OPTI 512 Midterm Due: October 16, 2019

1. Given the following function

$$f(x) = \frac{1}{X} rect\left(\frac{x}{d}\right) * comb\left(\frac{x}{X}\right),$$

a) What is the maximum value of d such that the rects() don't overlap?

The comb function are a series of delta function spaced by a distance X. Therefore $d \leq X$ will give rect functions that don't overlap on convolution.

b) What is the value of d such that the rects() are spaced by $2d$?

The spacing between the delta functions now needs to be $X = 3d$, which is $2d$ for the spacing plus d for the width of the rect.

c) What is the Fourier transform $F(\xi)$ when $d = X$? Simplify and *plot* your answer.

$$f(x) = \frac{1}{X} rect\left(\frac{x}{X}\right) * comb\left(\frac{x}{X}\right)$$

$$F(\xi) = X sinc(X\xi) comb(X\xi)$$

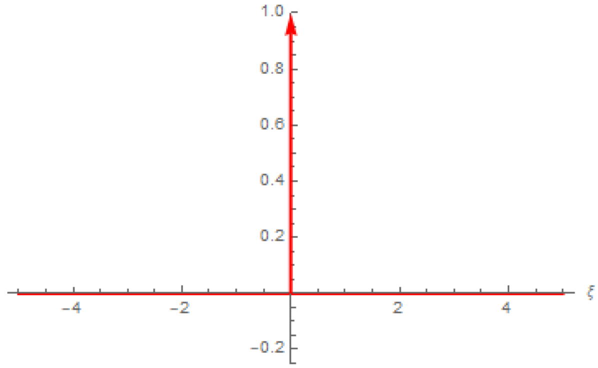
From the definition of the comb function

$$F(\xi) = \sum_{n=-\infty}^{\infty} sinc(X\xi)\delta\left(\xi - \frac{n}{X}\right)$$

Using the multiplication property of delta functions

$$F(\xi) = \sum_{n=-\infty}^{\infty} sinc(n)\delta\left(\xi - \frac{n}{X}\right)$$

But $sinc(n) = 0$ for $n \neq 0$, so $F(\xi) = \delta(\xi)$. This should make sense since the width of the rect functions is the spacing between the delta functions and the input reduces to $f(x) = 1$.



2. An LSI system has an impulse response of $h(x) = \delta(x) - 3sinc(3x)$.

a) What is the transfer function $H(\xi)$ of the system?

$$H(\xi) = 1 - rect\left(\frac{\xi}{3}\right)$$

This is an ideal high pass filter.

b) For the input $f(x) = cos(2\pi\xi_o x)$, what is the output $g(x)$ of the system?

The input spectrum is

$$F(\xi) = \frac{1}{2}[\delta(\xi - \xi_o) + \delta(\xi + \xi_o)]$$

$H(\xi)$ *is 0 for* $|\xi| < 1.5$*, and 1 for* $|\xi| > 1.5$ *and technically ½ for* $|\xi| = 1.5$*, although I didn't mark off for this. The output spectrum is*

$$G(\xi) = F(\xi)H(\xi) = \begin{cases} 0 & |\xi| < \xi_o \\ \frac{1}{4}[\delta(\xi - \xi_o) + \delta(\xi + \xi_o)] & |\xi| = \xi_o \\ \frac{1}{2}[\delta(\xi - \xi_o) + \delta(\xi + \xi_o)] & |\xi| > \xi_o \end{cases}$$

The inverse transform now gives

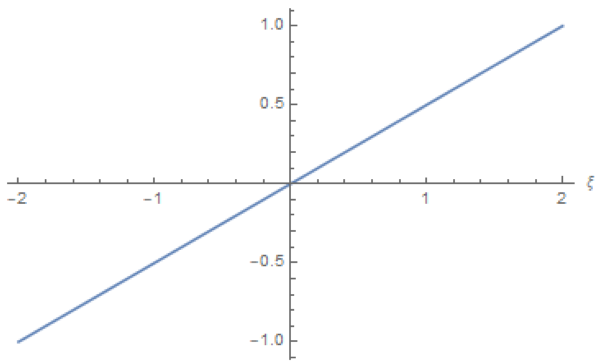
$$g(x) = \begin{cases} 0 & |\xi_o| < 1.5 \\ \frac{1}{2}cos(2\pi\xi_o x) & |\xi_o| = 1.5 \\ cos(2\pi\xi_o x) & |\xi_o| > 1.5 \end{cases}$$

c) What frequencies ξ_o pass through the system?

Only frequencies with $|\xi_o| \geq 1.5$ *pass through the system.*

3. Compute the complex Fourier series for the function $f(x) = \frac{x}{2}$ defined over the range $-2 \leq x < 2$, with period $X = 4$.

a) Sketch a plot of $f(x)$ over its range.



b) What is the fundamental frequency ξ_o of the series?

