OPTI 512 Midterm Due: October 16, 2019

1. Given the following function

$$f(x) = \frac{1}{X} rect\left(\frac{x}{d}\right) * comb\left(\frac{x}{X}\right),$$

   a) What is the maximum value of d such that the rects() don't overlap?

   b) What is the value of d such that the rects() are spaced by $2d$?

   c) What is the Fourier transform $F(\xi)$ when $d = X$? Simplify and *plot* your answer.

2. An LSI system has an impulse response of $h(x) = \delta(x) - 3sinc(3x)$.

   a) What is the transfer function $H(\xi)$ of the system?

   b) For the input $f(x) = cos(2\pi\xi_o x)$, what is the output $g(x)$ of the system?

   c) What frequencies ξ_o pass through the system?

3. Compute the complex Fourier series for the function $f(x) = \frac{x}{2}$ defined over the range $-2 \leq x < 2$, with period $X = 4$.

   a) Sketch a plot of $f(x)$ over its range.

   b) What is the fundamental frequency ξ_o of the series?

   c) Calculate the coefficients a_m of the series. Hint: $\int usin(u)du = -ucos(u) + sin(u)$.

4. Compute the following:

   a) A mask is made up of an annular region with inner radius a and outer radius b. The mask is opaque except in the region $a \leq r \leq b$, where the transmission is 1.0. Write an expression for the transmission $t(r)$ of the mask.

   b) For the annular mask, what is the 0^{th} order Hankel transform $\mathcal{H}_0\{t(r)\}$?

   c) The function $g(r) = \delta(r - a)$ describes ring of delta functions of radius a. What is the 0^{th} order Hankel transform $\mathcal{H}_0\{g(r)\}$?

Formula Sheet

$f(x,y) ** h(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h^*(x-\alpha, y-\beta)d\alpha d\beta$	Convolution
$f(x,y) \bigstar\bigstar h(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h(\alpha - x, \beta - y)d\alpha d\beta$	Correlation
$\gamma_{fh}(x,y) = f(x,y) \bigstar\bigstar h^*(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)h^*(\alpha - x, \beta - y)d\alpha d\beta$	Complex Correlation
$\mathcal{F}_{2D}\{f(x,y)\} = F(\xi,\eta) = \iint_{-\infty}^{\infty} f(x,y)exp[-i2\pi(\xi x + \eta y)]dxdy$	2D Fourier Transform
$\mathcal{F}_{2D}^{-1}\{f(\xi,\eta)\} = f(x,y) = \iint_{-\infty}^{\infty} F(\xi,\eta)exp[i2\pi(\xi x + \eta y)]d\xi d\eta$	Inverse Fourier Transform 2D
$\mathcal{H}_0\{f(r)\} = F(\rho) = 2\pi \int_0^{\infty} f(r)J_0(2\pi\rho r)rdr$	0^{th} Order Hankel Transform
$cos(a) = \frac{1}{2}[exp(ia) + exp(-ia)]$ $sin(a) = \frac{1}{2i}[exp(ia) - exp(-ia)]$	Trig Identities
$rect(x) = \begin{cases} 0 & \|x\| > 1/2 \\ 1/2 & \|x\| = 1/2 \\ 1 & \|x\| < 1/2 \end{cases}$ $tri(x) = \begin{cases} 0 & \|x\| \geq 1 \\ 1 - \|x\| & \|x\| < 1 \end{cases}$ $sinc(x) = \frac{sin(\pi x)}{\pi x}$ $Gaus(x) = exp[-\pi x^2]$ $\frac{1}{\|b\|} comb\left(\frac{x - x_o}{b}\right) = \sum_{n=-\infty}^{\infty} \delta(x - x_o - nb)$ $cyl(r) = \begin{cases} 0 & \|r\| > 1/2 \\ 1/2 & \|r\| = 1/2 \\ 1 & 0 \leq \|r\| < 1/2 \end{cases}$ $somb(r) = \frac{2J_1(\pi r)}{\pi r}$	Common Special Functions

$\delta(x - x_o) = 0 \ \ for\ x \neq x_o$ $\int_{x_1}^{x_2} f(\alpha)\delta(\alpha - x_o)d\alpha = f(x_o) \ \ for\ x_1 < x_o < x_2$ $\delta\left(\frac{x - x_o}{b}\right) = \lvert b \rvert \delta(x - x_o)$ $f(x)\delta(x - x_o) = f(x_o)\delta(x - x_o)$ $\int_{-\infty}^{\infty} exp[-i2\pi(\xi - \xi_o)x]dx = \delta(\xi - \xi_o)$	Properties of Delta Functions
$f(x) = \sum_{m=-\infty}^{\infty} a_m exp[i2\pi m \xi_o x] \quad with\ \xi_o = \frac{1}{X}\ and\ X = period$ $where\ a_m = \frac{1}{X}\int_{-X/2}^{X/2} f(x)\ \ exp[-i2\pi m \xi_o x]dx$	Complex Fourier Series
$\mathcal{F}_{2D}\{rect(x,y)\} = sinc(\xi,\eta)$ $\mathcal{F}_{2D}\{tri(x,y)\} = sinc^2(\xi,\eta)$ $\mathcal{F}_{2D}\{Gaus(x,y)\} = Gaus(\xi,\eta)$ $\mathcal{F}_{2D}\{comb(x,y)\} = comb(\xi,\eta)$ $\mathcal{F}_{2D}\{\delta(x \pm x_o, y \pm y_o)\} = exp[\pm i2\pi x_o \xi]exp[\pm i2\pi y_o \eta]$ $\mathcal{F}_{2D}\{exp[\pm i2\pi \xi_o x]exp[\pm i2\pi \eta_o y]\} = \delta(\xi \mp \xi_o, \eta \mp \eta_o)$ $\mathcal{F}_{2D}\{cos(2\pi \xi_o x)\} = \frac{1}{2}[\delta(\xi - \xi_o) + \delta(\xi + \xi_o)]\delta(\eta)$ $\mathcal{F}_{2D}\{sin(2\pi \xi_o x)\} = \frac{1}{2i}[\delta(\xi - \xi_o) - \delta(\xi + \xi_o)]\delta(\eta)$ $\mathcal{H}_0\{cyl(r)\} = \frac{\pi}{4} somb(\rho)$	Common 2D Fourier Transforms

