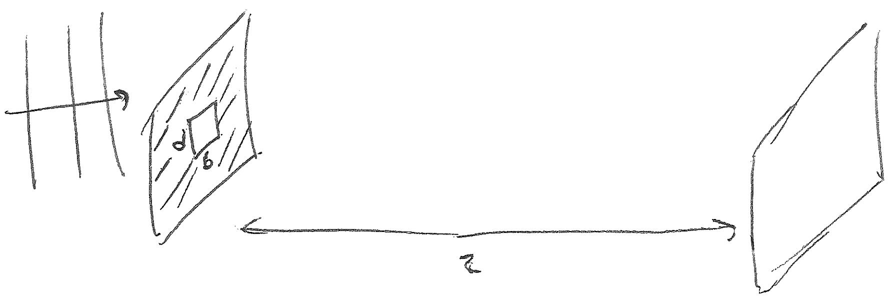
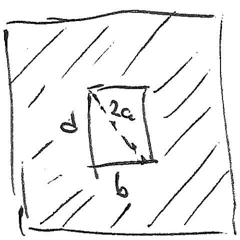
Example: Fraunhofer diffraction of a plane wave striking at rectangular aperture. Find ~~distance~~ irradiance pattern on screen a distance z away.



First let's figure out where the Fraunhofer approximation is valid.

We require

$$N_F = \frac{a^2}{\lambda z} \ll 1$$



The maximum extent of the aperture is along the diagonal so

$$4a^2 = b^2 + d^2$$

Require $\quad \dfrac{b^2+d^2}{4\lambda z} \ll 1 \quad \Rightarrow \quad z \gg \dfrac{b^2+d^2}{4\lambda}$

If $b = d = 1\text{mm}$ and $\lambda = 0.5\,\mu\text{m} \Rightarrow z \gg 1\text{m}$

If $b = d = 25\text{mm}$ and $\lambda = 0.5\,\mu\text{m} \Rightarrow z \gg 625\text{ m}$

The plane wave incident of the aperture has the form Ae^{ikz}, but this is just A when $z = 0$, so

$$U(x_0, y_0, 0) = A\,\text{rect}\left(\frac{x_0}{b}, \frac{y_0}{d}\right)$$

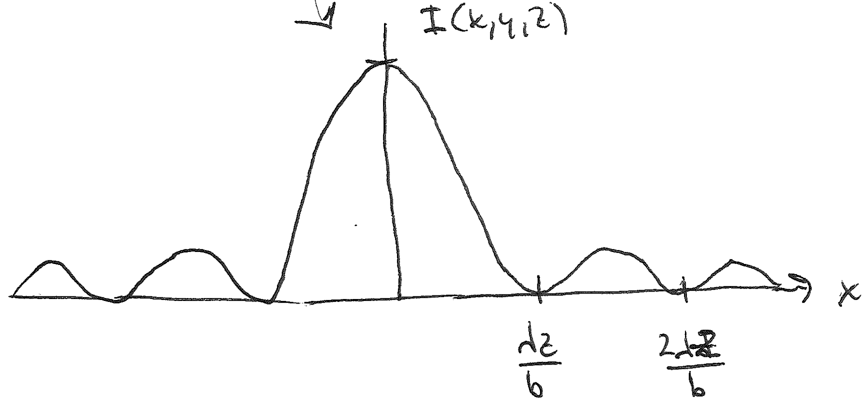
$$\mathcal{F}\{U(x_0, y_0, 0)\} = Abd\,\text{sinc}(b\xi, d\eta)$$

The spatial frequency variables $\xi = \frac{x}{\lambda z}$ and $\eta = \frac{y}{\lambda z}$. These basically convert the spatial frequency values into spatial coordinates on the plane a distance z away. From the Fraunhofer diffraction formula

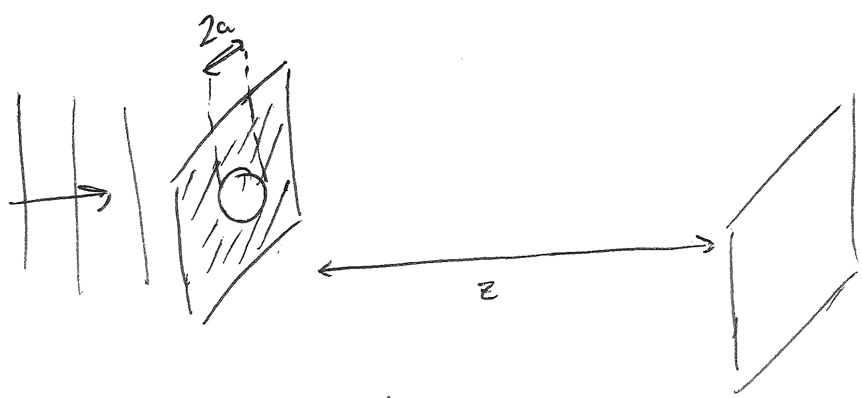
$$U(x,y,z) = Abd \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right] \operatorname{sinc}\left(\frac{bx}{\lambda z}, \frac{dy}{\lambda z}\right)$$

Irradiance

$$I(x,y,z) = |U(x,y,z)|^2 = \frac{A^2 b^2 d^2}{\lambda^2 z^2} \operatorname{sinc}^2\left(\frac{bx}{\lambda z}, \frac{dy}{\lambda z}\right)$$



Example: Fraunhofer diffraction of a plane wave illuminating a circular aperture. Find the irradiance pattern at distance z.



$z \gg \frac{a^2}{\lambda}$ to have valid approximation

$$U(x_0, y_0, 0) = A\,\mathrm{cyl}\left(\frac{r}{2a}\right)$$

$$\mathcal{F}\{U(x_0, y_0, 0)\} = (2a)^2 A \frac{\pi}{4}\,\mathrm{somb}(2a\rho) \qquad \text{where } \rho = \sqrt{\xi^2 + \eta^2}$$

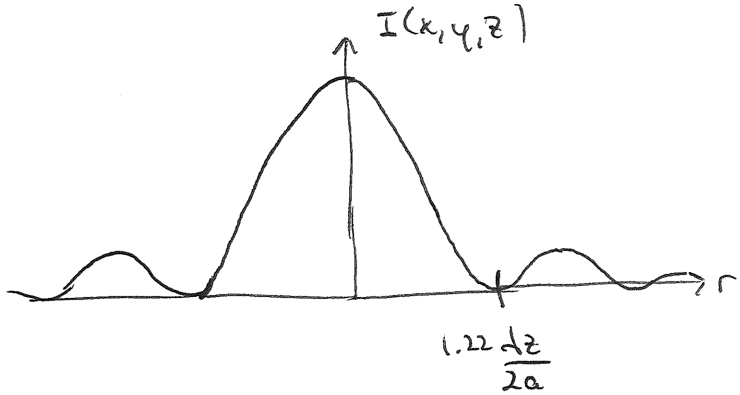
Replace $\xi = \frac{x}{\lambda z}$ $\eta = \frac{y}{\lambda z}$ $\Rightarrow \rho = \frac{r}{\lambda z}$ $\qquad r = \sqrt{x^2+y^2}$

Our field is

$$U(x,y,z) = a^2 A\pi\,\frac{\exp(ikz)}{i\lambda z}\exp\left(\frac{i\pi}{\lambda z}(x^2+y^2)\right)\mathrm{somb}\left(\frac{2ar}{\lambda z}\right)$$

Irradiance

$$I(x,y,z) = |U(x,y,z)|^2 = \left(\frac{a^2 A\pi}{\lambda z}\right)^2 \mathrm{somb}^2\left(\frac{2ar}{\lambda z}\right) \qquad \text{AIRY PATTERN}$$



