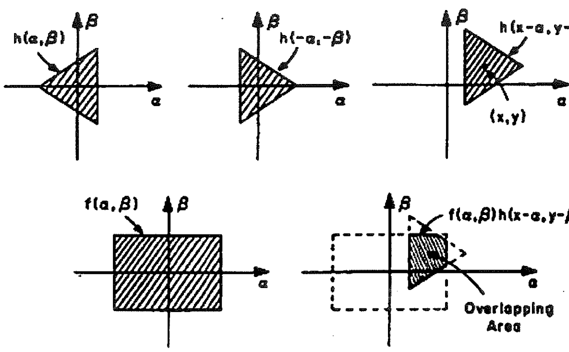
Convolution 2-D

Both convolution and the Fourier transform can be extended to two dimensions. In 2D, the convolution is

$$g(x,y) = f(x,y) ** h(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)\, h(x-\alpha, y-\beta)\, d\alpha\, d\beta$$

Note some authors use $*$ here



MUCH LIKE THE 1D CASE, h( ) IS FLIPPED AND x and y ARE ADJUSTED TO SLIDE h( ) OVER f( ). THE INTEGRAL FINDS THE VOLUME IN THE AREA OF OVERLAP.

2D Convolutions of separable functions

If both $f(x,y)$ and $h(x,y)$ are separable, then the 2D convolution reduces to the product of 1D convolutions

$$f(x,y) = f_1(x) f_2(y)$$
$$h(x,y) = h_1(x) h_2(y)$$

$$g(x,y) = \left[f_1(x) f_2(y)\right] ** \left[h_1(x) h_2(y)\right]$$

$$= \left[f_1(x) * h_1(x)\right]\left[f_2(y) * h_2(y)\right]$$

$$= g_1(x)\, g_2(y)$$

If only one function is separable

$$h(x,y) = h_1(x) h_2(y)$$

$$g(x,y) = f(x,y) ** \left[h_1(x) h_2(y)\right]$$

$$= f(x,y) * h_1(x) * h_2(y)$$

Properties of 2D Convolution - The 1D properties extend to 2D

$$f**h = h * f$$ Commutative property

$$f**[A_1 u + A_2 v] = A_1[f**u] + A_2[f**v]$$ Distributive property. A_1, A_2 constants

$$f**[u**v] = [f**u]**v$$ Associative property. Order doesn't matter

$$f(x-x_0, y-y_0)**h(x,y) = g(x-x_0, y-y_0)$$ Shift invariance

$$f\left(\frac{x}{b}, \frac{y}{b}\right)**h\left(\frac{x}{b}, \frac{y}{b}\right) = |b|^2 g\left(\frac{x}{b}, \frac{y}{b}\right)$$ Scaling

$$\iint_{-\infty}^{\infty} g(x,y)\,dx\,dy = \left[\iint_{-\infty}^{\infty} f(x,y)\,dx\,dy\right]\left[\iint_{-\infty}^{\infty} h(x,y)\,dx\,dy\right]$$ Volume of convolution is product of individual volumes.

Convolution with delta functions

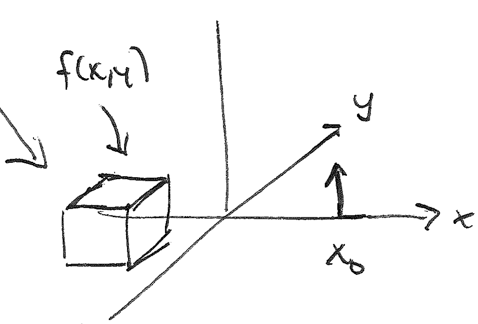
$$f(x,y) ** \delta(x-x_0, y-y_0) = f(x-x_0, y-y_0)$$

~~Define 1( ) "one" function equals 1 for all values of argument~~

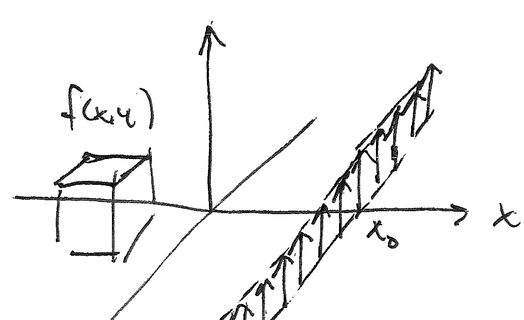
$$f(x,y) * \delta(x-x_0) = f(x-x_0, y)$$ ~~since $f(x,y) ** \delta(x-x_0) ** 1(y)$~~

But

$$f(x,y)**\delta(x-x_0) = \int_{-\infty}^{\infty} f(x-x_0, \beta)\,d\beta$$



1D convolution just reproduces f(x,y) at location of δ-function



Each delta function reproduces f(x,y) and they all combine.

