Spherical Waves

Returning back to the Helmholtz eq. on page 116

$$\nabla^2 U(x,y,z) + k^2 U(x,y,z) = 0$$

again assume $n=1$, but replace $k = \frac{2\pi}{\lambda}$ with $\frac{2\pi n}{\lambda}$ if in media.

Another solution to this equation are spherical waves

$$U(x,y,z) = \frac{A}{r} \exp(ikr)$$

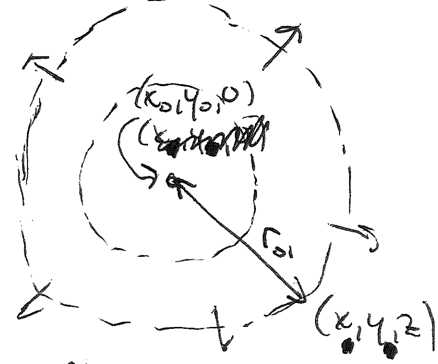
where r is the distance from the origin to the point (x,y,z). Note the amplitude is $\frac{A}{r}$ which means it falls off farther from the origin.

This solution is found by writing ∇^2 in spherical coordinates and the looking for solutions in the case when $\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \phi} = 0$ (i.e. the point source doesn't radiate in a prefered direction).

If the point source (origin of the spherical wave) is not located at the origin of the coordinate system, then we can simply translate the preceding expression so that

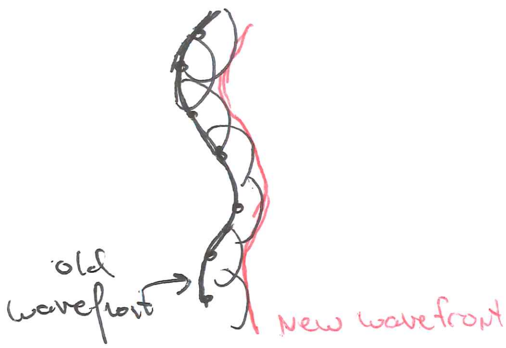
$$U(x,y,z) = \frac{A}{r_{01}} \exp(ikr_{01})$$

where $r_{01} = \left[(x_0-x)^2 + (y_0-y)^2 + z^2\right]^{1/2}$



~~for the point source located at $(x_0,y_0,0)$~~

As we did with plane waves, we want to represent a complex wavefront as a series of spherical waves and then propagate the spherical waves and reassemble. This concept was first described by Huygens and are known as wavelets.



Huygens Wavelets

A bunch of point sources are placed on the original wavefront. Each source propagates a spherical wavefront a fixed distance and the new wavefront is the envelope of these wavelets.

The concept is close to being correct, but a few modifications are needed to make it match reality. These modifications appear in the Rayleigh-Sommerfeld diffraction formula.

$$U(x_1, y_1, z) = \frac{1}{i\lambda} \iint_{-\infty}^{\infty} U(x_0, y_0, 0) \, \frac{z}{r_{01}} \, \frac{\exp(ikr_{01})}{r_{01}} \, dx_0 \, dy_0$$

This gives the field at the point (x_1, y_1, z) by (integrating) combining a bunch of spherical waves. The amplitude of each spherical wave is modulated by $\frac{|U(x_0, y_0, 0)|}{\lambda} \frac{z}{r_{01}}$. The $\frac{1}{\lambda}$ is a scale factor and $\frac{z}{r_{01}}$ gives preference to portions of the wavelet moving forward and reduces the effect of lateral propagation. The phase of spherical wave is adjusted by $\exp(-i\frac{\pi}{2}) \exp(i \operatorname{Arg}(U(x_0, y_0, 0)))$.

from $\frac{1}{i}$ in front.

