Signal Processing

Signal processing of LSI systems involves creating a custom impulse response for a system so that a desired output $g(x)$ is achieved for a given input $f(x)$. These custom impulse responses are called filters (or equivalently their corresponding transfer functions). The processing is typically done in the Fourier domain to take advantage of nice properties of LSI systems.

$$f(x) * h(x) = g(x)$$

$$F(\mathcal{F}) \downarrow F.I.T. \downarrow F.I.T.^{-1}$$

$$F(\mathcal{F}) \cdot H(\mathcal{F}) = G(\mathcal{F})$$

$$g(x) = \mathcal{F}^{-1} F(\mathcal{F}) H(\mathcal{F})$$

In general, $F(\mathcal{F})$, $G(\mathcal{F})$, and $H(\mathcal{F})$ will all be complex functions. They can be written as

$$F(\mathcal{F}) = A_F(\mathcal{F}) \exp(-i \Phi_F(\mathcal{F}))$$
$$G(\mathcal{F}) = A_G(\mathcal{F}) \exp(-i \Phi_G(\mathcal{F}))$$
$$H(\mathcal{F}) = A_H(\mathcal{F}) \exp(-i \Phi_H(\mathcal{F}))$$

Bipolar means that the functions can take on both positive and negative values. Note, this is different than the modulus of the complex function.

$$A_F(\mathcal{F}) = |A_F(\mathcal{F})| \text{ where } A_F(\mathcal{F}) \geq 0$$

$$A_H(\mathcal{F}) = |A_H(\mathcal{F})| e^{i \pi} \text{ where } A_H(\mathcal{F}) < 0$$

plus integer multiples of $2\pi$. 
The output of the filtered system is now given by

\[ g(x) = \mathcal{F}^{-1} \left\{ A_F(\xi) A_H(\xi) \exp \left[ -i (\Phi_F(\xi) + \Phi_H(\xi)) \right] \right\} \]

\( A_H(\xi) \) is the amplitude transfer function and it converts the input amplitude spectrum \( A_F(\xi) \) to the output amplitude spectrum via multiplication.

\[ A_G(\xi) = A_F(\xi) A_H(\xi) \]

Similarly \( \Phi_H(\xi) \) is the phase transfer function and it converts the input phase spectrum \( \Phi_F(\xi) \) to the output phase spectrum via addition.

\[ \Phi_G(\xi) = \Phi_F(\xi) + \Phi_H(\xi) \]

The most general transfer function is complex, but there are two special cases that are common: amplitude filters and phase filters.

**AMPLITUDE FILTERS** - These occur when \( \exp(-i \Phi_H(\xi)) = 1 \).

In this case,

\[ g(x) = \mathcal{F}^{-1} \left\{ A_F(\xi) A_H(\xi) \exp(-i \Phi_F(\xi)) \right\} \]

Some common amplitude filters are:

**Distortionless Amplitude Filter**

\[ A_H(\xi) = A = \text{constant} \]

\[ g(x) = A \mathcal{F}^{-1} \left\{ A_F(\xi) \exp(-i \Phi_F(\xi)) \right\} = A f(x) \]

Output is just a scaled version of input:

- \(|A| < 1\) attenuation filter
- \(|A| > 1\) amplification filter
**Binary Amplitude Filters** - only take on values of 0 and 1 depending on \( \omega \).

**Ideal Low Pass Filter**

\[
G(\omega) = \begin{cases} 
F(\omega) & \omega \leq \omega_0 \\
0 & \omega > \omega_0 
\end{cases}
\]

**Ideal High Pass Filter**

\[
G(\omega) = \begin{cases} 
0 & \omega \leq \omega_0 \\
F(\omega) & \omega > \omega_0 
\end{cases}
\]

**Ideal Bandpass Filter**

\[
G(\omega) = \begin{cases} 
F(\omega) & \omega_0 \leq \omega \leq \omega_0 + \Delta \omega \\
0 & \text{otherwise}
\end{cases}
\]

**Examples**

Input \( f(x) \)

Input Spectrum \( F(\omega) \)

\[
F(\omega) = \begin{cases} 
\chi & \omega \leq \omega_0 \\
0 & \omega > \omega_0 
\end{cases}
\]
Low Pass Example

\[ h(x) = 4 \text{sinc}(4x) \]
\[ f(x) = \text{Rect}(x) * \frac{1}{2} \text{comb}\left(\frac{x}{2}\right) \]
\[ H(f) = \text{Rect}\left(\frac{f}{4}\right) \]
\[ F(f) = \frac{1}{2} \text{sinc}(f) \text{comb}(2f) \]

\[ G(f) = \text{Rect}\left(\frac{f}{4}\right) \text{sinc}(f) \text{comb}(2f) \]
\[ G(f) = \frac{1}{2} \text{sinc}(f) + \frac{1}{2} \left[ \frac{1}{2\pi} \text{sinc}(f - \frac{1}{2}) + \frac{1}{2\pi} \text{sinc}(f + \frac{1}{2}) \right] 
+ \frac{1}{4} \left[ \frac{-2}{3\pi} \text{sinc}(f - \frac{1}{2}) - \frac{2}{3\pi} \text{sinc}(f + \frac{1}{2}) \right] \]

\[ G(x) = \frac{1}{2} + \frac{1}{4\pi} \cos(\pi x) - \frac{2}{3\pi} \cos\left(3\pi x\right) \]

In general, true binary filters are hard to make, so real filters will have some transition.