2D Cross Correlation and Autocorrelations

$$f(x,y) \star\star g(x,y) = \iint_{-\infty}^{\infty} f(\alpha,\beta)\, g(\alpha - x, \beta - y)\, d\alpha\, d\beta$$

2D Cross Correlation

$$= \iint_{-\infty}^{\infty} f(\alpha + x, \beta + y)\, g(\alpha,\beta)\, d\alpha\, d\beta$$

Note how the argument of $g()$ is reversed compared to convolution.

$$f(x,y) \star\star g(x,y) = f(x,y) ** g(-x,-y)$$

minus signs cancel flip in convolution

In general

$$f(x,y) \star\star g(x,y) \neq g(x,y) \star\star f(x,y)$$

Commutative property does not hold for correlation.

When $g(x,y) = f(x,y)$ then we call the cross correlation an auto correlation

These concepts can be extended to complex functions

$$\gamma_{fg}(x,y) = f(x,y) \star\star g^*(x,y)$$

Complex Cross Correlation

$$= f(x,y) ** g^*(-x,-y)$$

when $g(x,y) = f(x,y)$

$$\gamma_{ff}(x,y) = f(x,y) \star\star f^*(x,y) = \gamma_f(x,y)$$

Complex Autocorrelation

Properties

$$\gamma_f(x,y) = \gamma_f^*(-x,-y)$$

Hermitian

$$|\gamma_f(x,y)| \leq \gamma_f(0,0)$$

Max value at origin

FOURIER TRANSFORM 2D CARTESIAN COORDINATES

$$F(\xi,\eta) = \iint_{-\infty}^{\infty} f(\alpha,\beta)\, e^{-i2\pi(\alpha\xi + \beta\eta)}\, d\alpha\, d\beta$$

2D FOURIER TRANSFORM of $f(x,y)$

$$f(x,y) = \iint_{-\infty}^{\infty} F(\alpha,\beta)\, e^{i2\pi(\alpha x + \beta y)}\, d\alpha\, d\beta$$

2D INVERSE FOURIER TRANSFORM OF $F(\xi,\eta)$

Different shorthand notations

$$F(\xi,\eta) = \mathcal{F}\mathcal{F}\{f(x,y)\} \qquad f(x,y) \xrightarrow{\mathcal{F}\mathcal{F}} F(\xi,\eta)$$

$$f(x,y) = \mathcal{F}^{-1}\mathcal{F}^{-1}\{F(\xi,\eta)\} \qquad f(x,y) \xleftarrow{\mathcal{F}^{-1}\mathcal{F}^{-1}} F(\xi,\eta)$$

2D FOURIER TRANSFORM OF SEPARABLE FUNCTIONS

$$F(\xi,\eta) = \mathcal{F}\mathcal{F}\{f_1(x) f_2(y)\} = F_1(\xi)\, F_2(\eta) \qquad f(x,y) = f_1(x) f_2(y)$$

PROPERTIES OF 2D FOURIER TRANSFORM

$g(x,y) = \iint_{-\infty}^{\infty} G(\alpha',\beta') e^{j2\pi(\alpha' x + \beta' y)} d\alpha'\, d\beta'$	$G(\xi,\eta) = \iint_{-\infty}^{\infty} g(\alpha,\beta) e^{-j2\pi(\alpha\xi + \beta\eta)} d\alpha\, d\beta$
$f(\pm x, \pm y)$	$F(\pm\xi, \pm\eta)$
$f^*(\pm x, \pm y)$	$F^*(\mp\xi, \mp\eta)$
$F(\pm x, \pm y)$	$f(\mp\xi, \mp\eta)$
$F^*(\pm x, \pm y)$	$f^*(\pm\xi, \pm\eta)$
$f_1(x) f_2(y)$	$F_1(\xi) F_2(\eta)$
$f_1(x)$	$F_1(\xi)\delta(\eta)$
$f_2(y)$	$\delta(\xi) F_2(\eta)$
$f\left(\frac{x}{b}, \frac{y}{d}\right)$	$\lvert bd\rvert F(b\xi, d\eta)$
$f(x \pm x_0, y \pm y_0)$	$e^{\pm j2\pi x_0\xi} e^{\pm j2\pi y_0\eta} F(\xi,\eta)$
$e^{\pm j2\pi\xi_0 x} e^{\pm j2\pi\eta_0 y} f(x,y)$	$F(\xi \mp \xi_0, \eta \mp \eta_0)$
$e^{\pm j2\pi\xi_0 x} e^{\pm j2\pi\eta_0 y} f\left(\frac{x \pm x_0}{b}, \frac{y \pm y_0}{d}\right)$	$\lvert bd\rvert e^{\pm j2\pi x_0(\xi \mp \xi_0)} e^{\pm j2\pi y_0(\eta \mp \eta_0)} \times F[b(\xi \mp \xi_0), d(\eta \mp \eta_0)]$