The distance r_{01} can be rewritten as

$$r_{01} = z\left[1 + \frac{(x_0-x)^2 + (y_0-y)^2}{z^2}\right]^{1/2}$$

The Rayleigh-Sommerfeld formula is now

$$U(x,y,z) = \frac{1}{i\lambda z}\iint_{-\infty}^{\infty} U(x_0,y_0,0)\,\frac{\exp\left[ikz\left(1+\frac{(x-x_0)^2+(y-y_0)^2}{z^2}\right)^{1/2}\right]}{\left[1+\frac{(x-x_0)^2+(y-y_0)^2}{z^2}\right]}\,dx_0\,dy_0$$

This is just a convolution

$$U(x,y,z) = U(x,y,0) * h(x,y,z)$$

where $h(x,y,z) = \dfrac{1}{i\lambda z}\,\dfrac{\exp\left[ikz\left(1+\frac{x^2+y^2}{z^2}\right)^{1/2}\right]}{1+\frac{x^2+y^2}{z^2}}$ impulse response

what happens if we Fourier transform $h(x,y,z)$?

$$\mathcal{F}_{2D}\{h(x,y,z)\} = H(\xi,\eta;z) = \exp\left[i2\pi\sqrt{\frac{1}{\lambda^2}-\xi^2-\eta^2}\;z\right]$$

we've seen this before.

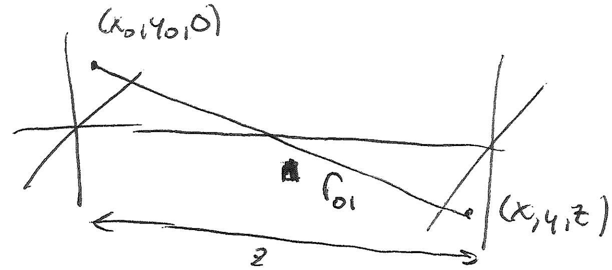
$$\begin{array}{ccccc} U(x,y,0) & * & h(x,y,z) & = & U(x,y,z) \\ \big\downarrow \mathcal{F}_{2D} & & \big\downarrow \mathcal{F}_{2D} & & \big\uparrow \mathcal{F}_{2D}^{-1} \\ A(\xi,\eta;0) & \cdot & H(\xi,\eta;z) & = & A(\xi,\eta;z) \end{array}$$

Convolution with spherical wavelets

Angular spectrum propagation

The Rayleigh-Sommerfeld formula is rigorous and gives excellent results as long as you're a couple wavelengths away from $z=0$. However, we would like to exploit Fourier transforms some more, so we can start making a few approximations. The first is

$$\frac{1}{i\lambda}\frac{z}{r_{01}^2} \simeq \frac{1}{i\lambda z}$$



Basically requires

$$\left[(x-x_0)^2+(y-y_0)^2\right]^{1/2} \ll |z|$$ Lateral displacement much smaller than the distance between planes.

The second approximation uses the binomial expansion of the square root in complex exponential

$$kr_{10} = kz + \frac{k}{2z}\left[(x_0-x)^2+(y_0-y)^2\right] - \frac{k}{8z^3}\left[(x_0-x)^2+(y_0-y)^2\right]^2 + \cdots$$

This last term is negligible if

$$|z|^3 \gg \frac{k}{8}\left[(x_0-x)^2+(y_0-y)^2\right]^2$$

These constraints on z define the <u>Fresnel region</u> and the phase can be approximated as

$$kr_{01} \simeq kz + \frac{k}{2z}\left[(x_0-x)^2+(y_0-y)^2\right]$$

Under these approximations, the Rayleigh Sommerfeld Diffraction Formula reduces to the Fresnel Diffraction Formula.

$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z}\iint_{-\infty}^{\infty} U(x_0,y_0,0)\exp\left[\frac{i\pi}{\lambda z}\left((x_0-x)^2+(y_0-y)^2\right)\right]dx_0\,dy_0$$

Collection of parabolic wavefronts

The Fresnel Diffraction formula is again a convolution.

$$U(x,y,z) = U(x,y,0) * h(x,y,z)$$

with $h(x,y,z) = \dfrac{\exp(ikz)}{i\lambda z} \exp\left(\dfrac{i\pi}{\lambda z}(x^2+y^2)\right)$ Impulse Response

Fourier transforming this gives

$$H(\xi,\eta;z) = \exp(ikz)\exp\left(-i\pi\lambda z(\xi^2+\eta^2)\right)$$ Fresnel Transfer Function

We saw this transfer function when deriving Gaussian beams, so the description of Gaussian beams is only valid when the Fresnel approximations are good.

