Let's look at the transfer function some more

$$H(\xi,\eta;z) = \exp\left(i2\pi\sqrt{\frac{1}{\lambda^2}-\xi^2-\eta^2}\;z\right) \quad \text{with } \xi = \frac{\alpha}{\lambda},\ \eta = \frac{\beta}{\lambda}$$

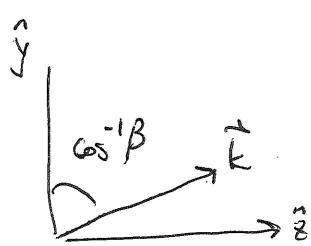
Here, we are assuming $n=1$, but if propagating in media then just replace all λ's by λ/n

Looking at the argument of the exponential

$$\frac{2\pi}{\lambda}\sqrt{1-(\alpha^2+\beta^2)}\;z \simeq \frac{2\pi}{\lambda}\left[1-\frac{\alpha^2+\beta^2}{2}\right]z$$

since $\sqrt{1+x} = 1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots$ binomial theorem

For this to hold, $|\alpha|$ and $|\beta|$ need to be small. Remember, these are direction cosines



$\Theta_y = \cos^{-1}\beta$ is angle between $\vec{k}$ and $\hat{y}$

$\cos\Theta_y = \beta$ so small β means Θ_y approaches $\frac{\pi}{2}$.

This approximation is good for ~~rays that~~ plane waves that are tending to propagate mainly in the z direction.

This approximation makes analytical calculations easier and is valid for many problems that appear in optics

Example: Gaussian Beams

Our approximated transfer function is now written as

$$H(\xi,\eta;z) = \exp\left(i2\pi\left(\frac{1}{\lambda} - \frac{\lambda(\xi^2+\eta^2)}{2}\right)z\right) = \exp(ikz)\exp\left(-i2\pi\left(\frac{\lambda(\xi^2+\eta^2)}{2}\right)z\right)$$

Let's use this to look at the following example.

A plane wave Ae^{ikz} is propagating along the z-axis direction and passes through an amplitude filter with a Gaussian profile

$$t(r) = \exp\left(-\frac{r^2}{w_0^2}\right) = \text{Gaus}\left(\frac{r}{\sqrt{\pi}\,w_0}\right)$$

The filter is located at $z=0$. What does the propagation look like after the plane wave passes through the filter?

$$U(x,y;0) = A\,\text{Gaus}\left(\frac{r}{\sqrt{\pi}\,w_0}\right)$$

Fourier transform to get angular spectrum

$$A(\xi,\eta;0) = A\sqrt{\pi}\,w_0\,\text{Gaus}(\sqrt{\pi}\,w_0\,\rho) \qquad \text{where } \rho^2 = \xi^2+\eta^2$$

Now propagate with approximated transfer function

$$A(\xi,\eta;z) = A(\xi,\eta;0)\,H(\xi,\eta;z)$$

$$A(\xi,\eta;z) = A\sqrt{\pi}\,w_0\,\text{Gaus}(\sqrt{\pi}\,w_0\,\rho)\exp(ikz)\exp\left(-i2\pi\left(\frac{\lambda\overbrace{(\xi^2+\eta^2)}^{\rho^2}}{2}\right)z\right)$$

$$U(x,y,z) = A\sqrt{\pi}\, w_0\, \mathcal{F}^{-1}_{2D}\left\{ \exp(ikz)\, \mathrm{Gaus}\left(\sqrt{\pi}\, w_0 \rho\right) \exp\left(-i2\pi\left(\frac{\lambda\rho^2}{2}\right)z\right)\right\}$$

$$U(x,y,z) = A b \exp(ikz)\, \mathcal{F}^{-1}_{2D}\left\{ \exp\left(-\pi b^2 \xi^2\right) \exp\left(-i\pi\lambda\xi^2 z\right) \cdot \exp\left(-\pi b^2 \eta^2\right) \exp\left(-i\pi\lambda\eta^2 z\right)\right\}$$

where $b = \sqrt{\pi}\, w_0$

Ok, it should be clear that this is separable into two 1D Fourier transforms with the same form.

