Signal Processing

Signal processing of LSI systems involves creating a custom impulse response for a system so that a desired output $g(x)$ is achieved for a given input $f(x)$. These custom impulse responses are called filters (or equivalently their corresponding transfer functions). The processing is typically done in the Fourier domain to take advantage of nice properties of LSI systems

$$\begin{array}{ccccc} f(x) & * & h(x) & = & g(x) \\ \downarrow \text{F.T.} & & \downarrow \text{F.T.} & & \uparrow \text{F.T.}^{-1} \\ F(\xi) & \cdot & H(\xi) & = & G(\xi) \end{array}$$

$$g(x) = \mathcal{F}\{F(\xi)H(\xi)\}$$

In general, $F(\xi)$, $G(\xi)$, and $H(\xi)$ will all be complex functions.

They can be written as

$$F(\xi) = A_F(\xi)\exp[-i\Phi_F(\xi)]$$

$$G(\xi) = A_G(\xi)\exp[-i\Phi_G(\xi)]$$

$$H(\xi) = A_H(\xi)\exp[-i\Phi_H(\xi)]$$

where both the A's and the Φ's are real-valued bipolar functions.

Bipolar means that the functions can take on both positive <u>and</u> negative values. Note, this is different than the modulus of the complex function.

$$A_H(\xi) = |A_H(\xi)| \quad \text{where } A_H(\xi) \geq 0$$

$$A_H(\xi) = |A_H(\xi)|e^{\pm i\pi} \quad \text{where } A_H(\xi) < 0$$

plus integer multiples of 2π.

The output of the filtered system is now given by

$$g(x) = \mathcal{F}^{-1}\{A_F(\xi)A_H(\xi)\exp[-i(\phi_F(\xi) + \phi_H(\xi)]\}$$

$A_H(\xi)$ is the amplitude transfer function and it converts the input amplitude spectrum $A_F(\xi)$ to the output amplitude Spectrum via multiplication.

$$A_G(\xi) = A_F(\xi) A_H(\xi)$$

Similary $\Phi_H(\xi)$ is the phase transfer function and it converts the input phase spectrum $\Phi_F(\xi)$ to the output phase spectrum via addition.

$$\Phi_G(\xi) = \Phi_F(\xi) + \Phi_H(\xi)$$

The most general transfer function is complex, but there are two special cases that are common: amplitude filters ad phase filters.

<u>AMPLITUDE FILTERS</u> - THESE occur when $\exp[-i\,\Phi_H(\xi)] = 1$

In this case, $g(x) = \mathcal{F}^{-1}\{A_F(\xi)A_H(\xi)\exp(-i\,\Phi_F(\xi))\}$

Some common amplitude filters are

<u>Distortionless Amplitude Filter</u>

$$A_H(\xi) = A = \text{constant}$$

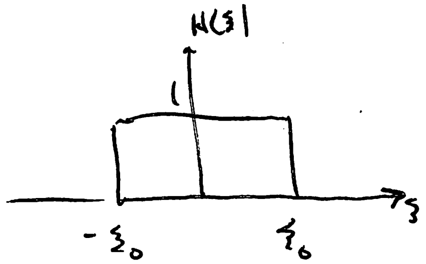
$$g(x) = A\,\mathcal{F}^{-1}\{A_F(\xi)\exp(-i\,\Phi_F(\xi))\} = Af(x)$$

Output is just a scaled version of input

$|A| \leq 1$ attenuation filter

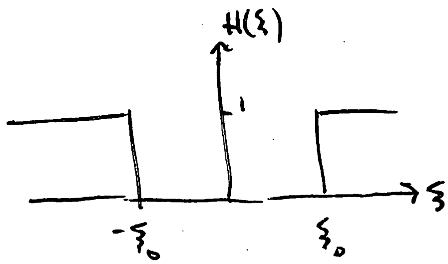
$|A| > 1$ amplification filter

BINARY AMPLITUDE FILTERS – only take on values of 0 and 1 depending on ξ.



