) = \Delta_2(0,0) \quad \text{Constant.}$$

$$\Delta_3(x,y) = \Delta_3(0,0) - \left(R_2 - \sqrt{R_2^2 - (x^2+y^2)}\right)$$

and $\Delta_0 = \Delta_1(0,0) + \Delta_2(0,0) + \Delta_3(0,0)$

Rewriting as

$$\Delta_1(x,y) = \Delta_1(0,0) - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right)$$

$$\Delta_3(x,y) = \Delta_3(0,0) + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

Combining gives

$$\Delta(x,y) = \Delta_0 - R_1\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_1^2}}\right) + R_2\left(1 - \sqrt{1 - \frac{x^2+y^2}{R_2^2}}\right)$$

Approximation with binomial expansion

$$\Delta(x,y) \simeq \Delta_0 - R_1\left(\cancel{1} - \left(\cancel{1} - \frac{x^2+y^2}{2R_1^2}\right)\right) + R_2\left(\cancel{1} - \left(\cancel{1} - \frac{x^2+y^2}{2R_2^2}\right)\right)$$

$$\Delta(x,y) \simeq \Delta_0 - \frac{x^2+y^2}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Phase Filter

$$t(x,y) = \exp[ikN\Delta_0]\exp\left[-ik(N-1)\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\right]$$

The formula
$$(N-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) = \frac{1}{f} \quad \text{Lensmaker's Formula}$$

A thin lens doesn't contain the cross terms

$$U_{OUT}(x,y,z) = \exp[ikN\Delta_0]\exp\left[\frac{-i\pi}{\lambda f}(x^2+y^2)\right]U_{IN}(x,y,z)$$

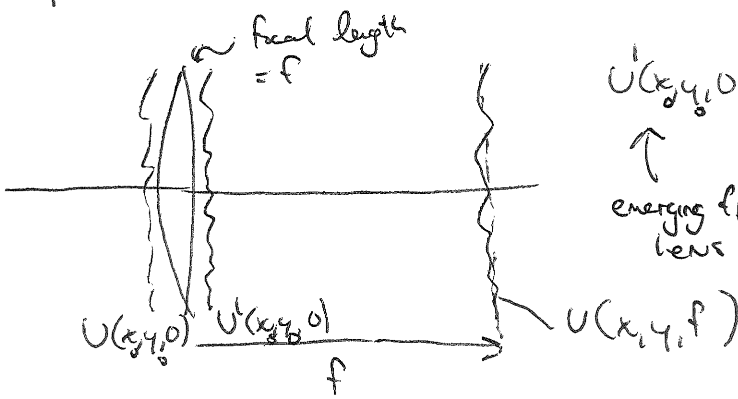
constant phase term (often ignored)

Quadratic phase term dependent upon the focal length of the lens.
Converging for $f>0$ and diverging for $f<0$

The effect of a thin lens is to introduce a quadratic phase factor that can be used to offset the quadratic phase factor in the Fresnel diffraction integral.

OBJECT PLACED AGAINST THE LENS

Here we'll look at the how the field changes when we know the field at the plane of the lens. We'll find the plane at the rear focal plane of the lens.



$$U'(x_0,y_0,0) = U(x_0,y_0,0)\,P(x_0,y_0)\exp\left[\frac{-i\pi}{\lambda f}(x_0^2+y_0^2)\right]$$

emerging from lens — incident on lens — pupil function

$P(x_0,y_0)$ = Pupil function

FRESNEL DIFFRACTION

$$U(x,y,f) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x^2+y^2)\right] \iint_{-\infty}^{\infty} U(x_0,y_0,0)\, P(x_0,y_0) \exp\left[\frac{-i\pi}{\lambda f}(x_0^2+y_0^2)\right]$$

$$\cdot \exp\left[\frac{i\pi}{\lambda f}(x_0^2+y_0^2)\right] \exp\left[-i2\pi\left(\frac{x}{\lambda z}x_0 + \frac{y}{\lambda z}y_0\right)\right] dx_0\, dy_0$$

if $P(x_0,y_0) \simeq 1$ when $U(x_0,y_0,0)$ has appreciable values.