$$\mathcal{F}^{-1}_{1D}\left\{ \exp\left(-\pi(b^2 + i\lambda z)\xi^2\right)\right\} = \frac{1}{\sqrt{b^2 + i\lambda z}} \exp\left[-\pi \frac{x^2}{b^2 + i\lambda z}\right]$$

This can be proved by writing the integral of the $\mathcal{F}^{-1}_{1D}$ and completing the square. We'll save this for a homework problem.

$$U(x,y,z) = \frac{Ab}{b^2 + i\lambda z} \exp\left[-\pi \frac{x^2 + y^2}{b^2 + i\lambda z}\right] \exp(ikz)$$

After a lot of painful math and with the following definitions

$$z_0 = \frac{k w_0^2}{2} = \frac{b^2}{\lambda}$$ Rayleigh Range

$$R(z) = z\left(1 + \frac{z_0^2}{z^2}\right)$$ Wavefront Radius

$$w(z) = w_0\left(1 + \frac{z^2}{z_0^2}\right)^{1/2}$$ Waist Radius

$$\Phi(z) = \tan^{-1}\left[\frac{z}{z_0}\right]$$ Phase Correction

$$\boxed{U(x,y,z) = A\frac{w_0}{w(z)}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]\exp\left[i\left(kz-\Phi(z)+k\frac{x^2+y^2}{2R(z)}\right)\right]}$$

KNOWN AS A GAUSSIAN BEAM

The amplitude of this beam is

$$A\frac{w_0}{w(z)}\exp\left[-\frac{x^2+y^2}{w^2(z)}\right]$$ which has a Gaussian profile

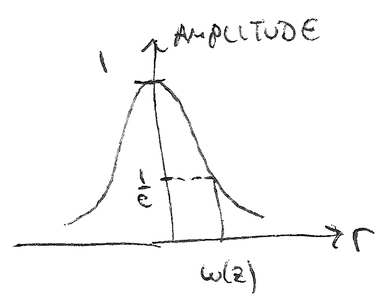
Note: at $z=0$, $w(0)=w_0$ and this just reduces to

$$A\exp\left[-\frac{r^2}{w_0^2}\right]$$ i.e. our starting amplitude.

~~Or~~ when $r=w_0$, the amplitude at the starting point is

$$A\exp[-1] = A\left(\frac{1}{e}\right)$$

So we're at the $1/e$ point of the Gaussian. This is a measure of the half width of the beam. Similarly, $w(z)$ describes the half width of the beam for locations away from $z=0$.



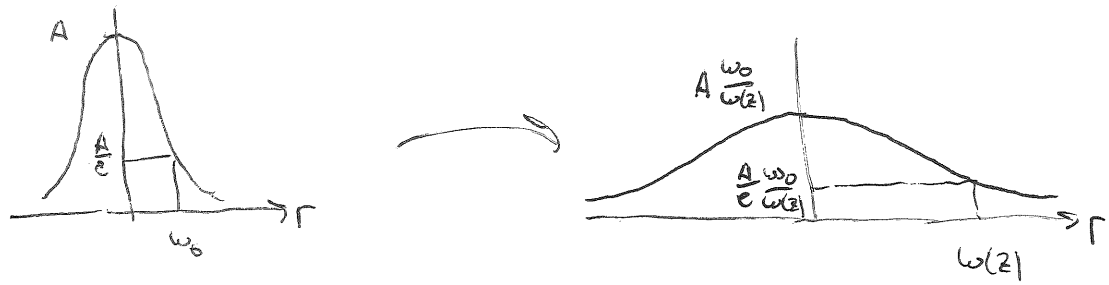
$$w(z) = w_0\left(1-\frac{z^2}{z_0^2}\right)^{1/2}$$ Find minimum half width

$$\frac{dw(z)}{dz} = \frac{w_0}{2}\frac{-2\frac{z}{z_0^2}}{\left(1-\frac{z^2}{z_0^2}\right)^{\frac{1}{2}}} = 0$$

Minimum occurs when $z=0$

So $w(0)=w_0$ is minimum half width called the "waist"

As the beam propagates away from $z=0$, the Gaussian amplitude spreads out. The $\frac{w_0}{w(z)}$ factor reduces the height of the Gaussian to keep the volume under the curve the same.