PROPERTIES CONTINUED

$h(x,y)$	$H(\xi,\eta)$
$A_1 f(x,y)+A_2 h(x,y)$	$A_1 F(\xi,\eta)+A_2 H(\xi,\eta)$
$f(x,y)**h(x,y)$	$F(\xi,\eta)H(\xi,\eta)$
$f(x,y)h(x,y)$	$F(\xi,\eta)**H(\xi,\eta)$
$f(x,y)\star\star h(x,y)$	$F(\xi,\eta)H(-\xi,-\eta)$
$f(x,y)h(-x,-y)$	$F(\xi,\eta)\star\star H(\xi,\eta)$
$\gamma_{fh}(x,y)=f(x,y)\star\star h^*(x,y)$	$F(\xi,\eta)H^*(\xi,\eta)$
$f(x,y)h^*(x,y)$	$\gamma_{FH}(\xi,\eta)=F(\xi,\eta)\star\star H^*(\xi,\eta)$
$\gamma_f(x,y)=f(x,y)\star\star f^*(x,y)$	$\|F(\xi,\eta)\|^2$
$\|f(x,y)\|^2$	$\gamma_F(\xi,\eta)=F(\xi,\eta)\star\star F^*(\xi,\eta)$

$g(x,y)=\iint_{-\infty}^{\infty} G(\alpha',\beta')e^{j2\pi(\alpha' x+\beta' y)}d\alpha'\,d\beta'$	$G(\xi,\eta)=\iint_{-\infty}^{\infty} g(\alpha,\beta)e^{-j2\pi(\alpha\xi+\beta\eta)}d\alpha\,d\beta$
$\delta(x,y)$	1
$\delta(x\pm x_0, y\pm y_0)$	$e^{\pm j2\pi x_0\xi}e^{\pm j2\pi y_0\eta}$
$e^{\pm j2\pi\xi_0 x}e^{\pm j2\pi\eta_0 y}$	$\delta(\xi\mp\xi_0, \eta\mp\eta_0)$
$\cos(2\pi\xi_0 x)$	$\frac{1}{2\|\xi_0\|}\delta\delta\left(\frac{\xi}{\xi_0}\right)\delta(\eta)$
$\sin(2\pi\eta_0 y)$	$\frac{j}{2\|\eta_0\|}\delta(\xi)\delta\delta\left(\frac{\eta}{\eta_0}\right)$
$\mathrm{rect}(x,y)$	$\mathrm{sinc}(\xi,\eta)$
$\mathrm{tri}(x,y)$	$\mathrm{sinc}^2(\xi,\eta)$
$\mathrm{Gaus}(x,y)$	$\mathrm{Gaus}(\xi,\eta)$
$\mathrm{comb}(x,y)$	$\mathrm{comb}(\xi,\eta)$
$x^k y^l$	$\left(\frac{1}{-j2\pi}\right)^{k+l}\delta^{(k,l)}(\xi,\eta)$
$\left(\frac{1}{j2\pi}\right)^{k+l}\delta^{(k,l)}(x,y)$	$\xi^k\eta^l$
$\exp[\pm j\pi(x^2+y^2)]$	$\pm j\exp[\mp j\pi(\xi^2+\eta^2)]$
$\exp\left[-\pi\left(\frac{x^2+y^2}{a+jc}\right)\right], a>0, a^2+c^2<\infty$	$(a+jc)\exp[-\pi(a+jc)(\xi^2+\eta^2)]$

TRANSFORMS OF COMMON FUNCTIONS

$$\iint_{-\infty}^{\infty} |f(x,y)|^2\,dx\,dy = \iint_{-\infty}^{\infty} |F(\xi,\eta)|^2\,d\xi\,d\eta$$ Rayleigh's Theorem in 2D

$$\mathcal{F}\{f(x,y)\star\star f^*(x,y)\} = |F(\xi,\eta)|^2$$ Wiener-Khinchine Theorem in 2D

Hankel Transform

In optics, we often encounter 2D functions with rotational or axial symmetry. A circular pupil is a common example. We'd like to analyze the properties of the 2D Fourier transform in the special case where $f(x,y) = f(r)$, where $r = \sqrt{x^2+y^2}$.

