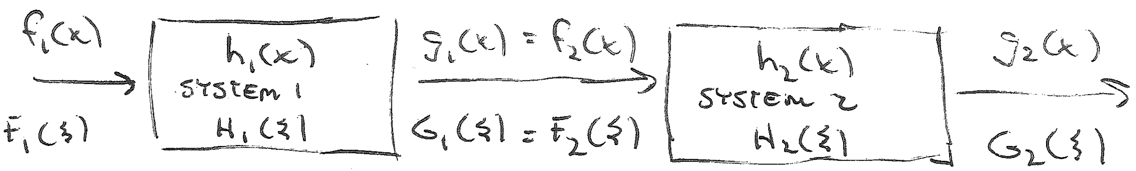
CASCADED SYSTEMS

We can string systems together so that the output of one system becomes the input of the next system.



For optical systems, cascading involves coupling the exit pupil of the first system to the entrance pupil of the next system.

Equalization OR INVERSE FILTERING

In the Fourier domain, we know that

$$G_1(\xi) = F_1(\xi)\, H_1(\xi)$$

Inverse filtering involves creating a second system with a transfer function

$$H_2(\xi) = \frac{1}{H_1(\xi)}$$

In this manner

$$G_2(\xi) = F_2(\xi)\, H_2(\xi) = \left(F_1(\xi)\, H_1(\xi)\right)\left(\frac{1}{H_1(\xi)}\right)$$

$$G_2(\xi) = F_1(\xi)$$ Perfect recovery of input

There are several potential pitfalls with this approach

(1) Wherever $H_1(\xi) = 0$, the filter goes to ∞.

(2) All processes include noise which will also be affected by filter.

(3) Physical filters might not be able to achieve $\frac{1}{H_1(\xi)}$ (e.g. transmission > 1)

WIENER FILTER - This filter attempts to combat some of the noise issues associated with inverse filtering. The model of the system incorporates additive noise.

$$g_i(x) = h_1(x) * f_i(x) + N(x)$$

$$G_i(\xi) = H_1(\xi) F_i(\xi) + N(\xi)$$

Here, we want to find a filter $H_2(\xi)$ that minimizes

$$\int_{-\infty}^{\infty} | F_i(\xi) - \hat{F}_i(\xi) |^2 \, d\xi$$

where $\hat{F}_i(\xi) = G_i(\xi) H_2(\xi)$ is an estimate of the input spectrum.

Note: By Rayleigh's Theorem

$$\int_{-\infty}^{\infty} | f_i(x) - \hat{f}_i(x) |^2 \, dx = \int_{-\infty}^{\infty} | F_i(\xi) - \hat{F}_i(\xi) |^2 \, d\xi$$

So minimizing the expression in the Fourier domain is the same as minimizing the difference in the spatial domain.

Plugging in for $\hat{F}_i(\xi)$

$$\int_{-\infty}^{\infty} \left| F_i(\xi) - \left(F_i(\xi) H_1(\xi) + N(\xi) \right) H_2(\xi) \right|^2 d\xi$$

GOAL: Find $H_2(\xi)$ which minimized this expression in a mean square sense so that $\hat{f}_i(x)$ looks like $f_i(x)$.

$$\int_{-\infty}^{\infty} \left| F_i(\xi)\left(1 - H_1(\xi) H_2(\xi)\right) - N(\xi) H_2(\xi) \right|^2 d\xi$$

multiply out and assuming cross terms average to zero meaning $F_i(\xi)$ and $N(\xi)$ are independent of each other

$$\int_{-\infty}^{\infty} \left[\left| F_i(\xi)\left(1 - H_1(\xi) H_2(\xi)\right) \right|^2 + \left| N(\xi) H_2(\xi) \right|^2 \right] d\xi$$

$$\int_{-\infty}^{\infty}\left[|F_1(\xi)|^2\,|1 - H_1(\xi)H_2(\xi)|^2 + |N(\xi)|^2\,|H_2(\xi)|^2\right]d\xi$$

The integral will be minimized when the integrand is minimized.
Take derivative with respect to $H_2(\xi)$ and set to zero.

$$|F_1(\xi)|^2\left[1 - H_1^*(\xi)H_2^*(\xi)\right]\left[-H_1(\xi)\right] + |N(\xi)|^2 H_2^*(\xi) = 0$$

$$-H_1(\xi)|F_1(\xi)|^2 + \cancel{|F(\xi)|^2}|H_1(\xi)|^2 H_2^*(\xi) + |N(\xi)|^2 H_2^*(\xi) = 0$$

$$H_2^*(\xi) = \frac{H_1(\xi)|F_1(\xi)|^2}{|H_1(\xi)|^2|F_1(\xi)|^2 + |N(\xi)|^2}$$

$$\boxed{H_2(\xi) = \frac{1}{H_1(\xi)}\cdot\frac{|H_1(\xi)|^2}{|H_1(\xi)|^2 + \dfrac{|N(\xi)|^2}{|F(\xi)|^2}}}$$ WIENER FILTER

$\dfrac{1}{H_1(\xi)}$ in front is the inverse filter, so still have problems when $H_1(\xi) = 0$.

$\dfrac{|N(\xi)|^2}{|F(\xi)|^2}$ may be unknown, but often have a model of the noise and can put $G(\xi)$ in for $F(\xi)$

$$\frac{|N(\xi)|^2}{|F(\xi)|^2} = \frac{1}{SNR}$$

Reciprocal of signal to noise.

If $N(\xi) = 0 \Rightarrow$ inverse filter

If $|N(\xi)|$ small compared to $|F(\xi)|$ boost ~~signal~~ $G(\xi)$

If $|N(\xi)|$ large compared to $|F(\xi)|$ don't boost $G(\xi)$

SIGNAL DETECTION - MATCHED FILTER

Here, we are not trying to recover the original input $f(x)$, but instead just trying to detect the presence of the signal without confusing it with other signals.

$s(x)$ desired signal

$N_1(x), N_2(x), N_3(x) \dots$ undesired signals

Want

$$g(x) = h(x) * s(x) \rightarrow \text{large}$$

$$g(x) = h(x) * N_i(x) \Rightarrow \text{small}$$

A matched filter has $h(x) = A s^*(-x)$ where A is a constant

$$g(x) = h(x) * f(x) \qquad \text{arbitrary input}$$

$$g(x) = A s^*(-x) * f(x)$$

$$g(x) = A \gamma_{fs}(x) \qquad \text{cross-correlation definition}$$

When $f(x) = s(x)$, then $\gamma_{fs}(x) = \gamma_s(x)$ i.e. auto correlation

In general $\gamma_s(0) \geq |\gamma_{fs}(x)|$

This says the peak of the autocorrelation is always larger than any value of the cross-correlation.

When we use $f(x) = N_i(x)$, then the output will be smaller than when $f(x) = s(x)$. A threshold can be created to confirm or reject the presence of $s(x)$.