1^{st} zero when

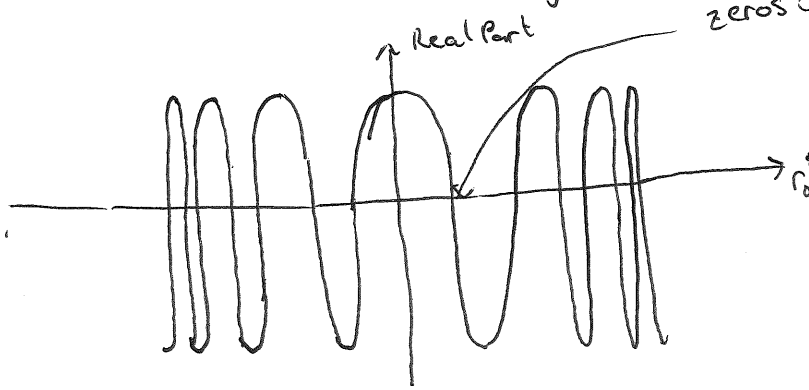
$$\frac{2ar}{\lambda z} = 1.22$$

$$r = 1.22\frac{\lambda z}{2a}$$

Fraunhofer diffraction requires large distances z which can quickly become unpractical. Let's try to understand the term

$$\exp\left[\frac{i\pi}{\lambda z}(x_0^2 + y_0^2)\right]$$

zeros when $\frac{\pi r_0^2}{\lambda z} = (2m+1)\frac{\pi}{2}$ $\quad$ m integer



$$r_0 = \sqrt{\frac{(2m+1)}{2}\lambda z}$$

Another way to handle the quadric phase factor is to try and cancel it with a component of the incident field. With Fraunhofer, we just assumed we're really far away. Now we can move into the Fresnel region if an input field is

$$U(x_0, y_0, 0) = U'(x_0, y_0, 0) \exp\left[\frac{-i\pi}{\lambda z}(x_0^2 + y_0^2)\right]$$

with $r_0^2 = x_0^2 + y_0^2$, the Fresnel diffraction integral becomes

$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z} r^2\right] \mathcal{F}_{2D}\left\{ U'(x_0, y_0, 0) \right\}$$

with $\xi = \frac{x}{\lambda z}$ $\eta = \frac{y}{\lambda z}$

What is this term $\exp\left(\frac{-i\pi}{\lambda z} r_0^2\right]$?

When we looked at spherical waves, we had

$$U(x,y,z) = \frac{A}{r} \exp(ikr)$$

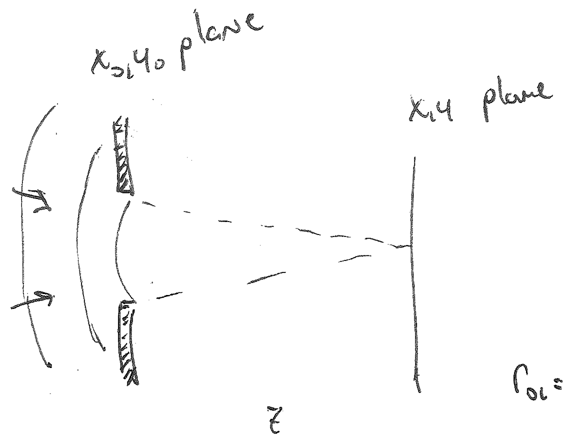
which is an expanding wavefront leaving the origin. We know it's expanding because if we include the time dependence

$$U(x,y,z;t) = \frac{A}{r} \exp(i(kr - \omega t))$$

As t increases, r has to increase to keep the phase of this function constant. A surface of constant phase is just the wavefront. If instead we have

$$U(x,y,z;t) = \frac{A}{r} \exp(-i(kr + \omega t)) = \frac{A}{r} \exp(i(-kr - \omega t))$$

then as t increases, then r must decrease to keep the phase term constant. This is a converging spherical wave.



$$\frac{A}{r_{01}} \exp\left(-ikr_{01}\right)$$

$$r_{01} = \left[x_0^2 + y_0^2 + z^2 \right]^{1/2}$$

$$r_{01} = z\left[1 + \frac{x_0^2 + y_0^2}{z^2}\right]^{1/2}$$

Converging spherical wave approximation

$$\frac{A}{z} \exp(-ikz) \exp\left[\frac{-i\pi}{\lambda z}(x_0^2 + y_0^2)\right]$$ parabolic wavefront converging to $(0,0,z)$

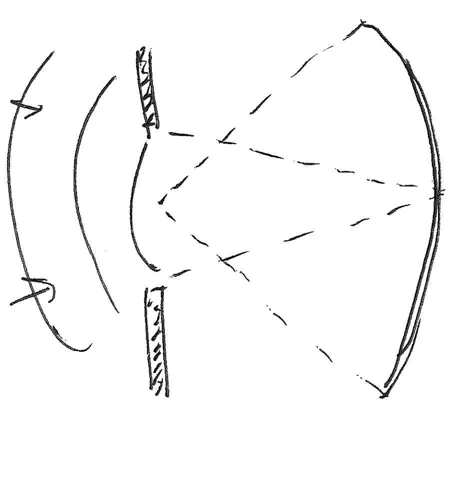
Fresnel Diffraction becomes

field introduced at plane $z=0$

$$U(x,y,z) = \frac{A}{i\lambda z^2} \exp\left[\frac{i\pi}{\lambda z} r^2\right] \mathcal{F}_{2D}\left\{ U(x_0, y_0, 0) \right\}$$

Note A/z ~ constant over aperture = Amplitude of spherical wave

So we get something that is related to the Fourier transform, but with quadratic phase in observation plane



If we curved the observation plane, the Fresnel diffraction pattern would just be Fourier transform. Often we don't even care about quadratic phase pattern since we are looking at $|U(x,y,z)|^2$.

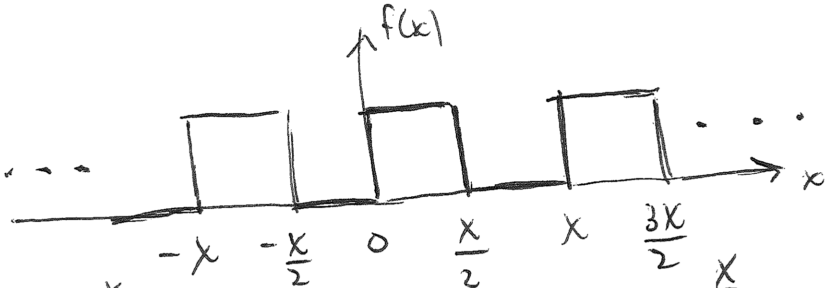
COMPLEX FOURIER SERIES REVISITED

Recall from page 34 of the notes the definition of the complex Fourier series

$$f(x) = \sum_{m=-\infty}^{\infty} a_m \exp\left(i2\pi m \xi_0 x\right)$$

$$a_m = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\exp\left[-i2\pi m \xi_0 x\right] dx \qquad \text{where } X = \frac{1}{\xi_0} = \text{period}$$

Let's find the Fourier series of



$$a_{m'} = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\exp\left[-i2\pi m' \xi_0 x\right] dx = \frac{1}{X}\int_{0}^{\frac{X}{2}} \exp\left[-i2\pi m' \xi_0 x\right] dx$$

use m' for the moment.

$$a_{m'} = \frac{1}{X}\left.\frac{\exp\left[-i2\pi m' \xi_0 x\right]}{-i2\pi m' \xi_0}\right|_0^{\frac{X}{2}} = \frac{1}{-i2\pi m'}\left[\exp\left[-im'\pi\right] - 1\right]$$

$$a_{m'} = \frac{\exp\left[-im'\frac{\pi}{2}\right]}{-i2\pi m'}\left(\exp\left[-im'\frac{\pi}{2}\right] - \exp\left[im'\frac{\pi}{2}\right]\right)$$

$$\boxed{a_{m'} = \frac{1}{2}\exp\left[-im'\frac{\pi}{2}\right]\operatorname{sinc}\left(\frac{m'}{2}\right)}$$

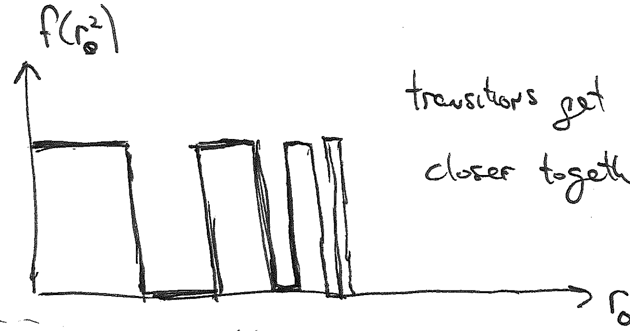
So we can write this example as

$$f(x) = \frac{1}{2}\sum_{m=-\infty}^{\infty} \exp\left(+im\frac{\pi}{2}\right)\operatorname{sinc}\left(\frac{m}{2}\right)\exp\left(-i2\pi m \xi_0 x\right)$$ let $m = -m'$