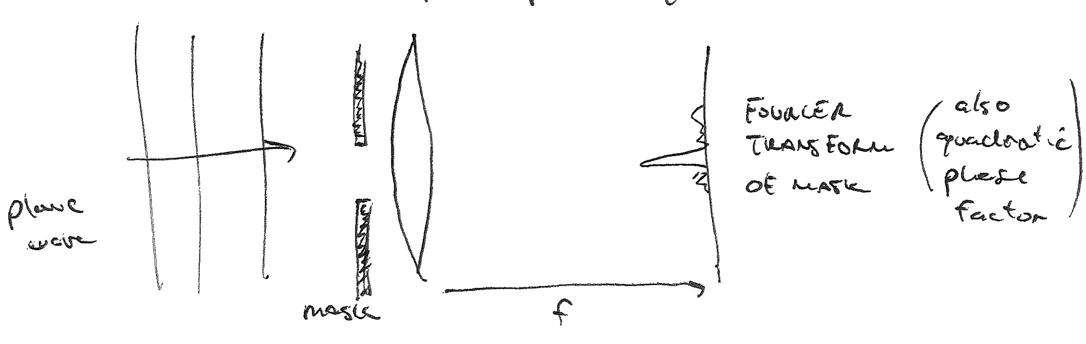
$$U(x,y,f) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x^2+y^2)\right] \mathcal{F}_{2D}\left\{U(x_0,y_0,0)\right\} \qquad \xi = \frac{x}{\lambda f} \quad \eta = \frac{y}{\lambda f}$$

~~Eb~~ Field in rear focal plane proportional to Fourier transform. This is essentially the case we saw when illuminating with a converging spherical wavefront because of extra quadratic phase term in front.

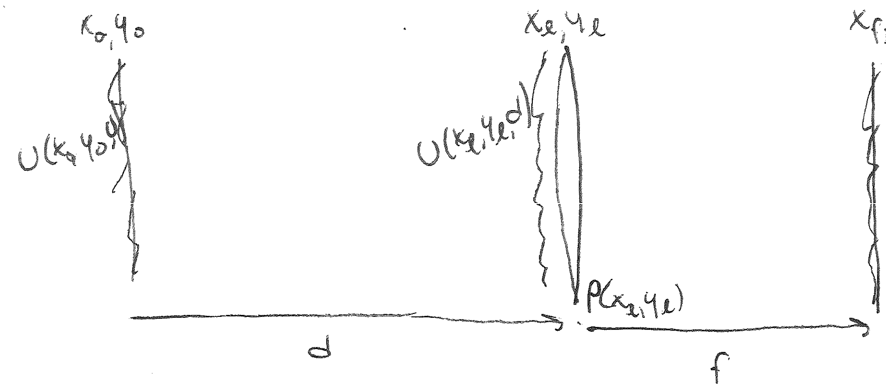
If $P(x_0,y_0)$ approximation doesn't hold

$$\boxed{U(x,y,f) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x^2+y^2)\right] \mathcal{F}_{2D}\left\{U(x_0,y_0,0)\, P(x_0,y_0)\right\} \qquad \xi = \frac{x}{\lambda f} \quad \eta = \frac{y}{\lambda f}}$$

When $P(x_0,y_0)$ is ~ 1 over the input field, we get essentially the same result as Fraunhofer diffraction. Thus, the lens can be thought of as imaging a distant plane (required for Fraunhofer approximation) to the rear focal point of the lens.



OBJECT PLACED IN FRONT OF LENS



put subscripts on x, y to keep planes straight.