In this case, the 2D Fourier transform becomes

$$F(\xi,\eta) = \iint_{-\infty}^{\infty} f\left(\sqrt{\alpha^2+\beta^2}\right) e^{-i2\pi(\alpha\xi+\beta\eta)}\, d\alpha\, d\beta$$

Let's use our typical polar coordinate definitions

$x = r\cos\theta$	$\xi = \rho\cos\phi$
$y = r\sin\theta$	$\eta = \rho\sin\phi$
$r = \sqrt{x^2+y^2}$	$\rho = \sqrt{\xi^2+\eta^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$	$\phi = \tan^{-1}\left(\frac{\eta}{\xi}\right)$

The integral becomes

$$F(\rho\cos\phi, \rho\sin\phi) = \int_0^{\infty}\int_0^{2\pi} f(r)\, e^{-i2\pi\rho r\cos(\theta-\phi)}\, r\, dr\, d\theta$$

From integral tables $\int_0^{2\pi} \exp\left[-ia\cos(\theta-\phi)\right] d\theta = 2\pi J_0(a)$

So

$$F(\rho\cos\phi, \rho\sin\phi) = 2\pi\int_0^{\infty} f(r)\, J_0(2\pi\rho r)\, r\, dr = F(\rho)$$

since no ϕ dependence in integral

2D FOURIER TRANSFORM OF ROTATIONALLY SYMMETRIC FUNCTION $f(r)$ IS ROTATIONALLY SYMMETRIC.

$$\boxed{\begin{aligned} F(\rho) &= 2\pi \int_0^{\infty} f(r)\, J_0(2\pi \rho r)\, r\, dr \\ f(r) &= 2\pi \int_0^{\infty} F(\rho)\, J_0(2\pi \rho r)\, \rho\, d\rho \end{aligned}}$$

0^{th} order Hankel transform of $f(r)$

0^{th} order inverse Hankel transform of $F(\rho)$

Different shorthand notations

$$H_0\{f(r)\} = F(\rho) \qquad H_0\{F(\rho)\} = f(r) \qquad f(r) \overset{H_0}{\longleftrightarrow} F(\rho)$$

Some basic properties

$$H_0\left\{f\left(\frac{r}{b}\right)\right\} = |b|^2 F(b\rho) \qquad \text{scaling}$$

Scaling also works for different scale factors in x and y direction

Given $H_0\{f(r)\} = F(\rho)$

$$H_0\left\{f\left(\sqrt{\left(\frac{x}{b}\right)^2 + \left(\frac{y}{d}\right)^2}\right)\right\} = |bd|\, F\left(\sqrt{(b\xi)^2 + (d\eta)^2}\right)$$

Shifting

$$H_0\left\{f\left(\sqrt{(x \pm x_0)^2 + (y \pm y_0)^2}\right)\right\} = \exp\left[\pm i2\pi x_0 \xi\right] \exp\left[\pm i2\pi y_0 \eta\right] F(\rho)$$

Example: cylinder function with unit diameter

$$H_0\{cyl(r)\} = 2\pi \int_0^{\infty} cyl(r)\, J_0(2\pi \rho r)\, r\, dr$$

$$= 2\pi \int_0^{1/2} J_0(2\pi \rho r)\, r\, dr$$

For this example, we need to take advantage of the following integral

$$\int_0^a J_0(r)\, r\, dr = a J_1(a)$$

with a change of variables

$$H_0\{cyl(r)\} = \frac{2\pi}{(2\pi\rho)^2}\int_0^{\pi\rho} J_0(r')\, r'\, dr' \qquad \text{with } r' = 2\pi\rho r, \quad dr' = 2\pi\rho\, dr$$

$$= \frac{2\pi}{(2\pi\rho)^2}\,\pi\rho\, J_1(\pi\rho) = \frac{\pi^2\rho}{4\pi\rho}\,\frac{2J_1(\pi\rho)}{\pi\rho} = \frac{\pi}{4}\,\mathrm{somb}(\rho)$$

$$\boxed{H_0\{cyl(r)\} = \frac{\pi}{4}\,\mathrm{somb}(\rho)}$$

Example: cylinder function with diameter = 4

$$H_0\left\{cyl\left(\frac{r}{4}\right)\right\} = 4^2\left[\frac{\pi}{4}\,\mathrm{somb}(4\rho)\right]$$