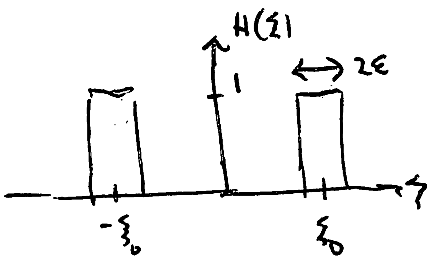
IDEAL LOW PASS FILTER

$$G(\xi) = \begin{cases} F(\xi) & \xi \le \xi_0 \\ 0 & \xi > \xi_0 \end{cases}$$



IDEAL HIGH PASS FILTER

$$G(\xi) = \begin{cases} 0 & \xi \le \xi_0 \\ F(\xi) & \xi > \xi_0 \end{cases}$$

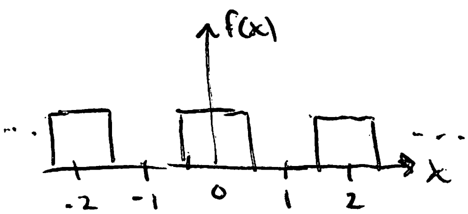


IDEAL BANDPASS FILTER

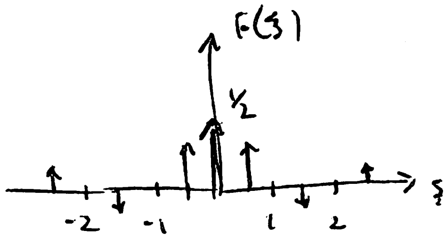
$$G(\xi) = \begin{cases} F(\xi) & \xi_0 - \varepsilon \le \xi \le \xi_0 + \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

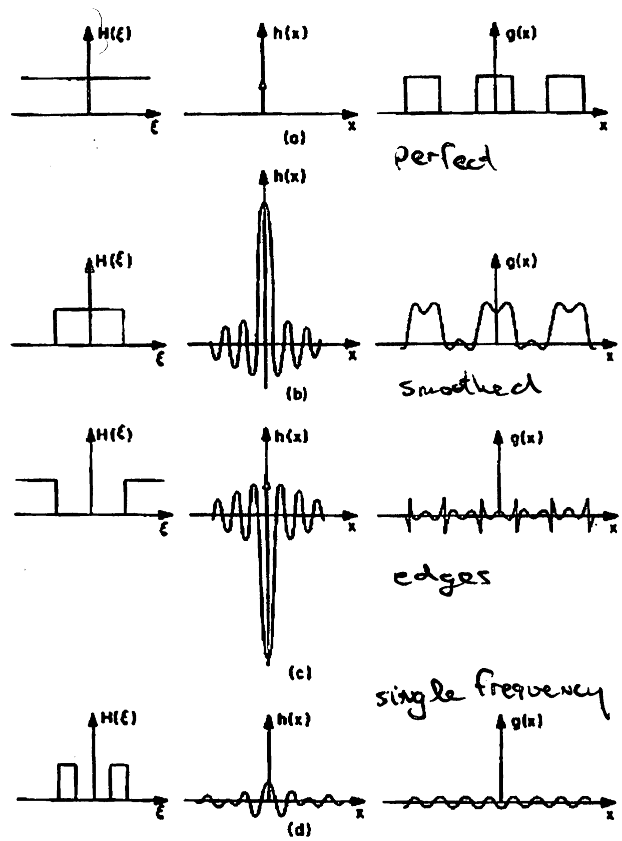
Examples

Input $f(x)$



Input Spectrum $F(\xi)$





LOW PASS EXAMPLE

$$h(x) = 4\,\mathrm{sinc}(4x) \qquad f(x) = \mathrm{rect}(x) * \frac{1}{2}\,\mathrm{comb}\left(\frac{x}{2}\right)$$

$$H(\xi) = \mathrm{rect}\left(\frac{\xi}{4}\right) \qquad F(\xi) = \frac{1}{2}\,\mathrm{sinc}(\xi)\,\mathrm{comb}(2\xi)$$

$$G(\xi) = \mathrm{rect}\left(\frac{\xi}{4}\right)\mathrm{sinc}(\xi)\,\mathrm{comb}(2\xi)$$

