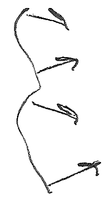
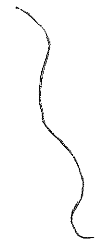
Propagation of ElectroMagnetic Waves

We are often interested in understanding the shape of a wavefront and its corresponding amplitude in some region of space given it properties at some other location.



$A_1(x,y,z)\,e^{-i\phi_1(x,y,z)}$



$A_2(x,y,z)\,e^{-i\phi(x,y,z)}$

There are a few simple waves that can be analyzed such as plane waves, spherical waves and Gaussian beams which are easy to propagate. In general though we are more interested in more complex wavefronts. The basic technique for analyzing these more complex wavefronts is to decompose them in to a set of one of the simple waves above, propagate the simple waves to a new location, and then reassemble the simple waves into the resultant wavefront. Below we will look at how this is done.

Maxwell's Equations - Describe the interactions of E- and M- fields

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

I'm assuming you have had exposure to these equations and are familiar with the notation.

For this class, we are only going to consider propagation through linear, homogeneous, isotropic materials which is a good model for materials like air, glass, liquid at reasonable fluence levels. This helps simplify Maxwell's equations. In this case

$\vec{D} = \varepsilon \vec{E}$ where ε is the electric permittivity

$\vec{B} = \mu \vec{H}$ where μ is the magnetic permeability

In addition we make a few other assumptions

ε = a scalar independent of λ, position and time
we can relax λ independence later.
Related to refractive index.

$\mu = \mu_0$ vacuum permeability. Production of magnetic field from electric current.

$\rho = 0$ No source charges

$\vec{J} = 0$ No currents

So Maxwell's Equations reduce to

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Taking the curl of the 3rd equation gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{H})$$

The 4th equation gives

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

There is a vector identity that enables us to simplify the preceding expression

$$\text{Identity:} \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E}$$

which gives

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

But $\vec{\nabla} \cdot \vec{E} = 0$ given our assumptions, so

$$\nabla^2 \vec{E} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

We can do similar ~~effects~~ steps to get a formula for the magnetic field.

$$\nabla^2 \vec{B} - \mu_0 \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

These two equations describe the propagation of EM waves through our assumed linear, isotropic, homogeneous material with no sources charges and currents. This means the $\vec{E}$-fields and $\vec{B}$-fields must satisfy the above equations to propagate.

A few further simplifications

$\vec{E}$ is a vector with components E_x, E_y, E_z

$\vec{B}$ is a vector with components B_x, B_y, B_z

Let's just use variable $u(x,y,z;t)$ for any one of these components.

Also, the refractive index is $n = \sqrt{\dfrac{\varepsilon}{\varepsilon_0}}$

and the speed of light is $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

So our differential equations have the form

$$\left(\nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) u(x,y,z;t) = 0 \qquad \text{Scalar wave equation}$$

For the time dependence assume a solution where

$$u(x,y,z;t) = A(x,y,z) \cos\left[2\pi\nu t - \phi(x,y,z)\right]$$

where $A(x,y,z)$ is the amplitude and $\phi(x,y,z)$ is the phase at any given point in space at time t.

Trig functions are a pain to work with so we usually use complex exponentials instead

$$u(x,y,z;t) = \mathrm{Re}\left[A(x,y,z) \exp\left[-i\left(2\pi\nu t - \phi(x,y,z)\right)\right]\right] \qquad \nu: \text{frequency}$$

Define $U(x,y,z) = A(x,y,z) \exp\left[+i\phi(x,y,z)\right]$

NOTE: SIGN OF "PHASE" changed to be consistent with Goodman. Need to make notes consistent throughout

$$u(x,y,z;t) = \mathrm{Re}\left[U(x,y,z) \exp\left[-i2\pi\nu t\right]\right]$$

We can just work with $U(x,y,z) \exp(-i2\pi\nu t)$ and then take real part at the end if needed. Often we are more interested in $|u(x,y,z;t)|^2$ since this is what detectors can measure.