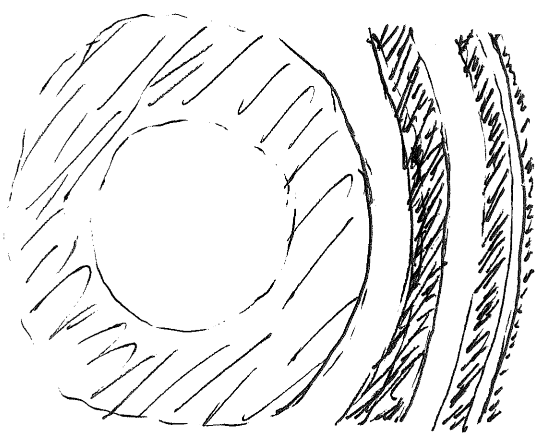
Suppose now I define a transmission mask $t(r)$ where

$$t(r_0) = f(r_0^2) = \frac{1}{2}\sum_{m=-\infty}^{\infty} \exp\left(+im\frac{\pi}{2}\right)\operatorname{sinc}\left(\frac{m}{2}\right)\exp\left(-i2\pi m \xi_0 r_0^2\right)$$

T	$r_0^2 = x$	r_0
$1 \Rightarrow 0$	$X/2$	$\sqrt{\frac{X}{2}}$
$0 \Rightarrow 1$	X	$\sqrt{X}$
$1 \Rightarrow 0$	$3X/2$	$\sqrt{\frac{3X}{2}}$
$0 \Rightarrow 1$	$2X$	$\sqrt{2X}$



transitions get progressively closer together



What happens when we put this mask into the Fresnel diffraction formula?

$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z}\exp\left[\frac{i\pi}{\lambda z}r^2\right]\cdot\frac{1}{2}\sum_{m=-\infty}^{\infty}\exp\left(+im\frac{\pi}{2}\right)\operatorname{sinc}\left(\frac{m}{2}\right)$$

$$\cdot\,\mathcal{F}_{2D}\left\{\exp\left(-i2\pi m\xi_0 r_0^2\right)\exp\left(\frac{i\pi}{\lambda z}r_0^2\right)\right\}$$

Ignore finite aperture for the time being.

with $\xi = \frac{x}{\lambda z}$ $\eta = \frac{y}{\lambda z}$

For the vast majority of terms, the quadratic phase factor will remain. However, there are certain cases where we can get one term to cancel.

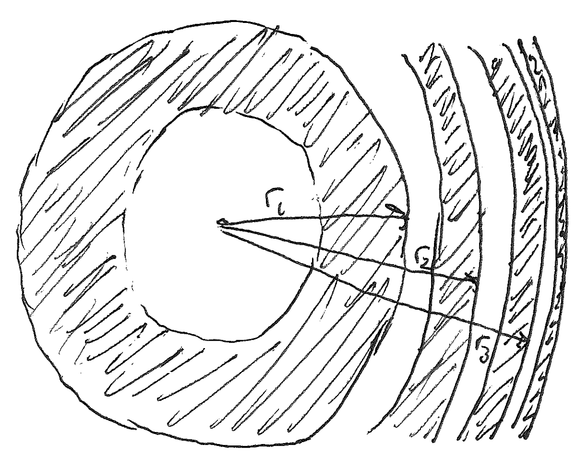
If $\cancel{2}\cancel{\pi} m_1^* \xi_0 \cancel{r_0^2} = \frac{\cancel{\pi}}{\lambda z} \cancel{r_0^2}$ then the terms cancel

$$\frac{1}{\xi_0} = X = 2 m_1^* \lambda z_{m_1^*}$$ m_1^* is a specific value of m

So if we build a zone plate with this constraint, the locations of the ~~rings~~ rings (one complete cycle of bright to dark in this case) are

$$r_j^2 = 2 j m_1^* \lambda z_{m_1^*} \rightarrow r_j = \sqrt{2 j m_1^* \lambda z_{m_1^*}}$$

j	r_j
0	0
1	$\sqrt{2 m_1^* \lambda z_{m_1^*}}$
2	$\sqrt{4 m_1^* \lambda z_{m_1^*}}$
3	$\sqrt{6 m_1^* \lambda z_{m_1^*}}$



each of these is called a Fresnel zone.

Fresnel Diffraction

m' term

$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z} \exp\left(\frac{i\pi}{\lambda z} r^2\right) \frac{1}{2} \Bigg[\mathcal{F}_{2D}\{1\} \ast \exp\left(i m' \frac{\pi}{2}\right) \mathrm{sinc}\left(\frac{m'}{2}\right) + \sum_{\substack{m=-\infty \\ m \neq m'}}^{\infty} \exp\left(i m \frac{\pi}{2}\right) \mathrm{sinc}\left(\frac{m}{2}\right) \mathcal{F}\left\{ \cancel{\exp\left(i m \frac{\pi}{2}\right) \mathrm{sinc}\left(\frac{m}{2}\right)} \exp\left(\frac{i\pi}{\lambda z}\left(1 - \frac{m}{m'}\right) r_0^2\right) \right\} \Bigg]$$

Fresnel Diffraction

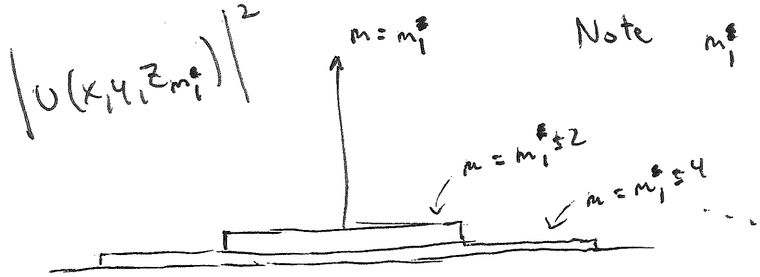
$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z} r^2\right] \frac{1}{2} \sum_{m=-\infty}^{\infty} \exp\left(im\frac{\pi}{2}\right) \text{sinc}\left(\frac{m}{2}\right) \mathcal{F}_{2D}\left\{\exp\left[\frac{i\pi}{\lambda}\left[\frac{m}{m_1^* z_{m_1^*}} - \frac{1}{z}\right] r_0^2\right]\right\}$$

examining the Fourier transform term

when $m = m_1^*$ and $z = z_{m_1^*}$, we have $\mathcal{F}_{2D}\{1\} = \delta\left(\frac{x}{\lambda z_{m_1^*}}, \frac{y}{\lambda z_{m_1^*}}\right)$

All other terms in series still have quadratic phase factor,

From a geometric viewpoint, the irradiance at $z = z_{m_1}$ looks like



Note m_1^* needs to be odd to get delta function because of $\text{sinc}\left(\frac{m}{2}\right)$

A portion of the incident light is getting focused to a point of the plane $z = z_{m_1^*}$. The remainder is an out of focus "halo" around the point.

If we include an aperture to limit the size of the Fresnel zone plate then this appears in the Fourier transform. For example, a circular aperture of diameter d has in the above

$$\mathcal{F}_{2D}\left\{\exp\left[\frac{i\pi}{\lambda}\left(\frac{m}{m_1^* z_{m_1^*}} - \frac{1}{z}\right] r_0^2\right] \text{cyl}\left(\frac{r_0}{d}\right)\right\}$$

when $m = m_1^*$ and $z = z_{m_1^*}$

$$\mathcal{F}_{2D}\left\{\text{cyl}\left(\frac{r_0}{d}\right)\right\} = \frac{d\pi}{4} \text{somb}\left(\frac{dr}{\lambda z_{m_1^*}}\right)$$

delta function becomes Airy pattern in irradiance.