From our previous result with the object against the lens (ignore $P(x_\ell, y_\ell)$ for the moment).

$$U(x_f, y_f, f+d) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x_f^2 + y_f^2)\right] \mathcal{F}_{2D}\left\{U(x_\ell, y_\ell, d)\right\} \qquad \xi_f = \frac{x_f}{\lambda f} \quad \eta_f = \frac{y}{\lambda f}$$

look at angular spectrum of input

$$A(\xi, \eta; 0) = \mathcal{F}_{2D}\left\{U(x_0, y; 0)\right\}$$

Propagate this within the Fresnel approximation to the lens.

$$A(\xi, \eta; d) = \exp(ikd) \exp\left(-i\pi\lambda d(\xi^2 + \eta^2)\right) A(\xi, \eta; 0)$$

But $A(\xi, \eta, d) = \mathcal{F}_{2D}\left\{U(x_\ell, y_\ell, d)\right\}$

So we can rewrite the field at the rear focal plane as

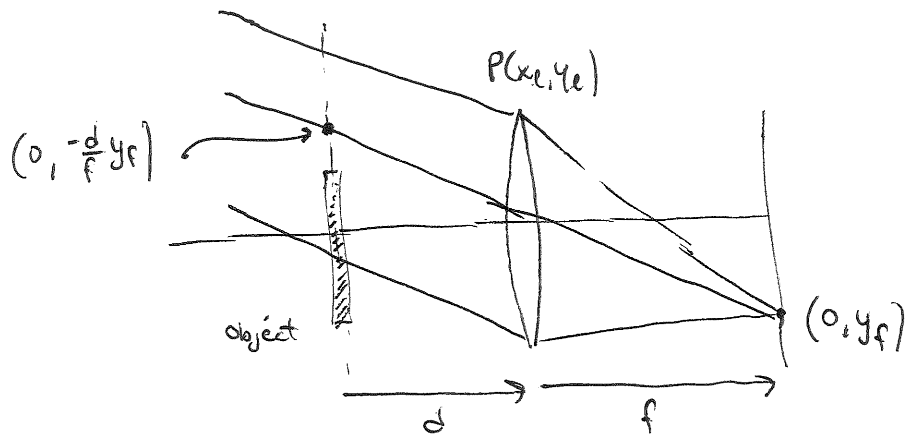
$$U(x_f, y_f, f+d) = \frac{\exp(ikf)}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}(x_f^2 + y_f^2)\right] A\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}; d\right)$$

with $A\left(\frac{x_f}{\lambda f}, \frac{y_f}{\lambda f}; d\right) = \exp(ikd) \exp\left(\frac{-i\pi}{\lambda f}\left(\frac{d}{f}\right)(x_f^2 + y_f^2)\right) A(\xi_f, \eta_f; 0)$

$$U(x_f, y_f, f+d) = \frac{\exp(ik(f+d))}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right] \mathcal{F}_{2D}\left\{U(x_0, y_0, 0)\right\} \quad \xi = \frac{x_f}{\lambda f} \quad \eta = \frac{y_f}{\lambda f}$$

ASSUMES ~~P~~ APERTURE OF LENS DOESN'T AFFECT ANYTHING

LET'S INCLUDE LENS APERTURE NOW FROM A GEOMETRIC POINT OF VIEW



vignetting

If d is too large or y_f is too large, the object can lay outside of the geometrical projection of $P(x_l, y_l)$ back onto the (x_0, y_0) plane.

To incorporate this effect into our results

$$U(x_f, y_f, f+d) = \frac{\exp(ik(f+d))}{i\lambda f} \exp\left[\frac{i\pi}{\lambda f}\left(1-\frac{d}{f}\right)(x_f^2+y_f^2)\right] \mathcal{F}_{2D}\left\{U(x_0, y_0, 0)\, P\left(x_0+\frac{d}{f}x_f,\ y_0+\frac{d}{f}y_f\right)\right\}$$