$\mathrm{sinc}\left(\frac{1}{2}\right) = \frac{2}{\pi}$

$\mathrm{sinc}(1) = 0$

$\mathrm{sinc}\left(\frac{3}{2}\right) = \frac{2}{3\pi}$

$$G(\xi) = \frac{1}{2}\delta(\xi) + \frac{1}{2}\left[\frac{2}{\pi}\delta\left(\xi - \frac{1}{2}\right) + \frac{2}{\pi}\delta\left(\xi + \frac{1}{2}\right)\right] + \frac{1}{2}\left[-\frac{2}{3\pi}\delta\left(\xi - \frac{3}{2}\right) - \frac{2}{3\pi}\delta\left(\xi + \frac{3}{2}\right)\right]$$

$$g(x) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi x) - \frac{2}{3\pi}\cos(3\pi x)$$

In general, true binary filters are hard to make, so real filters will have some transition.

PHASE FILTERS - requires $A_H(\xi) = 1$

$$g(x) = \mathcal{F}^{-1}\left\{A_F(\xi)\exp\left[-i\left(\Phi_F(\xi) + \Phi_H(\xi)\right)\right]\right\}$$

$$= \mathcal{F}^{-1}\left\{F(\xi)\exp\left[-i\Phi_H(\xi)\right]\right\}$$

Example: Linear Phase Filter

$$\Phi_H(\xi) = 2\pi x_0 \xi$$

In this case $g(x) = \mathcal{F}^{-1}\left\{F(\xi)\exp(-i2\pi x_0\xi)\right\}$

$$g(x) = f(x - x_0)$$

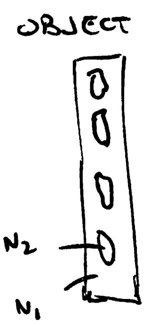
So, linear phase filters shift the input.

General Filter $H(\xi) = A_H(\xi) \exp[-i\Phi_H(\xi)]$

Example: Phase Contrast Microscopy

Phase contrast Microscopy is a technique for visualizing objects where $A_F(\xi) \simeq 1$ i.e. transparent, but $\Phi_F(\xi)$ varies usually due to variations in refractive index. A typical example of such an object is a series of biological cells which are transparent, but have a different refractive index than the surrounding media. The goal of the phase contrast technique is to create a filter $H(\xi)$ so that the phase $\Phi_F(\xi)$ gets converted to amplitude. In this manner, different refractive indices appear as shades of grey.

We need a few things first



$$f(x,y) = \exp[i(\phi_0 + \underbrace{\Delta\phi(x,y)}_{\text{Typically assumed small}})]$$

amplitude $= 1$ so transparent

ϕ_0 is average phase delay through object

$\Delta\phi(x,y)$ is spatially-varying phase dependent upon the optical path length at (x,y).

Note: the average of $\Delta\phi(x,y)$ is zero since ϕ_0 factored out. We'll use this later with the central ordinate theorem

We also need the series expansion

$$e^x = \sum_{N=0}^{\infty} \frac{x^N}{N!} = 1 + x + \frac{x^2}{2!} + \cdots$$

Consider the filter function

$$H(\xi,\eta) = \left[1 - \delta(\xi,\eta) + \sqrt{a}\exp\left(i\frac{\pi}{2}\right)\delta(\xi,\eta)\right]$$

a is a real Constant $0 \leq a \leq 1$

$$H(\xi,\eta) = \left[1 + \sqrt{a}\left(i - \frac{1}{\sqrt{a}}\right)\delta(\xi,\eta)\right]$$

We can rewrite the object as

$$f(x,y) = \exp[i\phi_0]\exp[i\Delta\phi(x,y)] \simeq \exp[i\phi_0](1 + i\Delta\phi(x,y))$$