Let's look at other distances

$$\frac{m}{m_1^* z_{m_1^*}} - \frac{1}{z} = 0$$

$$\boxed{z = \frac{m_1^* z_{m_1^*}}{m}}$$

Again m can only be odd or zero because of $\mathrm{sinc}\left(\frac{m}{2}\right)$

At distances that satisfy this expression, a portion of the light will come into focus. We can think of the Fresnel zone plate as a "lens" with multiple focal lengths. If $f_{m_1^*} = z_{m_1^*}$ is the primary focal length of the zone plate, then the other focal lengths will be

$$\boxed{f_m = \frac{m_1^*}{m} f_{m_1^*}}$$

Note f_m can be positive or negative depending upon the sign of m

Examples $m_1^* = 1$

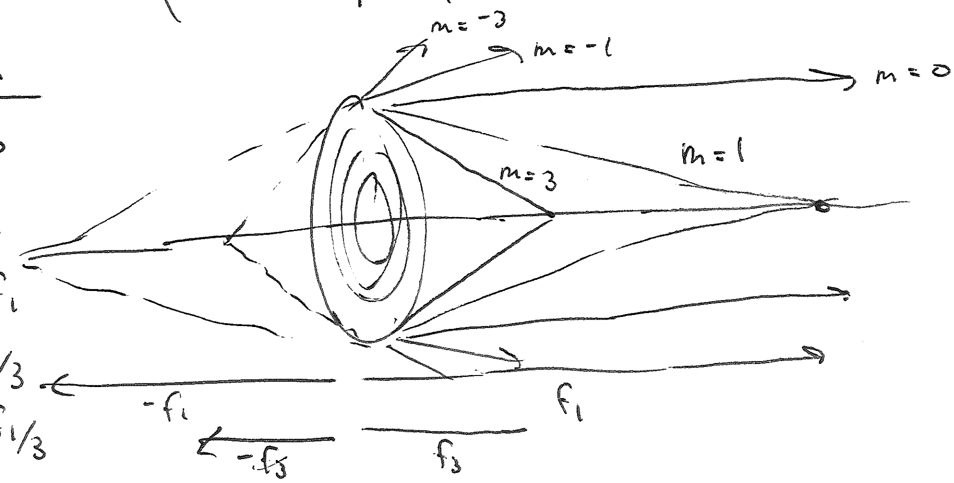
We have the 1^{st} zone of the Fresnel zone plate as

$$r_1 = \sqrt{2\lambda f_1} \Rightarrow f_1 = \frac{r_1^2}{2\lambda}$$

So physical dimensions of zone plate define the focal length.

& Possible values of $m = 0, \pm 1, \pm 3 \ldots$

m	f_m
0	∞
1	f_1
-1	$-f_1$
3	$f_1/3$
-3	$-f_1/3$



Each value of m corresponds to a diffraction order. Thus we can refer to the respective foci of the Fresnel zone plate by their diffraction order. The 0^{th} order $(m=0)$ is light that passes straight through the zone plate. The $+1^{st}$ order is the primary focal length f_1. The -1^{st} order has focal length $f_{-1} = -f_1$. The $+3^{rd}$ order has focal length $f_3 = \frac{1}{3} f_1$, etc.

Diffraction Efficiency

We now would like to know how much energy goes into each of these foci. Way back on page 71 of the notes, we described Parseval's theorem for Fourier Series, which says

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{m=-\infty}^{\infty} |a_m|^2 \quad \text{in our current notation}$$

The a_m here are the expansion coefficients of our complex Fourier Series (Note we have already switched $m' = -m$) in our description from page 143. Parseval's theorem is a statement about conservation of energy. Each term $|a_m|^2$ describes ~~to~~ the relative amount of energy going into the m^{th} diffraction order. The sum of all these terms represents the total energy.

For the Fresnel zone plate

DIFFRACTION EFFICIENCY η_m

$$a_m = \frac{1}{2} \exp\left[i m \frac{\pi}{2} \right] \operatorname{sinc}\left(\frac{m}{2} \right) \qquad \boxed{|a_m|^2 = \frac{1}{4} \operatorname{sinc}^2\left(\frac{m}{2} \right)}$$

m	η_m
± 5	0.004
± 3	0.011
± 1	0.101
0	0.25

For m even $\eta_m = 0$

25% of the light passes straight through unaffected

$\sum_{m=-\infty}^{\infty} |a_m|^2 = 0.5$ 50% of the light is blocked by opaque rings

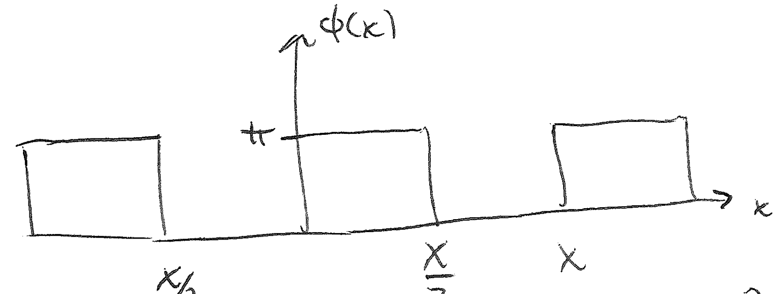
$\eta_1 = 0.101$ 10.1% of the light goes to primary focal point.

The problem with Fresnel zone plates are that they are highly inefficient, but may be the only choice is some cases like xrays where limited choices are available for lenses.

FRESNEL ZONE PHASE PLATE (PHASE REVERSAL ZONE PLATE)

Instead of modulating the amplitude, let's use a function that is periodic in phase.

$f(x) = \exp[i\phi(x)]$



SINCE exp(0) here

$$a_m = \frac{1}{X}\left[\int_{-\frac{X}{2}}^{0} \exp\left(-i2\pi m \xi_0 x\right] dx + \int_0 \exp(i\pi) \exp\left[-i2\pi m \xi_0 x\right] dx\right]$$

$$a_m = \frac{1}{\cancel{X}}\left[\frac{\exp\left[-i2\pi m \xi_0 x\right]}{-i2\pi m \cancel{\xi_0}}\Bigg|_{-\frac{X}{2}}^{0} - \frac{\exp\left[-i2\pi m \xi_0 x\right]}{-i2\pi m \cancel{\xi_0}}\Bigg|_{0}^{X/2}\right]$$

$$a_m = \frac{1}{-i2\pi m}\left[1 - \exp[im\pi] - (\exp[-im\pi] - 1)\right]$$

$$a_m = \frac{1}{-im\pi}\left[1 - \cos m\pi\right] = i\,\frac{m\pi}{2}\,\mathrm{sinc}^2\!\left(\frac{m}{2}\right)$$

The diffraction efficiency is now

$$\eta_m = |a_m|^2 = \left|\frac{m\pi}{2}\right|^2 \mathrm{sinc}^4\!\left(\frac{m}{2}\right)$$

m	η_m
± 5	0.016
± 3	0.045
± 1	0.405
0	0

For m even $\eta_m = 0$ including $\eta_0 = 0$

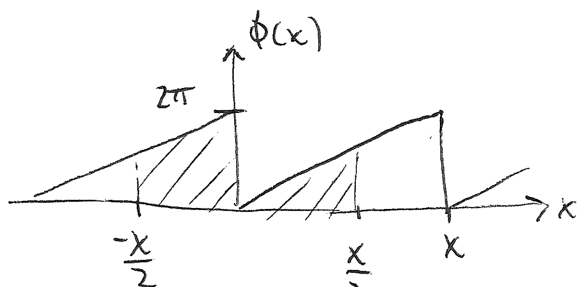
$\sum_{m=-\infty}^{\infty} |a_m|^2 = 1$ 100% passes through

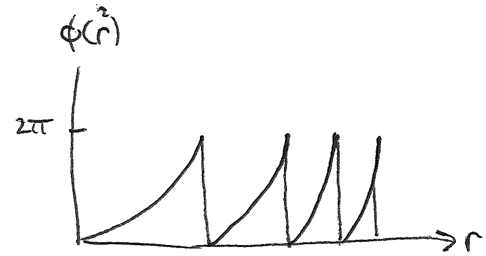
$\eta_1 = 0.405$ 40.5% of light goes to primary focal point, a 4x increase from Fresnel zone plate.

Can we do even better?