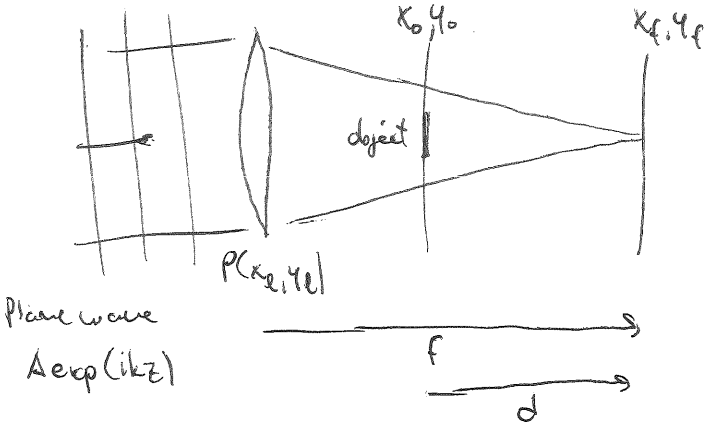
$$\xi = \frac{x_f}{\lambda f} \quad \eta = \frac{y_f}{\lambda f}$$

SPECIAL CASE: $d = f$ and $P \sim 1$ over object

$$U(x_f, y_f, f+d) = \frac{\exp(ik(f+d))}{i\lambda f} \mathcal{F}_{2D}\left\{U(x_0, y_0, 0)\right\} \quad \xi = \frac{x_f}{\lambda f} \quad \eta = \frac{y_f}{\lambda f}$$

FOURIER TRANSFORM AT THE SPEED OF LIGHT!

OBJECT PLACED BEHIND LENS



$$U'(x_0,y_0,0) = \underbrace{A\frac{f}{d}}_{\substack{\text{Amplitude adjusted}\\ \text{since it is being}\\ \text{concentrated}}}\ \underbrace{P\left(x_0\frac{f}{d},\, y_0\frac{f}{d}\right)}_{\substack{\text{Scaled pupil}\\ \text{of lens, geometric}\\ \text{projection to } (x_0,y_0)}}\ \underbrace{\exp\left[\frac{-i\pi}{\lambda d}(x_0^2+y_0^2)\right]}_{\substack{\text{spherical wave}\\ \text{converging to a}\\ \text{point distance } d\\ \text{away}}}\ \underbrace{U(x_0,y_0,0)}_{\substack{\text{effect of}\\ \text{the object on}\\ \text{incident field.}}}$$

$$U(x_f,y_f,d) = \frac{A\exp\left[\frac{i\pi}{\lambda d}(x_f^2+y_f^2)\right]}{i\lambda d}\,\frac{f}{d}\,\mathcal{F}_{2D}\left\{U(x_0,y_0,0)\,P\left(x_0\frac{f}{d},\, y_0\frac{f}{d}\right)\right\} \qquad \xi = \frac{x_f}{\lambda d} \quad \eta = \frac{y_f}{\lambda d}$$

since converging spherical wave cancels quadratic phase factor in the Fresnel propagation integral as shown previously on page 142

This configuration still has quadratic phase factor $\exp\left[\frac{i\pi}{\lambda d}(x_f^2+y_f^2)\right]$

but has the advantage that the size of the Fourier transform can be scaled by adjusting d relative to f.

where	QUADRATIC PHASE FACTOR	ADVANTAGE	DISADVANTAGE
AGAINST	YES	NO VIGNETTING FOR OBJECT SMALLER THAN LENS APERTURE	QUADRATIC PHASE
IN FRONT	NO, IF $d=f$	EXACT FOURIER TRANSFORM WHEN $d=f$	VIGNETTING RISK INCREASES
BEHIND	YES	FOURIER TRANSFORM CAN BE SCALED	VIGNETTING RISK INCREASES / QUADRATIC