$$\left(\nabla^2 - \frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} \right) U(x,y,z) \exp(-i2\pi\nu t) = 0$$

$$\nabla^2 U(x,y,z) \exp\left[-i2\pi\nu t\right] + \frac{4\pi^2 \nu^2 n^2}{c^2} U(x,y,z) \exp\left[-i2\pi\nu t\right] = 0$$

using $\quad c = \nu \lambda \quad$ c = speed of light

ν = frequency of light

λ = wavelength of light

Also define wave number $k = \dfrac{2\pi}{\lambda}$

$$\boxed{\nabla^2 U(x,y,z) + k^2 n^2 U(x,y,z) = 0} \qquad \text{Helmholtz eq.}$$

The spatial portion of our field needs to satisfy this equation in order for the field to propagate.

The SI units of $U(x,y,z)$ are $\dfrac{\sqrt{W}}{m}$ square root of watts per meter leading to $|U(x,y,z)|^2$ having units of W/m^2 (i.e. irradiance).

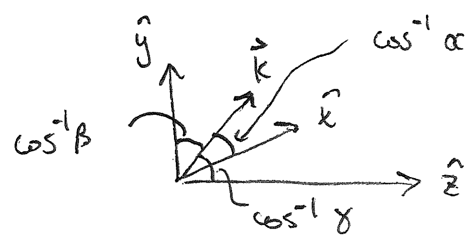
<u>Plane Waves</u> - One solution to the Helmholtz equation is a plane wave with the form

$$U(x,y,z) = A\exp(i\vec{k}\cdot\vec{r})$$

where $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is a vector describing the position (x,y,z) in Cartesian coordinate system. The vector $\vec{k}$ is

$$\vec{k} = \frac{2\pi n}{\lambda}\left(\alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}\right)$$

and is called the wave vector. The values α, β, γ are called direction cosines



The vector $\vec{k}$ makes ~~a~~ angles with the $\hat{x}, \hat{y}, \hat{z}$ axes. These angles are given by

$\Theta_x = \cos^{-1}\alpha$ angle between $\vec{k}$ and x-axis

$\Theta_y = \cos^{-1}\beta$ angle between $\vec{k}$ and y-axis

$\Theta_z = \cos^{-1}\gamma$ angle between $\hat{k}$ and z-axis

Putting this all together

$$U(x,y,z) = A\exp\left[i\frac{2\pi}{\lambda}(\alpha x + \beta y + \gamma z)\right]$$

The direction vectors satisfy

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

which can be rewritten as

$$\gamma = \sqrt{1 - \alpha^2 - \beta^2}$$

ASIDE: This comes from the angle between two vectors

$$\cos\Theta = \frac{(\vec{u}\cdot\vec{v})}{|\vec{u}||\vec{v}|}$$

$$\cos\Theta_x = \frac{\vec{k}\cdot\hat{x}}{|\vec{k}|}$$ since $\hat{x}$ is unit vector

$$|\vec{k}| = \sqrt{\left(\frac{2\pi}{\lambda}\right)^2(\alpha^2 + \beta^2 + \gamma^2)} = \frac{2\pi}{\lambda}\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}$$

$$\cos\Theta_y = \frac{\vec{k}\cdot\hat{y}}{|\vec{k}|} \qquad \cos\Theta_z = \frac{\vec{k}\cdot\vec{z}}{|\vec{k}|}$$

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = \frac{1}{|\vec{k}|^2}\left[(\vec{k}\cdot\hat{x})^2 + (\vec{k}\cdot\hat{y})^2 + (\vec{k}\cdot\hat{z})^2\right]$$

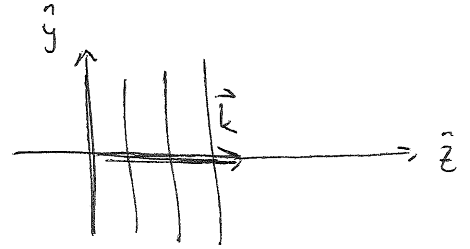
$$\alpha^2 + \beta^2 + \gamma^2 = \frac{1}{|\vec{k}|^2}\left[\underbrace{\left(\frac{2\pi N}{\lambda}\right)^2(\alpha^2+\beta^2+\gamma^2)}_{|\vec{k}|^2}\right]$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

