1. Prove that $\text{rect}_1(x) \ast \text{rect}_2(x) = \text{tri}(x)$ using the following procedure. Note the subscripts are just so we can keep track of the individual $\text{rect}(\cdot)$ functions.

(a) Write the convolution integral with these two functions.

$$\text{rect}_1(x) \ast \text{rect}_2(x) = \int_{-\infty}^{\infty} \text{rect}_1(\alpha)\text{rect}_2(x - \alpha)d\alpha.$$  

(b) Plot $\text{rect}_1(\alpha)$.

(c) Plot $\text{rect}_2(-\alpha)$. Note that this is just the second function flipped about the y-axis for the case when $x = 0$.

*This looks exactly the same since the function is already symmetric about the y axis.*

(d) Plot $\text{rect}_2(x - \alpha)$ for the cases where $x = -1, -1/2, 0, 1/2, 1$.

This shows how the second functions slides horizontally depending on the value of $x$. 
(e) What are the values of the areas of $rect_1(\alpha)rect_2(x - \alpha)$ for $x = -1, x = -1/2, x = 0, x = 1/2, x = 1$. The areas correspond to the integration of the product of the two functions.

For $x = -1$, the area of $rect_1(\alpha)rect_2(-1 - \alpha)$ is zero since the two functions don’t overlap.

For $x = -1/2$, the area of $rect_1(\alpha)rect_2(-1/2 - \alpha)$ is $1/2$ since the two functions half overlap and have a height of one.

For $x = 0$, the area of $rect_1(\alpha)rect_2(-\alpha)$ is $1$ since the two functions completely overlap and have a height of one.

For $x = 1/2$, the area of $rect_1(\alpha)rect_2(1/2 - \alpha)$ is $1/2$ since the two functions half overlap and have a height of one.

For $x = 1$, the area of $rect_1(\alpha)rect_2(1 - \alpha)$ is zero since the two functions don’t overlap.

(f) What is the general formula for the area of $rect_1(\alpha)rect_2(x - \alpha)$ for $-1 < x \leq 0$?

This will be a function of $x$.

The left edge of the overlap region is located at $\alpha = -1/2$ and the right edge of the overlap region is located at $\alpha = x + 1/2$, since the center of the second $rect()$ is located at $\alpha = x$. The area of the overlap region is therefore a width of $x + 1/2 - (-1/2)$ times a height one, leading to an area of $x + 1$. 
(g) What is the general formula for the area of $\text{rect}_1(\alpha)\text{rect}_2(x - \alpha)$ for $0 < x < 1$?

This will also be a function of $x$.

The left edge of the overlap region is located at $\alpha = x - 1/2$, since the center of the second $\text{rect}()$ is located at $\alpha = x$. The right edge of the overlap region is located at $\alpha = 1/2$. The area of the overlap region is therefore a width of $1/2 - (x - 1/2)$ times a height one, leading to an area of $1 - x$.

(h) Show that preceding results are consistent with the definition of $\text{tri}(x)$.

The preceding results can be summarized as

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = 0 \text{ for } x \leq -1. \]

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = x + 1 \text{ for } -1 < x \leq 0. \]

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = 1 - x \text{ for } 0 < x < 1. \]

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = 0 \text{ for } x \geq 1. \]
The first and last relationships can be combined as

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = 0 \text{ for } [x] \geq 1. \]

The middle two relationships can be combined as

\[ \text{rect}_1(x) \ast \text{rect}_2(x) = 1 - [x] \text{ for } [x] < 1, \text{ which is exactly the definition of } \text{tri}(x). \]

2. Use the Fourier integral to show that \( Gaus(\xi) = \mathcal{F}\{Gaus(x)\} \). Hint: Use completing the square and that

\[ \int_{-\infty}^{\infty} \exp[-\pi u^2] du = 1 \]

From our special function definitions

\[ Gaus(x) = \exp[-\pi x^2] \]

Plugging this into the Fourier integral gives

\[ F(\xi) = \int_{-\infty}^{\infty} \exp[-\pi x^2] \exp[-i2\pi \xi x] dx \]

Combining the exponents gives

\[ F(\xi) = \int_{-\infty}^{\infty} \exp[-\pi (x^2 + i2\xi x)] dx. \]

