Often, we don't know the full continuous function \( f(x, y) \), but instead only have a sampled version of it \( f_s(x, y) \).

A 1D example is measuring a time signal at uniformly spaced time intervals. A 2D example would be a digital image from your cell phone camera, where each pixel in the sensor samples the local intensity of the scene mapped onto it. A simplified model of this process is

\[
f_s(x, y) = \text{comb} \left( \frac{x}{x_s} \right) \text{comb} \left( \frac{y}{y_s} \right) f(x, y)
\]

This is just an array of delta functions spaced by \( x_s \) and \( y_s \) in the \( x \) and \( y \) directions, respectively. The areas of each delta function correspond to the local value of \( f(x, y) \).

Fourier transforming gives

\[
F_X(f, i) = x_s y_s \text{comb} \left( \frac{f}{x_s} \right) \text{comb} \left( \frac{i}{y_s} \right) F(f, i)
\]

and from the convolution theorem

\[
F_X(f, i) = \sum_{n, m} F \left( f - \frac{n}{x_s}, i - \frac{m}{y_s} \right)
\]

so the spectrum of a sampled signal looks like a bunch of shifted versions of the continuous spectrum \( F(f, i) \), with spacings of \( x_s \) and \( y_s \) in the \( f \) and \( i \) directions, respectively.
Band-limited Functions

The spectrum of a band-limited function is only non-zero over a finite region of Fourier space.

\[ F(\xi, \eta) \]

The spectrum of the sampled version looks like

\[ F_s(\xi, \eta) = \sum_{n,m} F \left( \frac{\xi}{B_x}, \frac{n}{B_y}, \frac{\eta}{B_z}, \frac{m}{B_z} \right) \]

Let's now multiply the sampled spectrum by a mask that blocks all the outside spectra and passes just the center one. \( \text{rect} \left( \frac{\xi}{B_x}, \frac{n}{B_y}, \frac{\eta}{B_z}, \frac{m}{B_z} \right) \) will work.

\[ F_s(\xi, \eta) \text{rect} \left( \frac{\xi}{B_x}, \frac{n}{B_y}, \frac{\eta}{B_z}, \frac{m}{B_z} \right) = F(\xi, \eta) \]

Inverse transform both sides.

\[ f(x, y) = \mathcal{F}^{-1} \left[ F_s(\xi, \eta) \text{rect} \left( \frac{\xi}{B_x}, \frac{n}{B_y}, \frac{\eta}{B_z}, \frac{m}{B_z} \right) \right] \]

In general, the mask can have any shape as long as it passes the center spectrum and blocks the remaining spectra.
What happens when $X$ and $Y$ are increased? First, this means that the separation between samples in $(X,Y)$ space becomes larger.

In Fourier space, increased $X_s$ and $Y_s$ means spectra get closer together.

There is a minimum sampling frequency $\frac{1}{X_{\text{min}}}$ and $\frac{1}{Y_{\text{min}}}$ where the spectra do not overlap.

\[ \frac{1}{X_{\text{min}}} = B_s \quad \frac{1}{Y_{\text{min}}} = B_n \]

So to avoid overlap for a bandwidth-limited function

\[ X_s \leq \frac{1}{B_s} \quad \text{and} \quad Y_s \leq \frac{1}{B_n} \]
The Nyquist Frequencies are defined as

\[ N_f = \frac{B_x}{2} \quad N_h = \frac{B_n}{2} \]

which are just the maximum absolute frequencies of the spectra.

i.e. spectra goes from \[ -\frac{B_x}{2} \leq \xi \leq \frac{B_x}{2} \], so \[ |\xi| \leq \frac{B_x}{2} \leq N_f \]

which means

\[ \frac{1}{X_{\min}} = 2N_f \quad \frac{1}{Y_{\min}} = 2N_h \]

So, you will often hear

\[ \frac{1}{X} \geq 2N_f \quad \text{and} \quad \frac{1}{Y} \geq 2N_h \]

as “the sampling frequency needs to be at least twice the Nyquist frequency.”

**Aliasing** - what happens when the sampling frequency is less than twice the Nyquist frequency?

i.e. \[ \frac{1}{X_S} < 2N_f \quad \text{and} \quad \frac{1}{Y_S} < 2N_h \]

The spectra begin to overlap, so there is no mask that can isolate just one of the spectra. Consequently, the original function \( f(x,y) \) cannot be perfectly recovered.