So our plane wave can be written in the form

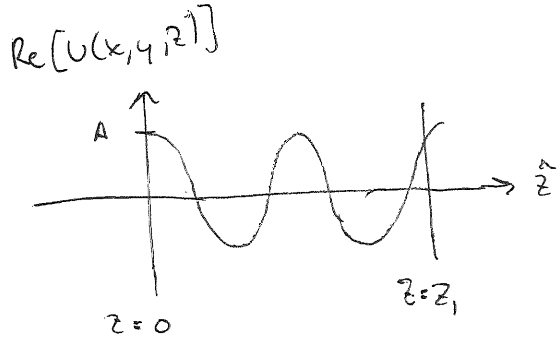
$$\boxed{U(x,y,z) = A\exp\left[i\frac{2\pi N}{\lambda}(\alpha x + \beta y)\right]\exp\left[i\frac{2\pi N}{\lambda}\sqrt{1-\alpha^2-\beta^2}\,z\right]}$$

<u>Example:</u> $\alpha = \beta = 0 \rightarrow \theta_x = \theta_y = 90°$



$$U(x,y,z) = A\exp\left[i\frac{2\pi N}{\lambda}z\right] = A\exp[ikNz]$$

plane wave traveling in $+z$ direction.



$$U(x,y,0) = A$$

$$U(x,y,z_1) = A\exp[ikNz_1]$$

So phase at $z=z_1$ is just $\phi = kNz_1$

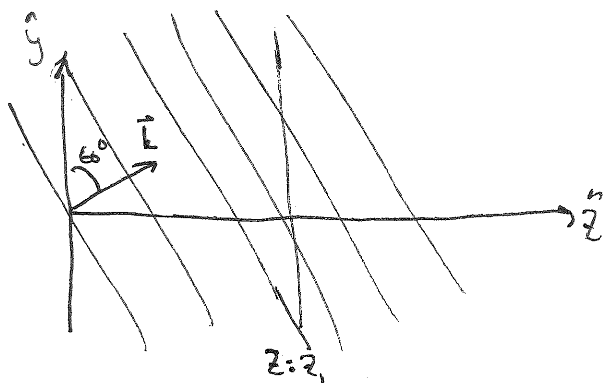
$$\phi = kNz_1 = 2\pi\left(\frac{Nz_1}{\lambda}\right)$$

Nz_1 is the optical path length between $z=0$ and $z=z_1$

$\frac{Nz_1}{\lambda}$ is the # of oscillations

Example $\alpha = 0$ $\beta = \frac{1}{2}$ $\rightarrow$ $\Theta_x = 90°$ $\Theta_y = 60°$

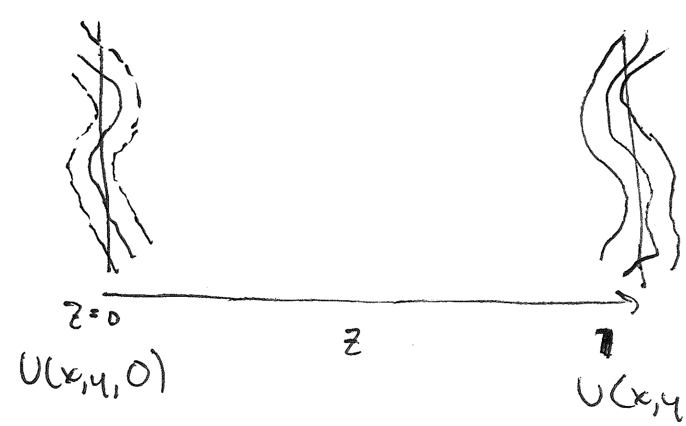
$$U(x,y,z_1) = A \exp\left[ikn\frac{y}{2}\right] \exp\left[ikn\frac{\sqrt{3}}{2}z_1\right]$$



We like plane waves because they stay planar with propagation and its easy to calculate. U at any plane z.