Kinoform

$$f(x) = \exp\left[i\phi(x)\right]$$





Let's make our lives a little easier and integrate from 0 to X instead.

$$a_m = \frac{1}{X}\int_0^X \exp\left(i2\pi\xi_0 x\right)\exp\left(-i2\pi m\xi_0 x\right)dx$$

$$a_m = \frac{1}{\cancel{X}}\;\frac{\exp\left(-i2\pi(m-1)\xi_0 x\right)}{-i2\pi(m-1)\cancel{\xi_0}}\Bigg|_0^X$$

$$a_m = \frac{1}{-i2\pi(m-1)}\left[\exp(-i2\pi(m-1)) - 1\right]$$

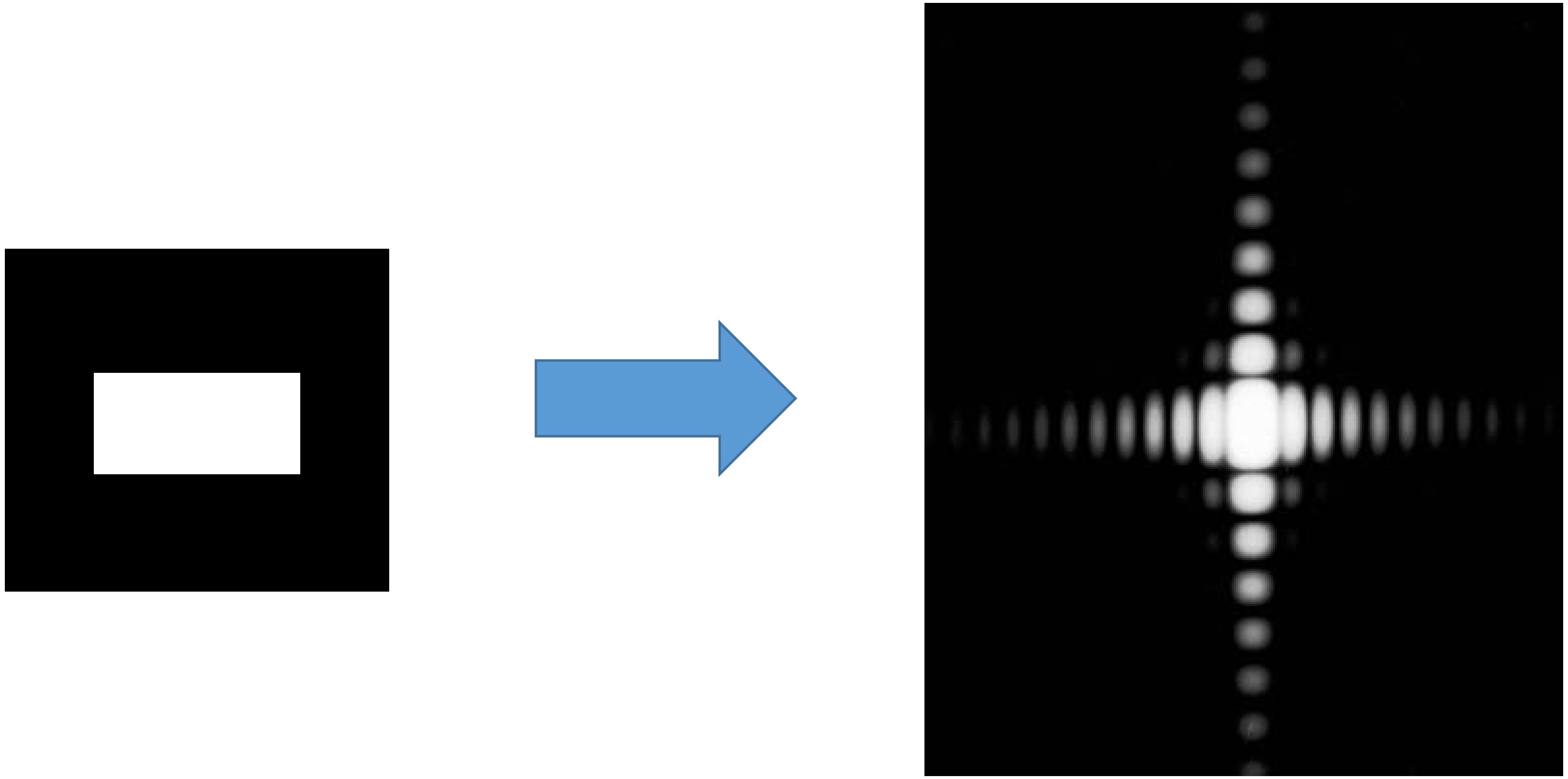
$$a_m = \exp\left[-i\pi(m-1)\right] \operatorname{sinc}(m-1)$$

The diffraction efficiency is now

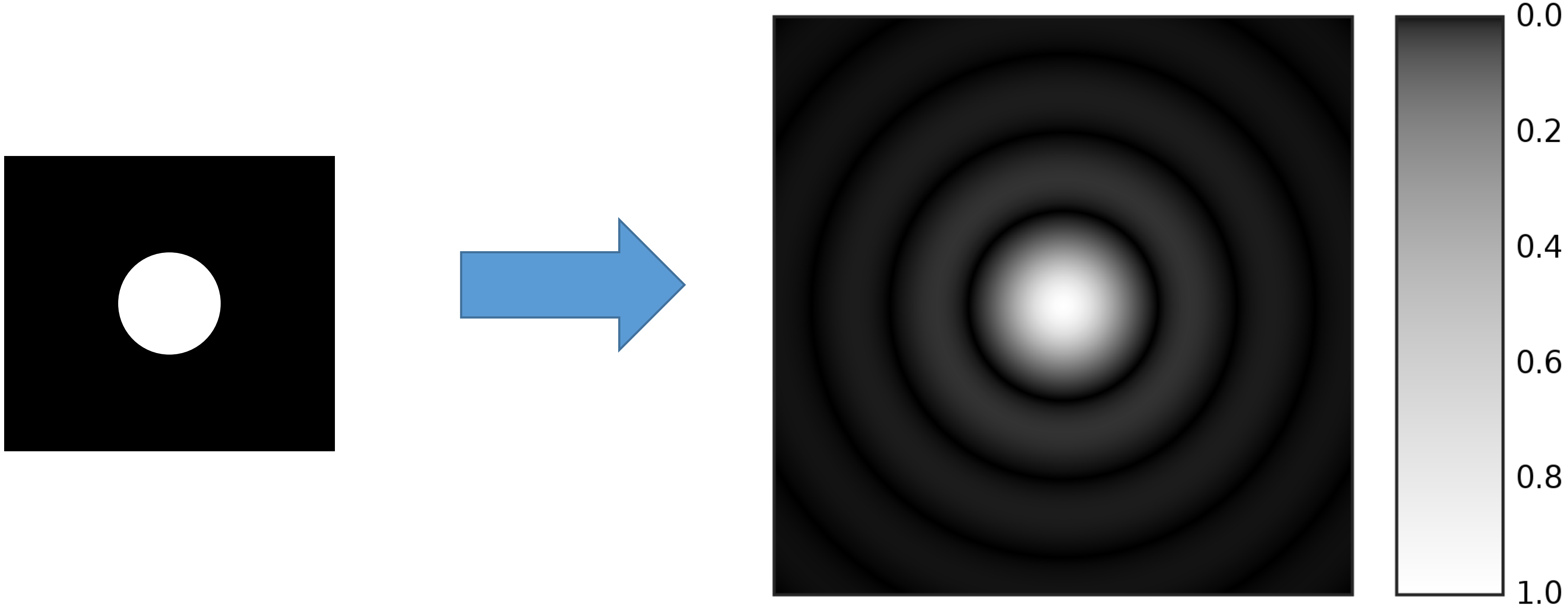
$$\eta_m = |a_m|^2 = \operatorname{sinc}^2(m-1) = \begin{cases} 1 & \text{for } m=1 \\ 0 & \text{otherwise} \end{cases}$$

All of the light goes into primary focal point!

