1. Given the following function

\[ f(x) = \frac{1}{X} \text{rect} \left( \frac{x}{d} \right) * \text{comb} \left( \frac{x}{X} \right), \]

a) What is the maximum value of \( d \) such that the rects() don’t overlap?
b) What is the value of \( d \) such that the rects() are spaced by \( 2d \)?
c) What is the Fourier transform \( F(\xi) \) when \( d = X \)? Simplify and plot your answer.

2. An LSI system has an impulse response of \( h(x) = \delta(x) - 3\text{sinc}(3x) \).

a) What is the transfer function \( H(\xi) \) of the system?
b) For the input \( f(x) = \cos(2\pi \xi_o x) \), what is the output \( g(x) \) of the system?
c) What frequencies \( \xi_o \) pass through the system?

3. Compute the complex Fourier series for the function \( f(x) = \frac{x}{2} \) defined over the range \(-2 \leq x < 2\), with period \( X = 4 \).

a) Sketch a plot of \( f(x) \) over its range.
b) What is the fundamental frequency \( \xi_o \) of the series?
c) Calculate the coefficients \( a_m \) of the series. Hint: \( \int u \sin(u) du = -u \cos(u) + \sin(u) \).

4. Compute the following:

a) A mask is made up of an annular region with inner radius \( a \) and outer radius \( c \). The mask is opaque except in the region \( a \leq r \leq b \), where the transmission is 1.0. Write an expression for the transmission \( t(r) \) of the mask.
b) For the annular mask, what is the 0th order Hankel transform \( \mathcal{H}_0\{t(r)\} \)?
c) The function \( g(r) = \delta(r - a) \) describes ring of delta functions of radius \( a \). What is the 0th order Hankel transform \( \mathcal{H}_0\{g(r)\} \)?
### Formula Sheet

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x, y) \ast h(x, y) = \int_{-\infty}^{\infty} f(\alpha, \beta) h^*(x - \alpha, y - \beta) d\alpha d\beta )</td>
<td>Convolution</td>
</tr>
<tr>
<td>( f(x, y) \star \star h(x, y) = \int_{-\infty}^{\infty} f(\alpha, \beta) h(\alpha - x, \beta - y) d\alpha d\beta )</td>
<td>Correlation</td>
</tr>
<tr>
<td>( \gamma_{fh}(x, y) = f(x, y) \star \star h^*(x, y) )</td>
<td>Complex Correlation</td>
</tr>
<tr>
<td>( \mathcal{F}<em>{2D}[f(x, y)] = F(\xi, \eta) = \int</em>{-\infty}^{\infty} f(x, y) \exp[-i2\pi(\xi x + \eta y)] dx dy )</td>
<td>2D Fourier Transform</td>
</tr>
<tr>
<td>( \mathcal{F}<em>{2D}^{-1}[F(\xi, \eta)] = f(x, y) = \int</em>{-\infty}^{\infty} F(\xi, \eta) \exp[i2\pi(\xi x + \eta y)] d\xi d\eta )</td>
<td>Inverse Fourier Transform 2D</td>
</tr>
<tr>
<td>( \mathcal{H}<em>0[f(r)] = F(\rho) = 2\pi \int</em>{0}^{\infty} f(r) J_0(2\pi \rho r) r dr )</td>
<td>0th Order Hankel Transform</td>
</tr>
<tr>
<td>( \cos(a) = \frac{1}{2} [\exp(ia) + \exp(-ia)] )</td>
<td>Trig Identities</td>
</tr>
<tr>
<td>( \sin(a) = \frac{1}{2i} [\exp(ia) - \exp(-ia)] )</td>
<td></td>
</tr>
<tr>
<td>( \text{rect}(x) = \begin{cases} 0 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>( \text{tri}(x) = \begin{cases} 0 &amp;</td>
<td>x</td>
</tr>
<tr>
<td>( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} )</td>
<td></td>
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<tr>
<td>( \text{Gaus}(x) = \exp[-\pi x^2] )</td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{</td>
<td>b</td>
</tr>
<tr>
<td>( \text{cyl}(r) = \begin{cases} 0 &amp;</td>
<td>r</td>
</tr>
<tr>
<td>( \text{somb}(r) = \frac{2J_1(\pi r)}{\pi r} )</td>
<td></td>
</tr>
</tbody>
</table>
\[
\delta(x - x_0) = 0 \text{ for } x \neq x_0
\]
\[
\int_{x_1}^{x_2} f(\alpha) \delta(\alpha - x_0) \, d\alpha = f(x_0) \text{ for } x_1 < x_0 < x_2
\]
\[
\delta \left( \frac{x - x_0}{b} \right) = |b| \delta(x - x_0)
\]
\[
f(x) \delta(x - x_0) = f(x_0) \delta(x - x_0)
\]
\[
\int_{-\infty}^{\infty} \exp[-i2\pi(\xi - \xi_0)x] \, dx = \delta(\xi - \xi_0)
\]
\[
f(x) = \sum_{m=-\infty}^{\infty} a_m \exp[i2\pi m \xi_0 x] \quad \text{with } \xi_0 = \frac{1}{X} \text{ and } X = \text{period}
\]
\[
\text{where } a_m = \frac{1}{X} \int_{-X/2}^{X/2} f(x) \exp[-i2\pi m \xi_0 x] \, dx
\]
\[
\mathcal{F}_{2D}\{\text{rect}(x, y)\} = \text{sinc}(\xi, \eta)
\]
\[
\mathcal{F}_{2D}\{\text{tri}(x, y)\} = \text{sinc}^2(\xi, \eta)
\]
\[
\mathcal{F}_{2D}\{\text{Gaus}(x, y)\} = \text{Gaus}(\xi, \eta)
\]
\[
\mathcal{F}_{2D}\{\text{comb}(x, y)\} = \text{comb}(\xi, \eta)
\]
\[
\mathcal{F}_{2D}\{\delta(x \pm x_0, y \pm y_0)\} = \exp[\pm i2\pi x_0 \xi] \exp[\pm i2\pi y_0 \eta]
\]
\[
\mathcal{F}_{2D}\{\exp[\pm i2\pi \xi_0 x] \exp[\pm i2\pi \eta_0 y]\} = \delta(\xi \mp \xi_0, \eta \mp \eta_0)
\]
\[
\mathcal{F}_{2D}\{\cos(2\pi \xi_0 x)\} = \frac{1}{2} [\delta(\xi - \xi_0) + \delta(\xi + \xi_0)] \delta(\eta)
\]
\[
\mathcal{F}_{2D}\{\sin(2\pi \xi_0 x)\} = \frac{1}{2i} [\delta(\xi - \xi_0) - \delta(\xi + \xi_0)] \delta(\eta)
\]
\[
\mathcal{H}_0\{\text{cyl}(r)\} = \frac{\pi}{4} \text{somb}(\rho)
\]
\begin{align*}
\mathcal{F}_{2D}(f(\pm x, \pm y)) &= F(\pm \xi, \pm \eta) \\
\mathcal{F}_{2D}(f^*(\pm x, \pm y)) &= F^*(\mp \xi, \mp \eta) \\
\mathcal{F}_{2D}(F(\pm x, \pm y)) &= f(\mp \xi, \mp \eta) \\
\mathcal{F}_{2D}(F^*(\pm x, \pm y)) &= F^*(\mp \xi, \mp \eta)
\end{align*}