$$\mathcal{F}_{2D}\{f(\pm x,\pm y)\} = F(\pm\xi,\pm\eta)$$ $$\mathcal{F}_{2D}\{f^*(\pm x,\pm y)\} = F^*(\mp\xi,\mp\eta)$$ $$\mathcal{F}_{2D}\{F(\pm x,\pm y)\} = f(\mp\xi,\mp\eta)$$ $$\mathcal{F}_{2D}\{F^*(\pm x,\pm y)\} = f^*(\pm\xi,\pm\eta)$$ $$\mathcal{F}_{2D}\{f_1(x)f_2(y)\} = \mathcal{F}_{1D}\{f_1(x)\}\mathcal{F}_{1D}\{f_2(y)\} = F_1(\xi)F_2(\eta)$$ $$\mathcal{F}_{2D}\{f_1(x)\} = F_1(\xi)\delta(\eta)$$ $$\mathcal{F}_{2D}\{f_2(y)\} = \delta(\xi)F_2(\eta)$$ $$\mathcal{F}_{2D}\left\{f\left(\frac{x}{b},\frac{y}{d}\right)\right\} = \lvert bd\rvert F(b\xi,d\eta)$$ $$\mathcal{F}_{2D}\{f(x\pm x_o,y\pm y_o)\} = exp[\pm i2\pi x_o\xi]exp[\pm i2\pi y_o\eta]F(\xi,\eta)$$ $$\mathcal{F}_{2D}\left\{exp[\pm i2\pi\xi_o x]exp[\pm i2\pi\eta_o y]f\left(\frac{x\pm x_o}{b},\frac{y\pm y_o}{b}\right)\right\} = \lvert bd\rvert exp[\pm i2\pi x_o(\xi\mp\xi_o)]exp[\pm i2\pi y_o(\eta\mp\eta_o)] \times F\big(b(\xi\mp\xi_o),d(\eta\mp\eta_o)\big)$$ $$\iint_{-\infty}^{\infty} f(x,y)dxdy = F(0,0) \ and \ \iint_{-\infty}^{\infty} F(\xi,\eta)d\xi d\eta = f(0,0)$$ $$\mathcal{H}_0\left\{f\left(\frac{r}{b}\right)\right\} = \lvert b\rvert^2 F(b\rho)$$	General Properties of 2D Fourier & Hankel Transforms
$$\mathcal{F}_{2D}\{f(x,y) ** h(x,y)\} = F(\xi,\eta)H(\xi,\eta)$$ $$\mathcal{F}_{2D}\{f(x,y)h(x,y)\} = F(\xi,\eta) ** H(\xi,\eta)$$ $$\mathcal{F}_{2D}\{f(x,y)\bigstar\bigstar h(x,y)\} = F(\xi,\eta)H(-\xi,-\eta)$$ $$\mathcal{F}_{2D}\{f(x,y)h(-x,-y)\} = F(\xi,\eta)\bigstar\bigstar H(\xi,\eta)$$ $$\mathcal{F}_{2D}\{\gamma_{fh}(x,y)\} = \mathcal{F}_{2D}\{f(x,y)\bigstar\bigstar h^*(x,y)\} = F(\xi,\eta)H^*(\xi,\eta)$$ $$\mathcal{F}_{2D}\{f(x,y)h^*(x,y)\} = F(\xi,\eta)\bigstar\bigstar H^*(\xi,\eta) = \gamma_{FH}(\xi,\eta)$$ $$\mathcal{H}_0\{f(r) ** h(r)\} = \mathcal{H}_0\{f(r)\bigstar\bigstar h(r)\} = F(\rho)H(\rho)$$ $$\mathcal{H}_0\{f(r)h(r)\} = F(\rho) ** H(\rho) = F(\rho)\bigstar\bigstar H(\rho)$$ $$\mathcal{H}_0\{\gamma_{fh}(r)\} = \mathcal{H}_0\{f(r)\bigstar\bigstar h^*(r)\} = F(\rho)H^*(\rho)$$	2D Transforms of Products, Correlations and Convolutions