Fraunhofer Diffraction

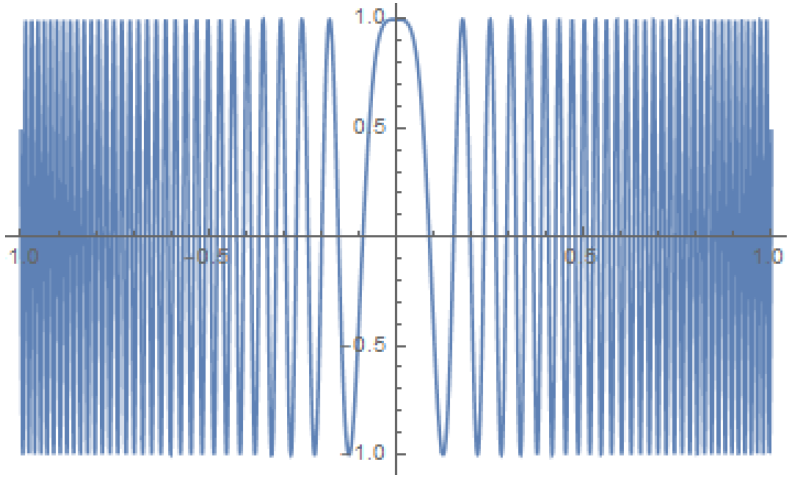


Fraunhofer Diffraction

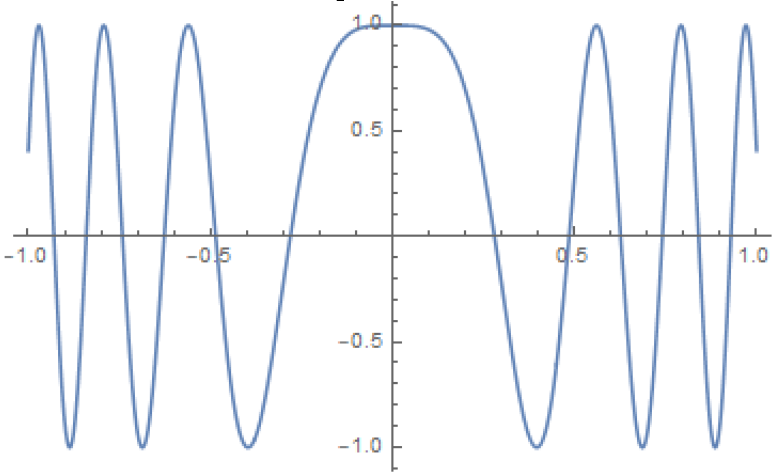


Quadratic Phase Factor

$N_F = 200$

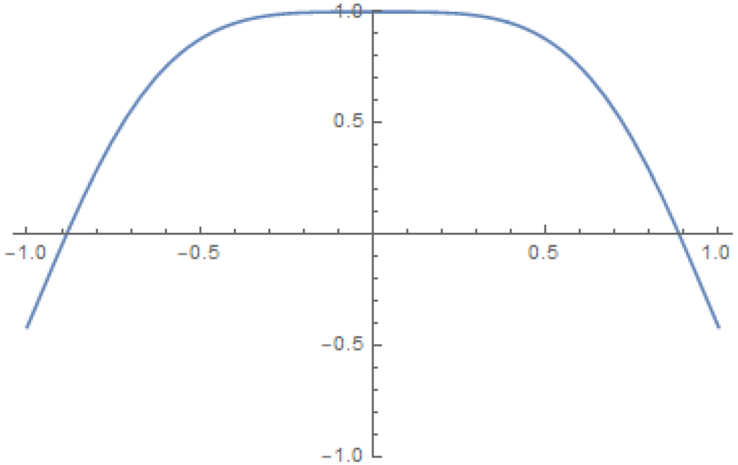


$N_F = 20$

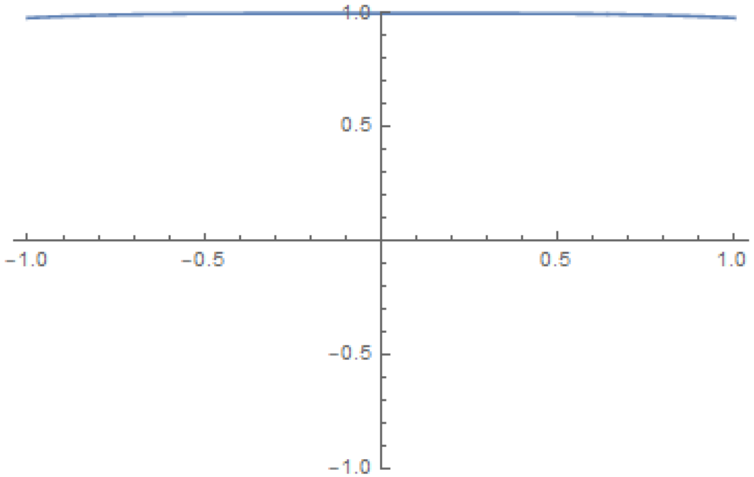


In general, moving into the Fresnel region means we need to calculate the fields numerically. In this case, you need to be wary of the sampling of this quadratic phase exponential to ensure accurate calculations.