Ok, what about the variable z_0 that shows up repeatedly?

$$z_0 = \frac{k w_0^2}{2}$$

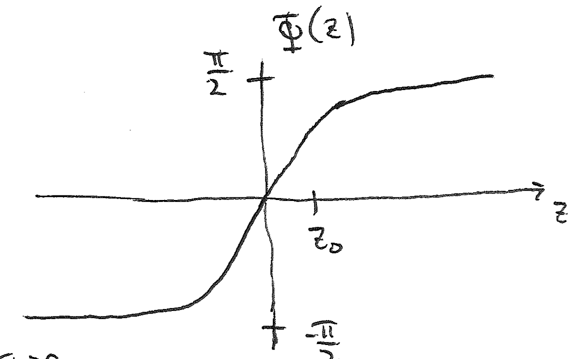
$$w(z_0) = w_0 \left(1+1\right)^{1/2} = \sqrt{2}\, w_0$$

z_0 is the location where the half width of the beam has increased by a factor of $\sqrt{2}$, z_0 is called the Rayleigh Range.

Now for the phase terms

$\exp[ikz]$ just accounts for the propagation of the initial plane wave as if there was no mask present.

$$\exp\left[-i\Phi(z)\right] = \exp\left[-i\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$



Phase shift of initial plane wave. Rapid change in Rayleigh Range and then converges towards $\frac{\pi}{2}$ shift for $z \gg z_0$. Gouy Phase Shift

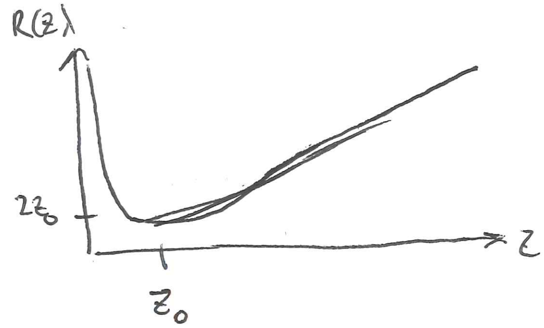
$$\exp\left(ik\frac{x^2+y^2}{2R(z)}\right) = \exp\left[i\frac{\pi r^2}{\lambda R(z)}\right]$$

The wavefront is approximately spherical (parabolic approximation) with radius $R(z)$.

Find minimum of $R(z)$

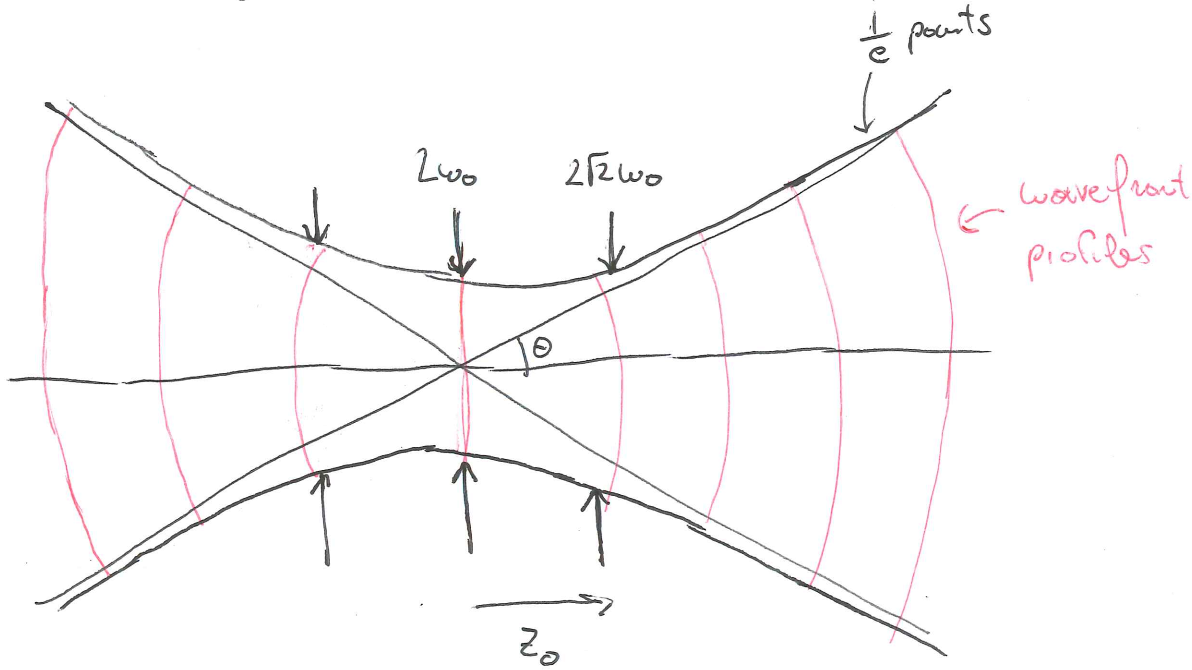
$$\frac{dR(z)}{dz} = 1 - \frac{z_0^2}{z^2} = 0 \Rightarrow \text{minimum when } z = z_0$$