One technique for analyzing complex wavefronts is to represent them as a sum of many plane waves, propagate each plane wave individually, and the combine all the propagated plane waves to get the resultant wavefront.

ANGULAR SPECTRUM



given $U(x,y,0)$

what $U(x,y,z)$ at some plane a distance z away

Let's consider a 2D Fourier transform of the field at z = 0

$$A(\xi,\eta;0) = \iint_{-\infty}^{\infty} U(x,y,0) \exp\left[-i2\pi(\xi x + \eta y)\right] dx\,dy$$

The inverse transform is

$$U(x,y,0) = \iint_{-\infty}^{\infty} A(\xi,\eta;0)\exp\left[i2\pi(\xi x + \eta y)\right] d\xi\, d\eta$$

The interpretation of this is that we are representing the complex wavefront $U(x,y,0)$ as a bunch of complex exponentials (sines and cosines) and that $A(\xi,\eta;0)$ are the weights or amounts of each of the complex exponentials.

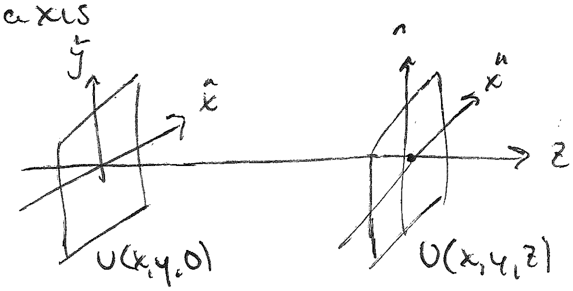
Let's now let the spatial frequency variables $\xi = \frac{\alpha}{\lambda}$ and $\eta = \frac{\beta}{\lambda}$.

We now have

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right) = \iint_{-\infty}^{\infty} U(x,y,0)\exp\left[-i\frac{2\pi}{\lambda}(\alpha x + \beta y)\right] dx\, dy$$

which is called the angular spectrum of $U(x,y,0)$.

We now want to know $U(x,y,z)$ at some distance z along the z-axis in a plane perpendicular to the z-axis



We'll call $A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)$ the angular spectrum of $U(x,y,z)$ with

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right) = \iint_{-\infty}^{\infty} U(x,y,z)\exp\left[-i\frac{2\pi}{\lambda}(\alpha x + \beta y)\right] dx\, dy$$

If we can find a connection between $A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right)$ and $A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right)$ we can exploit this connection to find $U(x,y,z)$ from $U(x,y,0)$.

$$U(x,y,z) = \iint_{-\infty}^{\infty} A\left(\frac{\alpha}{\lambda}; \frac{\beta}{\lambda}; z\right) \exp\left[i2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)\right] d\frac{\alpha}{\lambda}\, d\frac{\beta}{\lambda}$$

This field needs to satisfy the Helmholtz equation

$$\nabla^2 U + k^2 N^2 U = 0$$

Plugging in leads to

$$\frac{d^2}{dz^2} A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) + \left(\frac{2\pi}{\lambda}\right)^2 \left[1 - \alpha^2 - \beta^2\right] A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) = 0$$

The solution to this differential equation is

$$A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right) = A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right) \exp\left(i\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\, z\right)$$

So our propagation algorithm is

1. FOURIER TRANSFORM $U(x,y,0)$ TO GET $A(\xi, \eta; 0)$ with ξ, η equal to $\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}$ respectively.
2. multiply $A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; 0\right)$ by $\exp\left(i\frac{2\pi}{\lambda}\sqrt{1-\alpha^2-\beta^2}\, z\right)$ to get $A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right)$
3. Inverse Fourier Transform $A\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}; z\right)$ to get $U(x,y,z)$