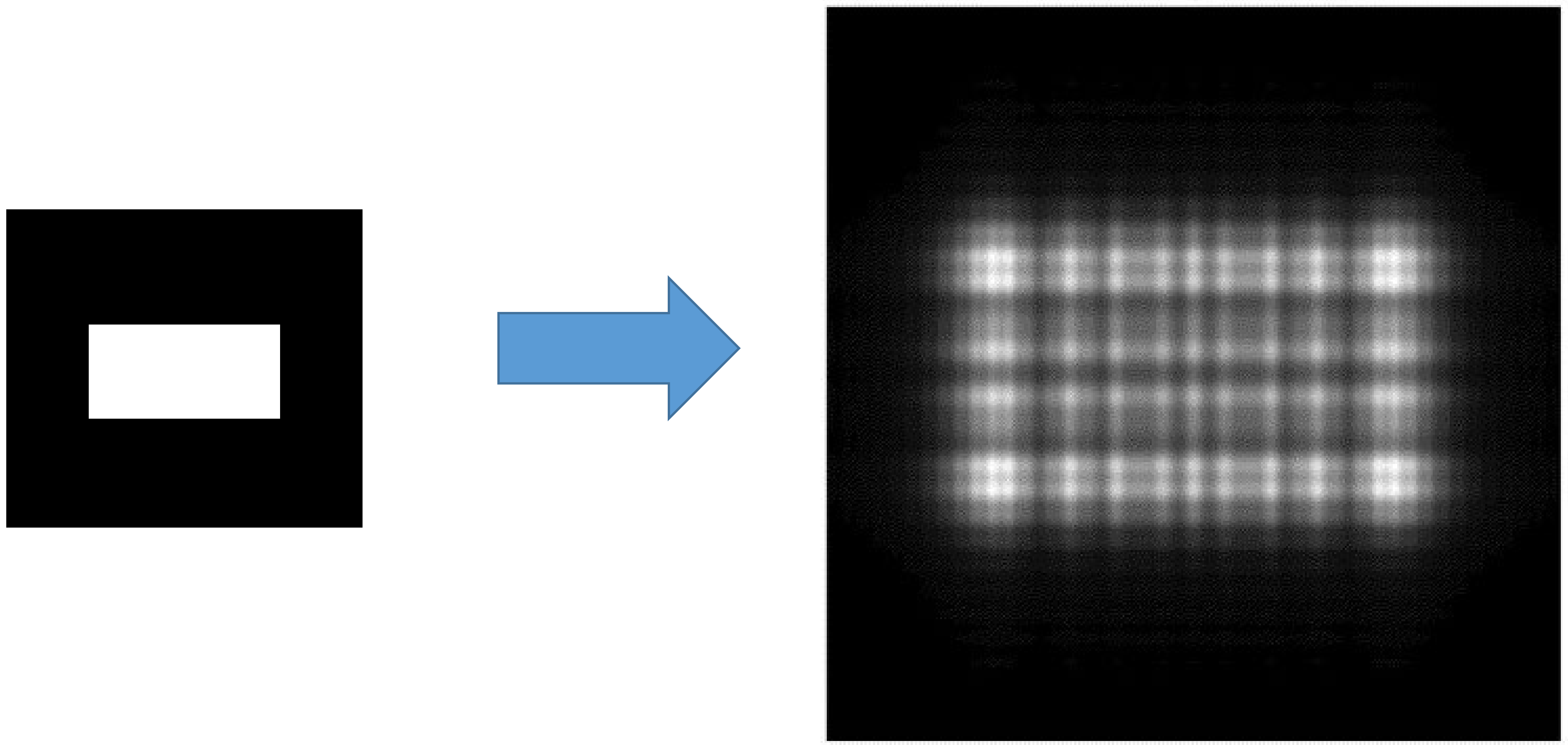
$N_F = 2$



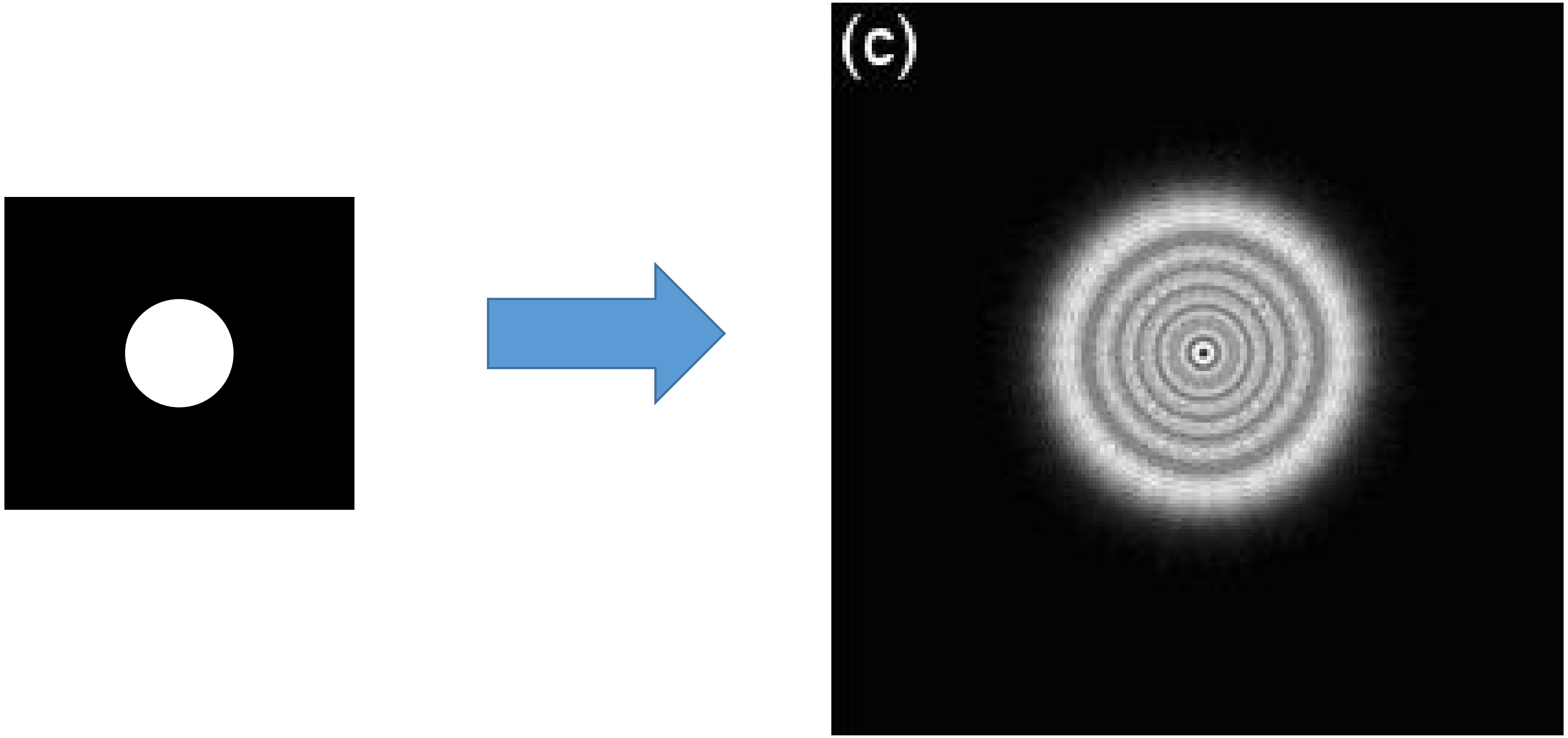
$N_F = 0.2$



Fresnel Diffraction

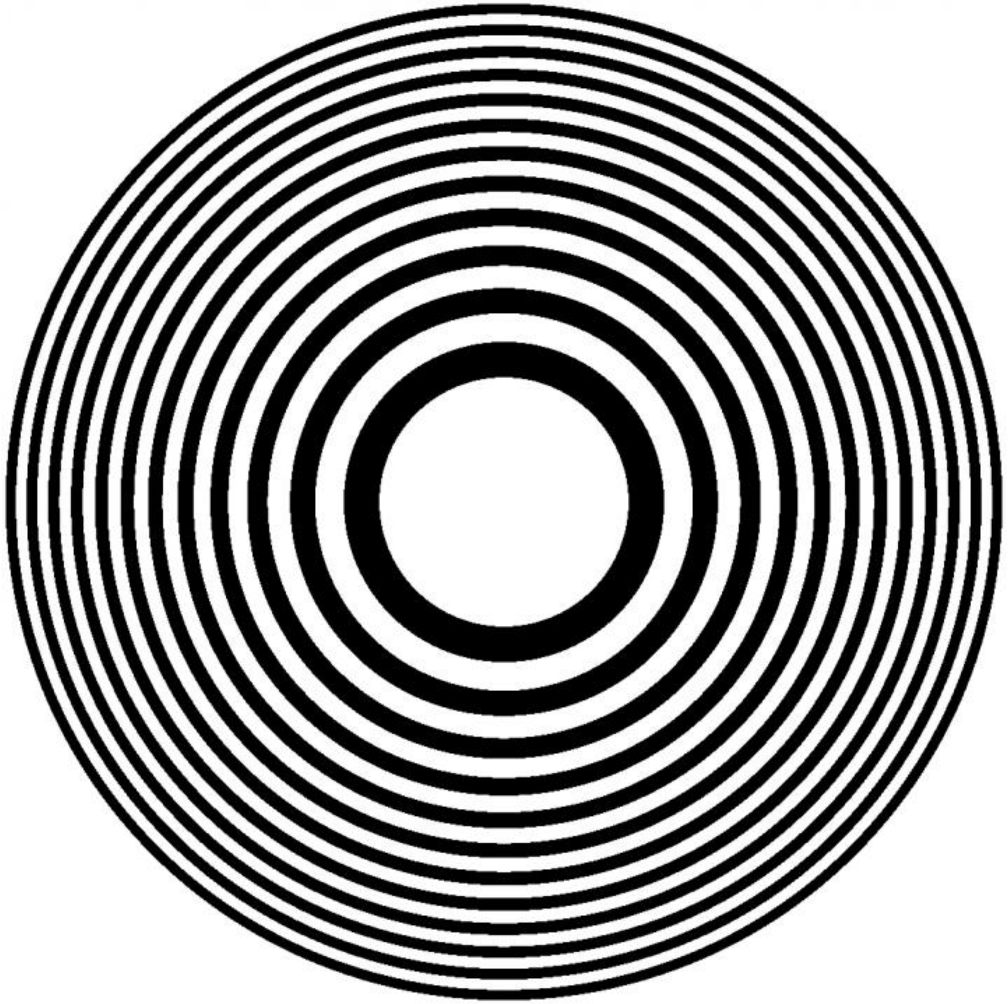


Fresnel Diffraction

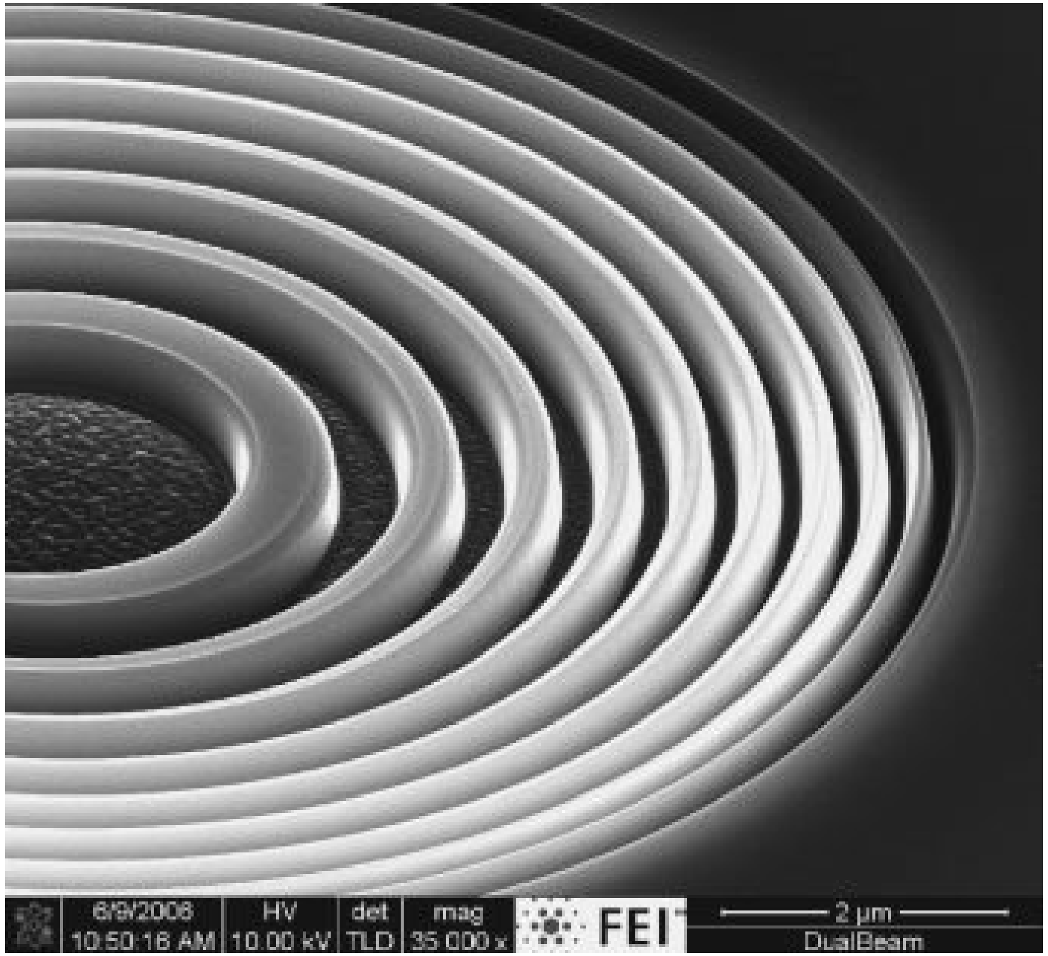


Fresnel Diffraction