$R(z_0) = 2z_0$



When z is large, $R(z) \to z$.

Putting this all together



$\Theta = \dfrac{\lambda}{\pi w_0}$ is related to the spread of the beam as it propagates.

Important features of Gaussian beams

① Phase (wavefront) is flat at beam waist.

② Far from Rayleigh range, the Gaussian beam looks like a spherical wave emanating from a point source located at the beam waist.

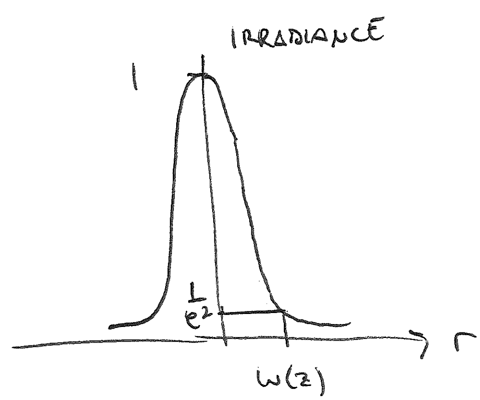
③ In the Rayleigh range, the beam is approximately collimated and approximately the same width.

④ Gaussian beams have a Gaussian amplitude everywhere in space. These equations also hold for $z < 0$.

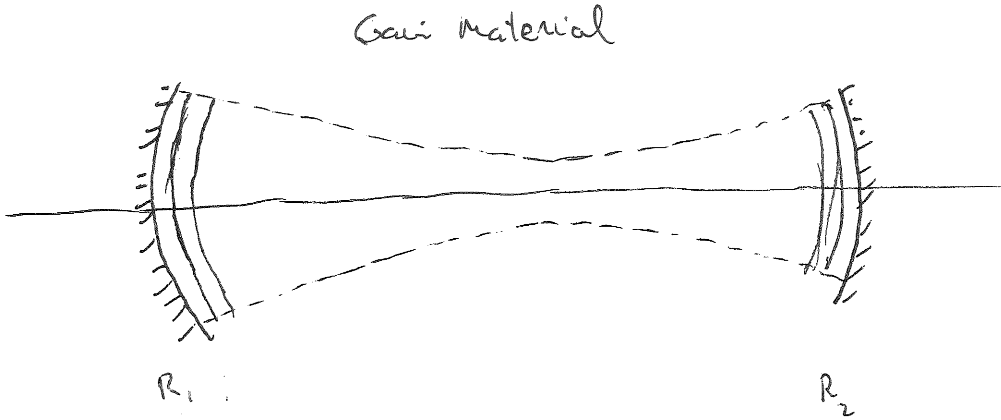
⑤ Irradiance is proportional to $|U(x,y,z)|^2 = I(x,y,z)$

so $$I(x,y,z) = A^2 \left(\frac{w_0}{w(z)}\right)^2 \exp\left(-2\frac{r^2}{w^2(z)}\right)$$

when we look at irradiance, when $r = w(z)$ we are at the $\frac{1}{e^2}$ point of the Gaussian beam.



We usually associate Gaussian beams with rotationally symmetric laser cavities. To form a laser cavity, the gain medium lies between to spherical mirrors.



The beam that forms must have radii that matches the radii of the two mirrors and the waist forms within the cavity. The beam emerging from the laser will be Gaussian and appear to have this internal waist.

Finally, once you know the waist w_0, you know everything about the Gaussian beam propagation.