**Phase Filters** - requires \( \Theta_h(f) = 1 \)

\[ g(x) = \mathcal{F}_c^{-1} \left\{ A_f(f) \exp\left[ -i \left( \Theta_f(f) + \Theta_h(f) \right) \right] \right\} \]
\[ = \mathcal{F}_c^{-1} \left\{ F(f) \exp\left[ -i \Theta_h(f) \right] \right\} \]

**Example:** Linear Phase Filter

\[ \Theta_h(f) = 2\pi \delta_0 f \]

In this case \( g(x) = \mathcal{F}_c^{-1} \left\{ F(f) \exp\left( -i2\pi \delta_0 f \right) \right\} \)
\[ g(x) = f(x - \delta_0) \]

So, linear phase filters shift the input.
Example: Phase Contrast Microscopy

Phase contrast microscopy is a technique for visualizing objects where \( \Phi_f(x) \approx 1 \), i.e., transparent, but \( \Phi_f(x) \) varies usually due to variations in refractive index. A typical example of such an object is a series of biological cells which are transparent but have a different refractive index than the surrounding media. The goal of the phase contrast technique is to create a filter \( H(x) \) so that the phase \( \Phi_f(x) \) gets converted to amplitude. In this manner, different refractive indices appear as shades of grey.

\[
P(x, y) = \exp \left[ i \left( \phi_0 + \Delta \phi(x, y) \right) \right]
\]

Typically assumed small amplitude \( \approx 1 \) so transparent

\( \phi_0 \) is average phase delay through object

\( \Delta \phi(x, y) \) is spatially-varying phase dependent upon the optical path length at \((x, y)\).

Note: the average of \( \Delta \phi(x, y) \) is zero since \( \phi_0 \) factored out.

We'll use this later with the central difference theorem

We also need the series expansion

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots
\]
Consider the filter function
\[ H(x, y) = \left[ 1 - \eta(x, y) + \sqrt{a} \exp(i \frac{\pi}{2}) \delta(x, y) \right] \]
\[ H(x, y) = \left[ 1 + \sqrt{a} \left( i - \frac{1}{\sqrt{a}} \right) \delta(x, y) \right] \]

We can rewrite the object as
\[ f(x, y) = \exp[i \phi_0] \exp[i \Delta \phi(x, y)] \approx \exp[i \phi_0] \left( 1 + i \Delta \phi(x, y) \right) \]

The object spectrum is
\[ F(x, y) = \exp[i \phi_0] \left( \delta(x, y) + \text{const} \Delta \phi(x, y) \right) \]

The output spectrum is
\[ G(x, y) = F(x) H(x) \]
\[ = \exp[i \phi_0] \left( \delta(x, y) + \text{const} \Delta \phi(x, y) \right) + \sqrt{a} \left( i - \frac{1}{\sqrt{a}} \right) \delta^2(x, y) \]

Looks messy, but let's simplify things a little
Define \( \Delta \phi(x, y) = \int_{-\infty}^{\infty} \Delta \phi(x, y) dx dy \)

\[ \Delta \phi(0, 0) \delta(x, y) = \Delta \phi(0, 0) \delta(x, y) \quad \text{multiplication by delta function} \]

But \( \int_{-\infty}^{\infty} \Delta \phi(x, y) dx dy = 0 \) since \( \Delta \phi(x, y) \) is zero mean, since we used \( \phi_0 \)

So last term of \( G(x, y) \) above is zero
Now look at $\delta'(x, n)$. Technically, this is undefined but we can make some arguments that it acts just like $\delta(x, n)$. So

$$G(x, n) = \exp \left[ i \phi_0 \right] \left[ \delta'(x, n) + \frac{1}{\alpha} i \Delta \phi(x, n) \right]$$

$$G(x, n) = \exp \left[ i \phi_0 \right] \left[ \frac{1}{\alpha} i \Delta \phi(x, n) \right]$$

The output is

$$g(x, y) = i \exp \left[ i \phi_0 \right] \left[ \frac{1}{\alpha} + \Delta \phi(x, y) \right]$$

In microscopy, we’re observing incoherence which is proportional to $|g(x, y)|^2$

$$|g(x, y)|^2 = \alpha + \frac{d\alpha}{d\phi} \Delta \phi(x, y) + \frac{d^2\alpha}{d\phi^2} \Delta \phi(x, y)^2$$

$$\text{Contrast} = \frac{d\alpha}{d\phi} \left( \Delta \phi(x, y)_{\text{max}} - \Delta \phi(x, y)_{\text{min}} \right)$$

As $\alpha$ gets smaller, the contrast increases.