We can also expand the squares in the exponent and rewrite the Fresnel Diffraction formula as

$$U(x,y,z) = \frac{\exp(ikz)}{i\lambda z}\exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right]\iint_{-\infty}^{\infty} U(x_0,y_0,0)\exp\left[\frac{i\pi}{\lambda z}(x_0^2+y_0^2)\right] \cdot \exp\left[-i2\pi\left(\frac{x}{\lambda z}x_0 + \frac{y}{\lambda z}y_0\right)\right]dx_0\,dy_0$$

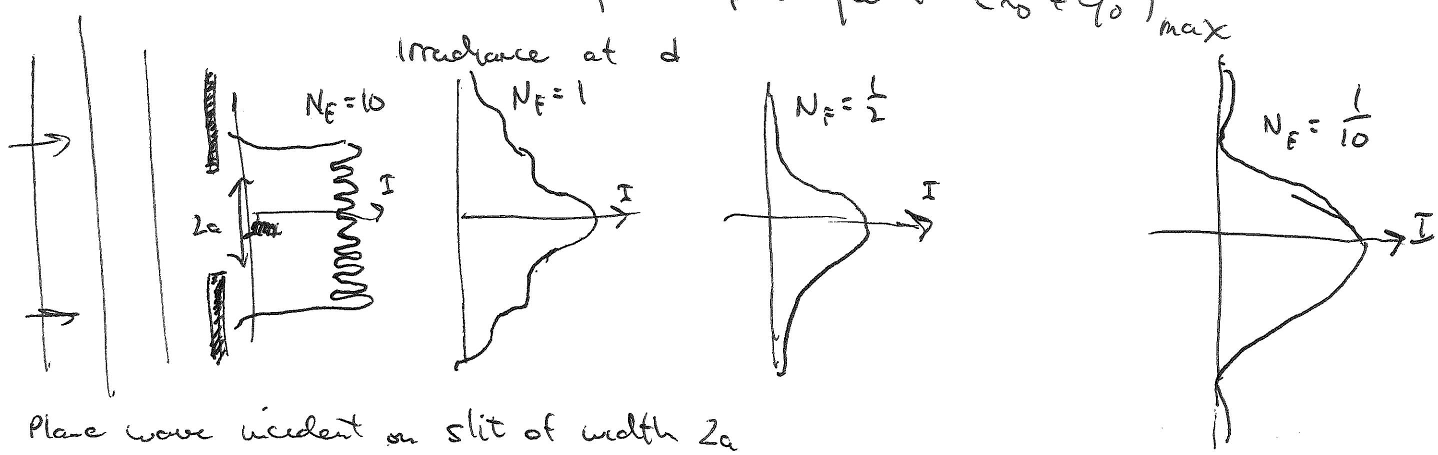
FRESNEL DIFFRACTION

$$\boxed{U(x,y,z) = \frac{\exp(ikz)}{i\lambda z}\exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right]\mathcal{F}_{2D}\left\{U(x_0,y_0,0)\exp\left[\frac{i\pi}{\lambda z}(x_0^2+y_0^2)\right]\right\} \quad \text{with } \xi = \frac{x}{\lambda z} \text{ and } \eta = \frac{y}{\lambda z}}$$

So output is obtained by first multiplying input by complex exponential and then Fourier transforming.

The Fresnel Number is a useful concept for understanding the effects of the complex exponential in the Fourier transform.

$$N_F = \frac{a^2}{\lambda z}$$ where a is the maximum radial extent of the input field $(x_0^2 + y_0^2)_{max}$



Plane wave incident on slit of width $2a$

~~The complex exponential~~

Rule of thumb

$N_F >> 1$ use angular spectrum approach

$N_F \simeq 1$ Fresnel diffraction good approximation

What happens if $N_F << 1$?

In this case $\exp\left[\frac{i\pi}{\lambda z}(x_0^2 + y_0^2)\right] \simeq 1$

FRAUNHOFER DIFFRACTION

$$\boxed{U(x,y,z) = \frac{\exp(ikz)}{i\lambda z} \exp\left[\frac{i\pi}{\lambda z}(x^2+y^2)\right] \mathcal{F}_{2D}\left\{U(x_0,y_0,0)\right\} \quad \text{with } \xi = \frac{x}{\lambda z} \text{ and } \eta = \frac{y}{\lambda z}}$$

WHEN $N_F << 1$ the output field is proportional to the Fourier transform of the input field.