Completing the square gives

\[ F(\xi) = \exp[-\pi \xi^2] \int_{-\infty}^{\infty} \exp[-\pi (x^2 + i2\xi x - \xi^2)] dx. \]

This reduces to

\[ F(\xi) = Gaus(\xi) \int_{-\infty}^{\infty} \exp[-\pi (x + i\xi)^2] dx. \]

Substituting \( u = x + i\xi \) leads to
\[ F(\xi) = \text{Gaus}(\xi) \int_{-\infty}^{\infty} \exp[-\pi u^2] du = \text{Gaus}(\xi) \]

*Based on the integral in the hint.*

3. Based on the results of question 2, using the scaling and shifting properties of Fourier transforms to calculate

(a) \( \mathcal{F}\{\text{Gaus}(4x)\} \). Plot \( \text{Gaus}(4x) \) and its Fourier transform for a horizontal range of -6 to 6 and a vertical range from 0 to 1.

*The previous problem showed that \( \text{Gaus}(\xi) = \mathcal{F}\{\text{Gaus}(x)\} \). Let’s first rewrite our Gaussian function in its standard form:

\[ \text{Gaus}\left(\frac{x}{1/4}\right). \]

*The scaling property says:

\[ \mathcal{F}\left\{ f\left(\frac{X}{b}\right) \right\} = [b]F(b\xi), \]

*So this says that:

\[ \mathcal{F}\left\{ \text{Gaus}\left(\frac{X}{b}\right) \right\} = [b]\text{Gaus}(b\xi). \]

*Since \( b = 1/4 \) in the case,

\[ \mathcal{F}\{\text{Gaus}(4x)\} = \frac{1}{4}\text{Gaus}\left(\frac{\xi}{4}\right). \]

Plotting
(b) $\mathcal{F}\{\text{Gaus}\left(\frac{x}{4}\right)\}$. Plot $\text{Gaus}\left(\frac{x}{4}\right)$ and its Fourier transform for a horizontal range of -6 to 6 and a vertical range from 0 to 4.

*From part (a)*

$$\mathcal{F}\{\text{Gaus}\left(\frac{x}{b}\right)\} = [b]\text{Gaus}(b\xi).$$

*Since $b = 4$ in the case,*

$$\mathcal{F}\{\text{Gaus}\left(\frac{x}{4}\right)\} = 4\text{Gaus}(4\xi).$$

*Plotting*

(c) Based on (a) and (b) how are the widths of the Gaussian function and its Fourier transform related?

*When the width of the Gaussian in $x$-space is narrow, its width in $\xi$-space is large and vice versa. This is true for all Fourier transforms.*
(d) $\mathcal{F}\{Gaus(4x - 2)\}$. Plot $Gaus(4x - 2)$ and the magnitude and phase of its Fourier transform. Furthermore, plot the real and imaginary parts of the transform. How does the magnitude and phase of this function compare to the transform in part (a)? How does the envelope of the real and imaginary parts relate to the magnitude? Use a horizontal range of -6 to 6 for all plots. Use a vertical range from 0 to 1 for $Gaus(4x - 2)$ and the modulus if its transform. Use a vertical range from -1 to 1 for the real and imaginary parts of the Fourier transform.

*Let’s put this into standard form*

$$Gaus(4x - 2) = Gaus\left(4\left(x - \frac{1}{2}\right)\right) = Gaus\left(\frac{x - \frac{1}{2}}{\frac{1}{4}}\right)$$

*Combining the shifting and scaling properties of Fourier transforms says*

$$\mathcal{F}\left\{f\left(\frac{x \pm x_o}{b}\right)\right\} = |b| \exp[\pm i2\pi x_o \xi] F(b \xi),$$

*So*

$$\mathcal{F}\left\{Gaus\left(\frac{x - \frac{1}{2}}{\frac{1}{4}}\right)\right\} = \frac{1}{4} \exp[-i\pi \xi] Gaus\left(\frac{\xi}{4}\right).$$

*The plots of the function, the modulus and phase of its Fourier transform and the real and imaginary parts of its Fourier transform are below. The magnitude is unchanged from part (a), but now the phase is linear instead of zero. Note that the envelope of real and imaginary parts of the Fourier transform is the magnitude.*