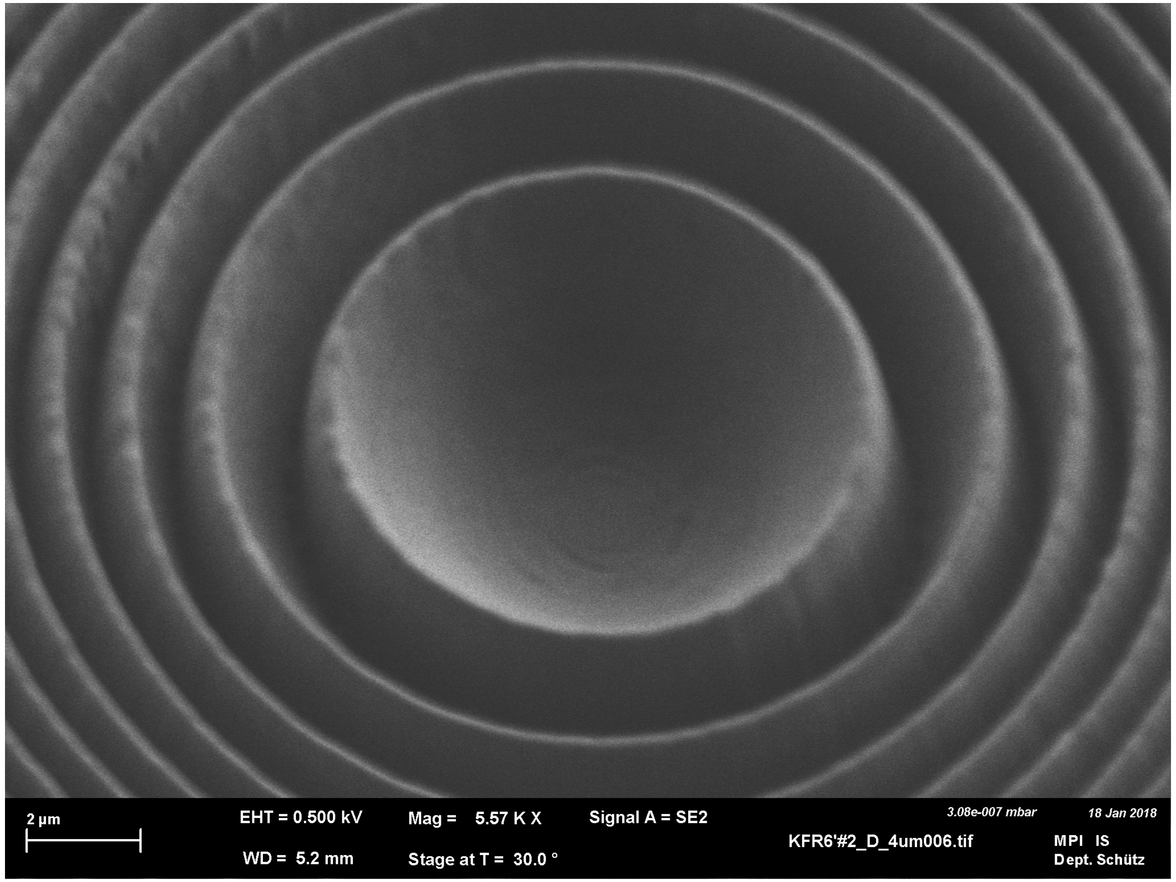
Fresnel Zone Plate



Phase Reversal Zone Plate

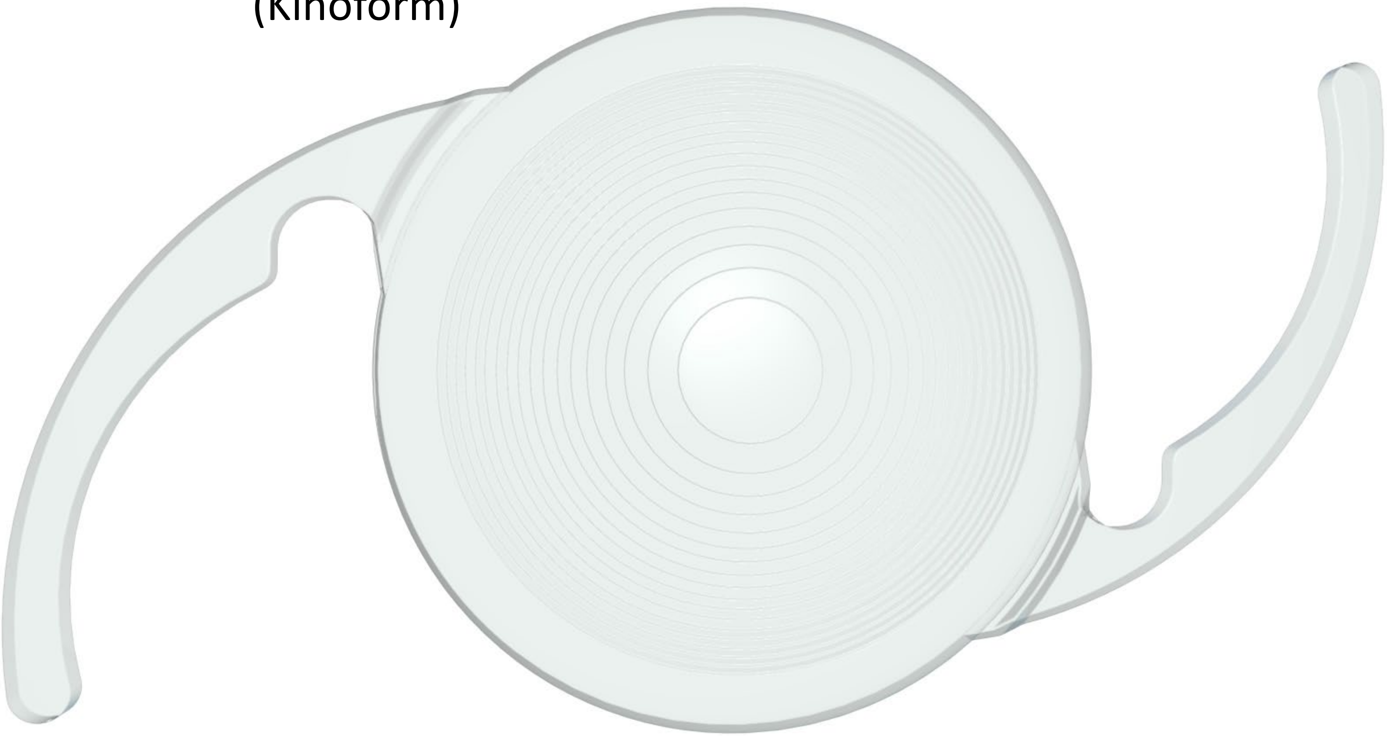


Kinoform Diffractive lens



These are not the same

Diffractive Lens
(Kinoform)



Fresnel Lens (Refractive)