The (input) object spectrum is

$$F(\xi,\eta) = \exp[i\phi_0]\left(\delta(\xi,\eta) + i\,\mathcal{F}_{2D}\{\Delta\phi(x,y)\}\right)$$

The output spectrum is

$$G(\xi,\eta) = F(\xi)H(\xi)$$

$$= \exp[i\phi_0]\Big(\delta(\xi,\eta) + i\,\mathcal{F}_{2D}\{\Delta\phi(x,y)\} + \sqrt{a}\left(i - \frac{1}{\sqrt{a}}\right)\delta^2(\xi,\eta)$$
$$+ i\sqrt{a}\left(i - \frac{1}{\sqrt{a}}\right)\mathcal{F}_{2D}\{\Delta\phi(x,y)\}\,\delta(\xi,\eta)\Big)$$

Looks messy, but let's simplify things a little

Define $\Delta\Phi(\xi,\eta) = \mathcal{F}_{2D}\{\Delta\phi(x,y)\}$

$\Delta\Phi(\xi,\eta)\,\delta(\xi,\eta) = \Delta\Phi(0,0)\,\delta(\xi,\eta)$ multiplication by delta function

$\Delta\Phi(0,0) = \iint_{-\infty}^{\infty}\Delta\phi(x,y)\,dx\,dy$ Central ordinate theorem

But $\iint_{-\infty}^{\infty}\Delta\phi(x,y)\,dx\,dy = 0$ since $\Delta\phi(x,y)$ is zero mean, since we used ϕ_0

So last term of $G(\xi,\eta)$ above is zero

Now look at $\delta^2(\xi,\eta)$. Technically, this is undefined but we can make some arguments that it acts just like $\delta(\xi,\eta)$. So

$$G(\xi,\eta) = \exp[i\phi_0]\left[\delta(\xi,\eta) + i\Delta\Phi(\xi,\eta) + \sqrt{a}\left(i - \frac{1}{\sqrt{a}}\right)\delta(\xi,\eta)\right]$$

$$G(\xi,\eta) = \exp[i\phi_0]\left[\sqrt{a}\, i\, \delta(\xi,\eta) + i\,\Delta\Phi(\xi,\eta)\right]$$

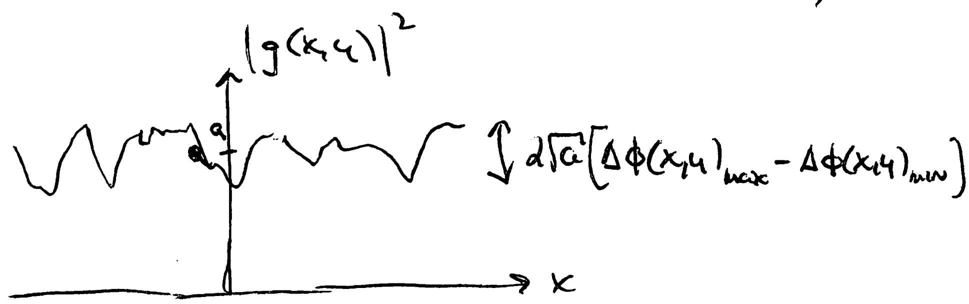
The output is

$$g(x,y) = i\exp[i\phi_0]\left[\sqrt{a} + \Delta\phi(x,y)\right]$$

In microscopy, we're observing irradiance which is proportional to $|g(x,y)|^2$

$$|g(x,y)|^2 = a + \alpha\sqrt{a}\,\Delta\phi(x,y) + \cancel{\Delta\phi^2(x,y)}$$

small



$$\text{Contrast} = \frac{\alpha\sqrt{a}\left[\Delta\phi(x,y)_{max} - \Delta\phi(x,y)_{min}\right]}{a}$$

$$\text{Contrast} = \frac{\alpha\left[\Delta\phi(x,y)_{max} - \Delta\phi(x,y)_{min}\right]}{\sqrt{a}}$$

As $\sqrt{a}$ gets smaller, the contrast increases.