Artifacts in the recovered function will appear and these artifacts are referred to as aliasing.
Example: \( f(\xi) = \cos(2\pi \xi_0 \xi) \) start with 1D example

\[
F(\xi) = \frac{1}{2} \left[ \delta(\xi - \xi_0) + \delta(\xi + \xi_0) \right]
\]

Band-limited since \( F(\xi) = 0 \) for \(|\xi| > \xi_0\)

Nyquist frequency: \( \xi_0 \)

\( B_x = 2\xi_0 \)

Let's start our sampling at \( 4\xi_0 \) four times the Nyquist frequency

\[
\frac{1}{X_s} = 4\xi_0
\]

\[
X_s = \frac{1}{4\xi_0}
\]

\[
f_s(\lambda) = \text{comb}(4\xi_0 \lambda) \cos(2\pi \xi_0 \lambda)
\]

\[
F_s(\xi) = \frac{1}{4\xi_0} \text{comb}(\frac{\xi}{4\xi_0}) \ast \frac{1}{2} \left[ \delta(\xi - \xi_0) + \delta(\xi + \xi_0) \right]
\]

\[
F_s(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left[ \delta(\xi - n\xi_0) + \delta(\xi + \xi_0) \right] \ast \delta(\xi - n\xi_0)
\]

\[
F_s(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left[ \delta(\xi - n\xi_0) + \delta(\xi + \xi_0) \right]
\]

\[
F_s(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left[ \delta(\xi - (4n+1)\xi_0) + \delta(\xi - (4n-1)\xi_0) \right]
\]

\[
... \Rightarrow \quad F_s(\xi)
\]
Now let's multiply $F_x(\xi)$ by a mask to capture the center spectrum. In general, we don't know the maximum spectral frequency $\xi_0$ of the signal. Instead, we only know the sampling frequency $4\xi_0$ and the sampling at $\xi = 2n\xi_0$.

If the signal is band-limited, then multiplying by $\text{rect}(\frac{\xi}{4\xi_0})$ is the best option to recover the original signal $f(x)$.

$$F_x(\xi) \text{ rect}(\frac{\xi}{4\xi_0}) = \frac{1}{2} \left[ \delta(\xi - \xi_0) + \delta(\xi + \xi_0) \right]$$

$$f(x) = \mathcal{F}^{-1} \left[ F_x(\xi) \text{ rect}(\frac{\xi}{4\xi_0}) \right] = \cos(2\pi \xi_0 x) \quad \text{Perfect recovery}$$

Let's reduce the sampling frequency to $2\xi_0$ or twice the Nyquist frequency.

$$\frac{1}{X_S} = 2\xi_0$$

or

$$X_S = \frac{1}{2\xi_0}$$

$$f_0(x) = \text{comb}(\frac{1}{2\xi_0} x) \cos(2\pi \xi_0 x)$$

$$F_x(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left[ \delta(\xi - (2n+1)\xi_0) + \delta(\xi - (2n-1)\xi_0) \right]$$

Similar steps as before

$$F_0(\xi) \text{ rect}(\frac{\xi}{2\xi_0}) = \frac{1}{2} \left[ \delta(\xi - \xi_0) + \delta(\xi + \xi_0) \right]$$

$$f(x) = \mathcal{F}^{-1} \left[ F_0(\xi) \text{ rect}(\frac{\xi}{2\xi_0}) \right] = \cos(2\pi \xi_0 x) \quad \text{Perfect recovery}$$
Let's reduce the sampling frequency to \( \frac{3}{2} f_0 \) or 1.5 \( x \) the Nyquist frequency.

So

\[
\frac{1}{X_s} = \frac{3f_0}{2}
\]

\[
X_s = \frac{2}{3f_0}
\]

\[
f_s(x) = \text{comb} \left( \frac{3f_0}{2} x \right) \cos \left( 2\pi f_0 x \right)
\]

\[
F_s(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2} \left[ \delta(\xi - \left( \frac{3}{2} n + 1 \right) f_0) + \delta(\xi - \left( \frac{3}{2} n - 1 \right) f_0) \right]
\]

\[
F_s(\xi) \text{ rect} \left( \frac{\xi}{3f_0} \right) = \frac{1}{2} \left[ \delta(\xi - \frac{f_0}{2}) + \delta(\xi + \frac{f_0}{2}) \right]
\]

\[
f(x) = \mathcal{F}^{-1} \left\{ F_s(\xi) \text{ rect} \left( \frac{\xi}{3f_0} \right) \right\} = \cos \left( 2\pi \left( \frac{f_0}{2} \right) x \right)
\]

Aliased signal maps to lower spectral frequency.