FROM SCALING

$$H_0\left\{f\left(\frac{r}{b}\right)\right\} = |b|^2 F(b\rho)$$

$$H_0\left\{cyl\left(\frac{r}{4}\right)\right\} = 4\pi\,\mathrm{somb}(4\rho)$$

PROPERTIES OF 0^{th} ORDER HANKEL TRANSFORM

$g(r)=2\pi\int_0^\infty G(\rho')J_0(2\pi r\rho')\rho'\,d\rho'$	$G(\rho)=2\pi\int_0^\infty g(r')J_0(2\pi\rho r')r'\,dr'$
$f(r)$	$F(\rho)$
$f\left(\frac{r}{b}\right)$	$\lvert b\rvert^2 F(b\rho)$
$h(r)$	$H(\rho)$
$A_1 f(r)+A_2 h(r)$	$A_1 F(\rho)+A_2 H(\rho)$
$f(r)**h(r)=f(r)\star\star h(r)$	$F(\rho)H(\rho)$
$f(r)h(r)$	$F(\rho)**H(\rho)=F(\rho)\star\star H(\rho)$
$\gamma_{fh}(r)=f(r)\star\star h^*(r)$	$F(\rho)H^*(\rho)$

PROPERTIES CONTINUED

$g(r)=2\pi\int_0^{\infty}G(\rho')J_0(2\pi r\rho')\rho'\,d\rho'$	$G(\rho)=2\pi\int_0^{\infty}g(r')J_0(2\pi\rho r')r'\,dr'$
$f(r)h^*(r)$	$\gamma_{FH}(\rho)=F(\rho)\star\star H^*(\rho)$
$\gamma_f(r)=f(r)\star\star f^*(r)$	$\lvert F(\rho)\rvert^2$
$\lvert f(r)\rvert^2$	$\gamma_F(\rho)=F(\rho)\star\star F^*(\rho)$

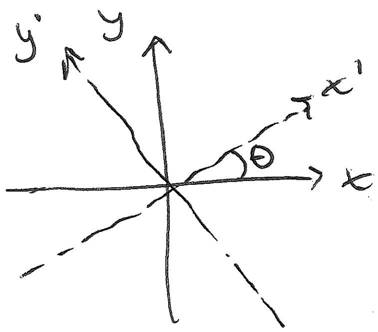
$g\left(\sqrt{x^2+y^2}\right)=\mathcal{F}\mathcal{F}\left\{G\left(\sqrt{\xi^2+\eta^2}\right)\right\}$	$G\left(\sqrt{\xi^2+\eta^2}\right)=\mathcal{F}^{-1}\mathcal{F}^{-1}\left\{f\left(\sqrt{x^2+y^2}\right)\right\}$
$f\left(\sqrt{x^2+y^2}\right)$	$F\left(\sqrt{\xi^2+\eta^2}\right)$
$f\left(\sqrt{\left(\frac{x}{b}\right)^2+\left(\frac{y}{d}\right)^2}\right)$	$\lvert bd\rvert F\left(\sqrt{(b\xi)^2+(d\eta)^2}\right)$
$f\left(\sqrt{(x\pm x_0)^2+(y\pm y_0)^2}\right)$	$e^{\pm j2\pi x_0\xi}e^{\pm j2\pi y_0\eta}F\left(\sqrt{\xi^2+\eta^2}\right)$
$e^{\pm j2\pi\xi_0 x}e^{\pm j2\pi\eta_0 y}f\left(\sqrt{x^2+y^2}\right)$	$F\left(\sqrt{(\xi\mp\xi_0)^2+(\eta\mp\eta_0)^2}\right)$