The fundamental frequency is

$$\xi_o = \frac{1}{X} = \frac{1}{4}$$

c) Calculate the coefficients a_m of the series. Hint: $\int u sin(u) du = -u cos(u) + sin(u)$.

$$a_m = \frac{1}{4} \int_{-2}^{2} \frac{x}{2} exp(-i2\pi m \xi_o x) dx$$

Expand into real and imaginary parts

$$a_m = \frac{1}{8}\left[\int_{-2}^{2} x cos(2\pi m \xi_o x) dx - i \int_{-2}^{2} x sin(2\pi m \xi_o x) dx\right]$$

The first term is $odd \times even = odd$*, so the integral goes to zero. The second term is* $odd \times odd = even$*, so the integral doubles.*

$$a_m = -\frac{i}{4}\int_0^2 x\sin\left(\frac{m\pi x}{2}\right)dx$$

This integral has the form of the integral in the hint with substitution $u = m\pi x/2$.

$$a_m = -\frac{i}{4}\left(\frac{2}{m\pi}\right)^2\int_0^{m\pi} u\sin(u)du$$

$$a_m = -\frac{i}{m^2\pi^2}[-m\pi\cos(m\pi) - 0]$$

The value of $\cos(m\pi)$ *just alternates between* 1 *and* -1*, so the coefficient can be written compactly as*

$$a_m = \frac{i}{m\pi}(-1)^m \qquad m \neq 0$$

We also need to handle the case where $m = 0$*. One application of L'Hopital's rule shows that* $a_0 = 0$*.*

Alternative Method (Integration by Parts)

With

$$a_m = \frac{1}{8}\int_{-2}^{2} x\exp(-i2\pi m\xi_o x)dx$$

make the substitutions

$$u = x \qquad dv = \exp(-i2\pi m\xi_o x)dx$$
$$du = dx \qquad v = \frac{\exp(-i2\pi m\xi_o x)}{-i2\pi m\xi_o}$$

then by integration by parts,

$$a_m = \frac{1}{8}\left[\frac{x\exp(-i2\pi m\xi_o x)}{-i2\pi m\xi_o}\Bigg|_{x=-2}^{x=2} - \int_{-2}^{2}\frac{\exp(-i2\pi m\xi_o x)}{-i2\pi m\xi_o}dx\right].$$

Continuing along

$$a_m = \frac{1}{8}\left[\frac{2exp(-i4\pi m\xi_o) - (-2)exp(i4\pi m\xi_o)}{-i2\pi m\xi_o} - \frac{exp(-i2\pi m\xi_o x)}{(-i2\pi m\xi_o)^2}\Bigg|\begin{matrix} x = 2 \\ \\ x = -2 \end{matrix}\right]$$

$$a_m = \frac{1}{8}\left[\frac{2}{-i\pi m\xi_o}cos(4\pi m\xi_o) - \frac{exp(-i4\pi m\xi_o) - exp(i4\pi m\xi_o)}{(-i2\pi m\xi_o)^2}\right]$$

$$a_m = \frac{1}{8}\left[\frac{2}{-i\pi m\xi_o}cos(4\pi m\xi_o) + \frac{sin(4\pi m\xi_o)}{2i(\pi m\xi_o)^2}\right]$$

With $\xi_o = 1/4$*,*

$$a_m = \frac{cos(m\pi)}{-i\pi m} + \frac{sin(m\pi)}{i(\pi m)^2} = \frac{icos(m\pi)}{m\pi} = \frac{i}{m\pi}(-1)^m \qquad m \neq 0$$

Again, we also need to handle the case where $m = 0$. *One application of L'Hopital's rule shows that* $a_0 = 0$.

4. Compute the following:

   a) A mask is made up of an annular region with inner radius a and outer radius b. The mask is opaque except in the region $a \leq r \leq b$, where the transmission is 1.0. Write an expression for the transmission $t(r)$ of the mask.

$$t(r) = cyl\left(\frac{r}{2b}\right) - cyl\left(\frac{r}{2a}\right)$$

Remember that the diameter describes the width of the cyl function.

   b) For the annular mask, what is the 0th order Hankel transform $\mathcal{H}_0\{t(r)\}$?

$$T(\rho) = \mathcal{H}_0\left\{cyl\left(\frac{r}{2b}\right) - cyl\left(\frac{r}{2a}\right)\right\} = \pi b^2 somb(2b\rho) - \pi a^2 somb(2a\rho)$$

since

$$\mathcal{H}_0\left\{f\left(\frac{r}{b}\right)\right\} = |b|^2F(b\rho) \text{ and } \mathcal{H}_0\{cyl(r)\} = \frac{\pi}{4}somb(\rho)$$

Brute Force

We can also revert to brute force if needed. From the definition of the Hankel transform

$$T(\rho) = 2\pi\int_0^b J_0(2\pi\rho r)rdr - 2\pi\int_0^a J_0(2\pi\rho r)rdr.$$

In the notes, we showed that

$$\mathcal{H}_0\{cyl(r)\} = 2\pi\int_0^{1/2} J_0(2\pi\rho r)rdr = \frac{\pi}{4}somb(\rho).$$

If we make the following substitutions: $r = 2br', dr = 2bdr'$ *in the first integral and* $r = 2ar', dr = 2adr'$ *in the second integral, then*

$$T(\rho) = (4b^2)2\pi\int_0^{1/2} J_0(2\pi(2b\rho)r')r'dr' - (4a^2)2\pi\int_0^{1/2} J_0(2\pi(2a\rho)r')r'dr'.$$

$$T(\rho) = (4b^2)\left(\frac{\pi}{4}somb(2b\rho)\right) - (4a^2)\left(\frac{\pi}{4}somb(2a\rho)\right)$$

$$T(\rho) = \pi b^2 somb(2b\rho) - \pi a^2 somb(2a\rho)$$

c) The function $g(r) = \delta(r-a)$ describes ring of delta functions of radius a. What is the 0^{th} order Hankel transform $\mathcal{H}_0\{g(r)\}$?

$$G(\rho) = \mathcal{H}_0\{\delta(r-a)\} = 2\pi\int_0^\infty \delta(r-a)J_0(2\pi\rho r)rdr$$

By sifting $G(\rho) = 2\pi a J_0(2\pi a\rho)$.