Finally, reduce the sampling frequency to \( f_0 \) i.e. the Nyquist frequency.

\[
\frac{1}{X_s} = f_0
\]

\[
X_s = \frac{1}{f_0}
\]

\[
f_s(x) = \text{comb} \left( f_0 x \right) \cos \left( 2\pi f_0 x \right) = \text{comb} \left( f_0 x \right)
\]
\[
F_s(\delta) = \frac{1}{\delta_0} \text{comb} \left( \frac{\delta}{\delta_0} \right)
\]

\[
F_s(3) \text{ rect} \left( \frac{3}{\delta_0} \right) = \delta(3)
\]

\[
f(x) = \mathcal{F}^{-1} \left\{ F_s(\delta) \text{ rect} \left( \frac{\delta}{\delta_0} \right) \right\} = 1 \quad \text{No cosine modulation}
\]

Aliasing is an error where high frequency patterns get mapped to low frequency patterns. For the examples above, when

\[
\frac{1}{X_s}
\]

<table>
<thead>
<tr>
<th>( \frac{4}{N_3} )</th>
<th>( \cos \left( 2\pi \frac{3}{2} \delta_0 x \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2N_3 )</td>
<td>( \cos \left( 2\pi \delta_0 x \right) )</td>
</tr>
<tr>
<td>( \frac{3}{2} N_3 )</td>
<td>( \cos \left( 2\pi \frac{3}{2} \delta_0 x \right) ) + half the real frequency</td>
</tr>
<tr>
<td>( N_3 )</td>
<td>1</td>
</tr>
</tbody>
</table>

No modulation

**SHOW IMAGES OF ALIASED PATTERNS**

Aliasing occurs when \( \frac{1}{X_s} \) is too large.

1. \( f(x) \) is band-limited by \( X_s \).
2. \( f(x) \) is not band-limited.

Sampling with \( \frac{1}{X_s} = 2N_3 \) does not ensure perfect recovery. This is just a fuzzy boundary between aliasing and non-aliasing.
Example: sampling frequency is $2\omega_0$ for an input $\sin(2\pi\omega_0 x)$.

\[ \frac{1}{\Delta x} = 2\omega_0 \text{ Nyquist frequency} \]

\[ x_s = \frac{1}{2\omega_0} \]

\[ f_s(x) = \text{comb}(2\omega_0 x) \sin(2\pi\omega_0 x) \]

Sample points occur at zeros of $\sin(2\pi\omega_0 x)$.

\[ F_s(\xi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\omega_0} \left[ \delta(\xi - (2\pi n)\omega_0) - \delta(\xi - (2\pi n-1)\omega_0) \right] \]

\[ F_s(\xi) \text{rect} \left( \frac{\xi}{2\omega_0} \right) = 0 \]

\[ f(x) = \mathcal{F}^{-1} \left\{ F_s(\xi) \text{rect} \left( \frac{\xi}{2\omega_0} \right) \right\} = 0 \]

Camera pixels: up to now, we have used sample S functions to sample points on $f(x, y)$. Camera pixels are slightly different in that the value they supply is an integration of the incidence over the aperture of the pixel. However, the same sampling issues arise.
pixel records (integrates) over its area
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta) \text{rect} \left( \frac{\alpha - x}{b}, \frac{\beta - y}{d} \right) \, d\alpha \, d\beta \]

This has the form of a convolution: \( f(x,y) \ast \text{rect} \left( \frac{x}{b}, \frac{y}{d} \right) \)

The sampled signal now looks like

\[
f_s(x,y) = \left[ f(x,y) \ast \text{rect} \left( \frac{x}{k_s}, \frac{y}{d} \right) \right] \text{comb} \left( \frac{x}{k_s} \right) \text{comb} \left( \frac{y}{d} \right)
\]

Recall that convolution is a smoothing operation, so our original object \( f(x,y) \) is "pre-smoothed" before it is sampled now.

\[
F_s(\xi, \eta) = b d X_s Y_s \left[ F(\xi, \eta) \text{sinc} \left( b \xi, d \eta \right) \right] \ast \text{comb} \left( X_s \right) \text{comb} \left( Y_s \right)
\]

\[
F_s(\xi, \eta) = b d \sum_{n,m} F(\xi - \frac{n}{X_s}, \eta - \frac{m}{Y_s}) \text{sinc} \left( b \left( \xi - \frac{n}{X_s} \right), d \left( \eta - \frac{m}{Y_s} \right) \right)
\]

The sinc function causes a roll off in the high frequencies of \( F(\xi, \eta) \). Where the sinc function is zero, the spectrum is lost.

Pixels also record (integrate) in the time domain so we can extend all these sampling concepts to inputs \( f(x,y,t) \) (e.g., image on a sensor changes with time). Video imaging is a good example. Temporal aliasing occurs when the frame rate of the recording is slower than the temporal variations in the scene. This leads to the "wagon wheel" effect.
Interferometry is routinely used for testing the shape of an optical surface. This is typically used to compare a surface being fabricated to a known surface to look for imperfections in shape. Two wavefronts, one from a known reference surface and one from the test surface, are combined. Anywhere the two wavefronts differ by an integer multiple of the wavelength, a fringe appears. Fringe patterns can be related to surface shape.

If the fringes get too close together compared to the size of the pixel on the sensor, then aliasing will occur. This is often the case when testing highly aspheric surfaces where the reference wavefront is typically a "best-fit" spherical shape.