\[ \mathcal{F}_{2D}\{f_1(x)f_2(y)\} = \mathcal{F}_{1D}\{f_1(x)\}\mathcal{F}_{1D}\{f_2(y)\} = F_1(\xi)F_2(\eta) \]

\[ \mathcal{F}_{2D}\{f_1(x)\} = F_1(\xi)\delta(\eta) \]

\[ \mathcal{F}_{2D}\{f_2(y)\} = \delta(\xi)F_2(\eta) \]

\[ \mathcal{F}_{2D}\left\{ f\left(\frac{x}{b}, \frac{y}{d}\right) \right\} = |bd|F(b\xi, d\eta) \]

\[ \mathcal{F}_{2D}\{f(x \pm x_0, y \pm y_0)\} = \exp[\pm i2\pi x_0 \xi] \exp[\pm i2\pi y_0 \eta] F(\xi, \eta) \]

\[ \mathcal{F}_{2D}\left\{ \exp[\pm i2\pi \xi_0 x] \exp[\pm i2\pi \eta_0 y] f\left(\frac{x \pm x_0}{b}, \frac{y \pm y_0}{b}\right) \right\} \]

\[ = |bd| \exp[\pm i2\pi x_0(\xi \mp \xi_0)] \exp[\pm i2\pi y_0(\eta \mp \eta_0)] \]

\[ \times F(b(\xi \mp \xi_0), d(\eta \mp \eta_0)) \]

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = F(0,0) \quad \text{and} \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\xi, \eta) \, d\xi \, d\eta = f(0,0) \]

\[ \mathcal{H}_0\left\{ f\left(\frac{r}{\rho}\right) \right\} = |b|^2 F(b\rho) \]

\begin{align*}
\mathcal{F}_{2D}\{f(x, y) * h(x, y)\} &= F(\xi, \eta)H(\xi, \eta) \\
\mathcal{F}_{2D}\{f(x, y) \overset{\star\star}{h}(x, y)\} &= F(\xi, \eta) \overset{\star\star}{H}(\xi, \eta) \\
\mathcal{F}_{2D}\{f(x, y) h(-x,-y)\} &= F(\xi, \eta) \overset{\star\star}{H}(\xi, \eta) \\
\mathcal{F}_{2D}\{f(y_h, x)\} &= \mathcal{F}_{2D}\{f(x, y) \overset{\star\star}{h}^*(x, y)\} = F(\xi, \eta)H^*(\xi, \eta) \\
\mathcal{F}_{2D}\{f(x, y) h^*(x, y)\} &= F(\xi, \eta) \overset{\star\star}{H}^*(\xi, \eta) = \gamma_{FH}(\xi, \eta) \\
\mathcal{H}_0\{f(r) * h(r)\} &= \mathcal{H}_0\{f(r) \overset{\star\star}{h}(r)\} = F(\rho)H(\rho) \\
\mathcal{H}_0\{f(r) h(r)\} &= F(\rho) * H(\rho) = F(\rho) \overset{\star\star}{H}(\rho) \\
\mathcal{H}_0\{\gamma_{f h}(r)\} &= \mathcal{H}_0\{f(r) \overset{\star\star}{h}(r)\} = F(\rho)H^*(\rho)
\end{align*}