THE 4F SYSTEM

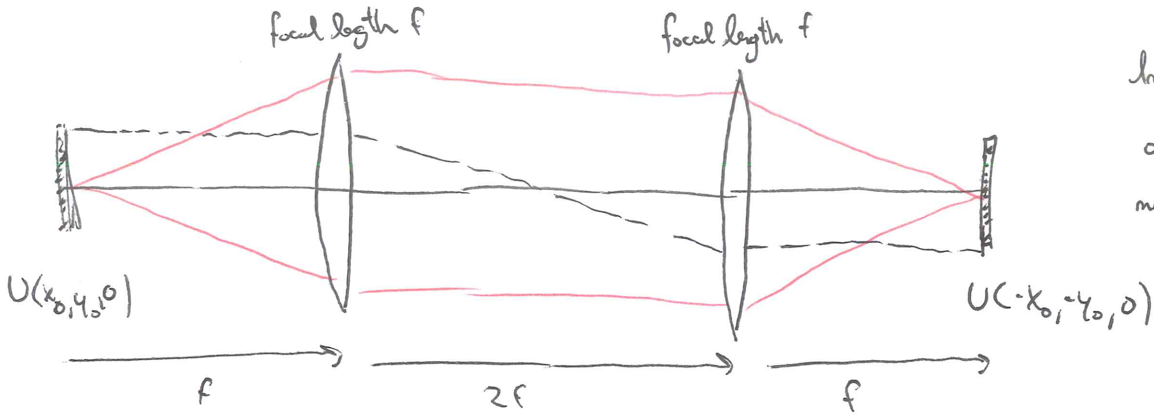
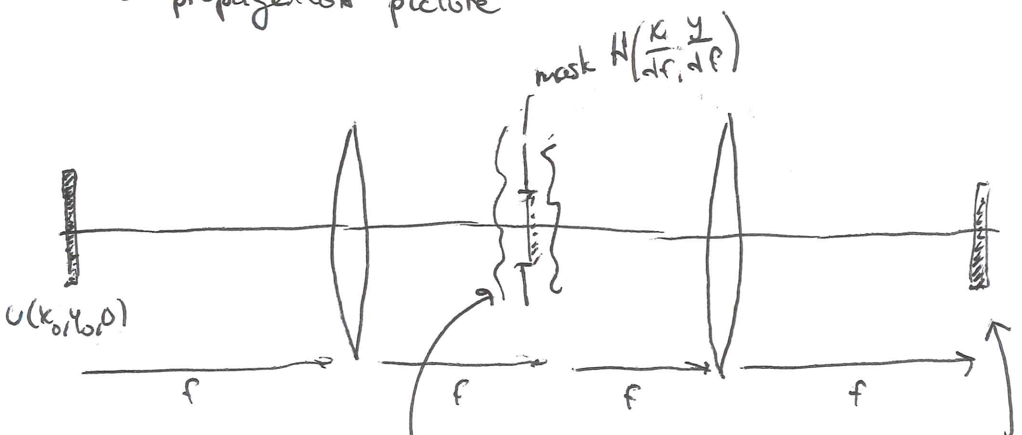


Image reproduces our object, but with magnification $m=-1$

Fresnel propagation picture



$\mathcal{F}_{2D}\{U(x_0,y_0,0)\}$

$\xi=\frac{x}{\lambda f}\quad \eta=\frac{y}{\lambda f}$

$U(-x_0,-y_0,0) ** \mathcal{F}_{2D}\left\{H\left(\frac{x}{\lambda f},\frac{y}{\lambda f}\right)\right\}$

equals

$U(-x_0,-y_0,0) ** h(-x_0,-y_0)$

THIS GIVES US A WAY TO DO THE SIGNAL PROCESSING DISCUSSED ON PAGES 83-89

NOTE $\quad H(\xi,\eta)=\mathcal{F}_{2D}\{h(x,y)\}$

second lens does

$$\mathcal{F}_{2D}\{H(\xi,\eta)\}=\mathcal{F}_{2D}\left[\mathcal{F}_{2D}\{h(x,y)\}\right]=h(-x,-y)$$

Conceptually, this should be pleasing since it's consistent with magnification $m=-1$.