$g(r)=2\pi\int_0^{\infty}G(\rho')J_0(2\pi r\rho')\rho'\,d\rho'$	$G(\rho)=2\pi\int_0^{\infty}g(r')J_0(2\pi\rho r')r'\,dr'$
$\frac{\delta(r)}{\pi r}$	1
$\delta(r-r_0),\quad r_0>0$	$2\pi r_0 J_0(2\pi r_0\rho)$
$\frac{1}{r}$	$\frac{1}{\rho}$
$\mathrm{cyl}(r)$	$\frac{\pi}{4}\,\mathrm{somb}(\rho)$
e^{-r}	$\frac{2\pi}{(4\pi^2\rho^2+1)^{3/2}}$
$\mathrm{Gaus}(r)$	$\mathrm{Gaus}(\rho)$
$\cos(\pi r^2)$	$\sin(\pi\rho^2)$
$e^{\pm j\pi r^2}$	$\pm je^{\mp j\pi\rho^2}$
$\exp\left\{-\pi\left(\frac{r^2}{a+jc}\right)\right\},\quad a\geqslant 0,\ a^2+c^2<\infty$	$(a+jc)e^{-\pi(a+jc)\rho^2}$

Some common 0th order Hankel transform pairs.

Radon Transform

The Radon transform takes a ~~series of~~ 2D function $f(x,y)$ and converts it to a collection of 1D functions. The collection is technically a 2D function, but it is part Cartesian and part polar (weird, I know). To begin to understand the Radon transform, let's first look at rotation of the Cartesian coordinate system.

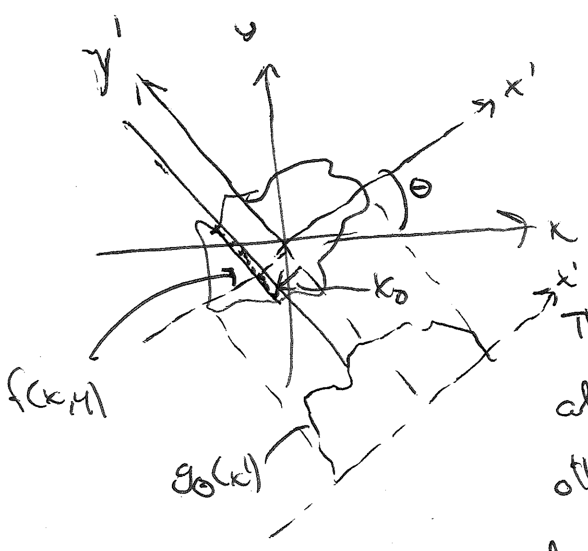


We can rotate the Cartesian coordinate system by an angle θ

The coordinates between the two reference frames are related by

$$x' = x\cos\theta + y\sin\theta \qquad x = x'\cos\theta - y'\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta \quad \text{or inverting} \quad y = x'\sin\theta + y'\cos\theta.$$

For the Radon transform, we're interested in the line integral along the y'-axis of some function $f(x,y)$



The line integral is just a 1D integration of the function $f(x,y)$ along the y' axis. In general, it depends on both x' and θ. This can be written as a 2D function $g(x'_0, \theta)$, although it is often written as $g_\theta(x'_0)$. In other words, for a fixed value of θ, the function is 1D $g_\theta(x'_0)$ in the coordinate x'_0.

Our typical application for the Radon transform is that we have a set of $g_\theta(x')$, ideally for continuous θ, but in actuality we have a limited # of angles θ. We want to take this set and figure out what $f(x,y)$ is.

$$g_\theta(x_0) = \int_{-\infty}^{\infty} f(x'\cos\theta - y'\sin\theta,\ x'\sin\theta + y'\cos\theta)\,\delta(x'-x_0)\,dy'$$

Convert to (x,y) coordinate system

$$g_\theta(x_0) = \iint_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - x_0)\,dx\,dy$$

Fourier transform (1D with respect to x_0)

$$G_\theta(\xi_0) = \mathcal{F}\{g_\theta(x_0)\} = \iiint_{-\infty}^{\infty} f(x,y)\,\delta(x\cos\theta + y\sin\theta - x_0)\exp[-i2\pi\xi_0 x_0]\cdot dx\,dy\,dx_0$$

$$G_\theta(\xi_0) = \iint_{-\infty}^{\infty} f(x,y)\exp[-i2\pi\xi_0(x\cos\theta + y\sin\theta)]\,dx\,dy$$

Now, let's compare this to $F(\xi,\eta)$

$$F(\xi,\eta) = \mathcal{F}\mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y)\exp[-i2\pi(\xi x + \eta y)]\,dx\,dy$$

$$G_\theta(\xi_0) = F(\xi_0\cos\theta,\ \xi_0\sin\theta)$$