Note: If $\alpha^2+\beta^2<1$ then

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)\exp\left(i\frac{2\pi}{\lambda}\sqrt{1-\alpha^2\pm\beta^2}\,z\right)$$

propagating wave

If $\alpha^2+\beta^2>1$

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};z\right)=A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)\exp\left(-\frac{2\pi}{\lambda}\sqrt{\alpha^2+\beta^2-1}\,z\right)$$

evanescent wave

exponential decay

SHORTHAND SUMMARY

$$\boxed{\begin{aligned}&U(x,y,z)=\mathcal{F}^{-1}_{2D}\left\{A(\xi,\eta;0)\exp\left(i2\pi\sqrt{\tfrac{1}{\lambda^2}-\xi^2-\eta^2}\,z\right)\right\}\\&\text{with } \xi=\frac{\alpha}{\lambda},\ \eta=\frac{\beta}{\lambda} \text{ and}\\&A(\xi,\eta;0)=\mathcal{F}_{2D}\{U(x,y,0)\}\end{aligned}}$$

LSI View

$$\begin{array}{ccccc}U(x,y,0) & * & h(x,y;z) & = & U(x,y,z)\\ \Big\downarrow \mathcal{F}_{2D} & & \Big\downarrow \mathcal{F}_{2D} & & \Big\uparrow \mathcal{F}^{-1}_{2D}\\ A(\xi,\eta;0) & & H(\xi,\eta;z) & = & A(\xi,\eta;z)\end{array}$$

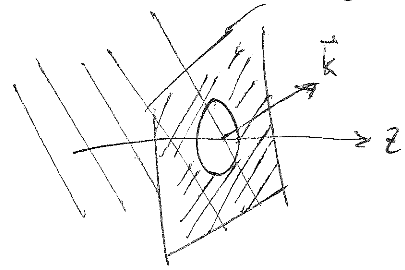
with $H(\xi,\eta;z)=\exp\left(i2\pi\sqrt{\frac{1}{\lambda^2}-\xi^2-\eta^2}\,z\right)$

Can assume $H(\xi,\eta;z)=0$ when $\xi^2+\eta^2>\frac{1}{\lambda^2}$ because of evanescent waves

Example: Before on page 119 we had a plane wave

$$U(x,y,z) = \cancel{A} \exp\left[ik \frac{y}{2}\right] \exp\left[ik \frac{\sqrt{3}}{2} z\right]$$ where we will assume $N=1$ and $A=1$ unit amplitude

Suppose this plane wave is incident on a circular aperture with diameter of 1mm.



$$t(r) = cyl\left(\frac{r}{1mm}\right) = cyl(r)$$

The field right after the aperture is

$$U_+(x,y,0) = U(x,y,0)\,t(r) = \cancel{A}\exp\left[ik\frac{y}{2}\right] cyl(r)$$

Find $A(\xi,\eta;10)$ ~~$A(\xi,\eta,10)$~~ i.e. 10 mm down stream

$$A(\xi,\eta,0) = \mathcal{F}_{2D}\left\{\exp\left[i2\pi\left(\frac{1}{2\lambda}\right)y\right] cyl(r)\right\} \qquad \xi = \frac{\alpha}{\lambda}, \quad \eta = \frac{\beta}{\lambda}$$

$$A(\xi,\eta;0) = \delta\left(\xi, \eta - \frac{1}{2\lambda}\right) * \frac{\pi}{4}\,\text{somb}(\rho) \quad \text{with } \rho^2 = \xi^2 + \eta^2$$

$$A(\xi,\eta;0) = \frac{\pi}{4}\,\text{somb}\left(\sqrt{\xi^2 + \left(\eta - \frac{1}{2\lambda}\right)^2}\right)$$

$$A(\xi,\eta;10) = \frac{\pi}{4}\,\text{somb}\left(\sqrt{\xi^2 + \left(\eta - \frac{1}{2\lambda}\right)^2}\right)\exp\left(i2\pi\sqrt{\frac{1}{\lambda^2} - \xi^2 - \eta^2}\; z\right)$$

Can numerically Fourier transform this to get result. $U(x,y,10)$

The mask causes the plane wave heading in a single direction ($\vec{k}$ in yz) to be converted to a bunch of plane waves heading in all different directions.