SPECIAL THEOREMS FOR FOURIER TRANSFORMS

There are ~~several~~ several special theorems that arise in Fourier optics. We have already seen the convolution theorem

$$\mathcal{F}\{f(x) * g(x)\} = F(\xi)\,G(\xi)$$

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See page 63 for derivation

There are a couple other theorems that are occasionally useful.

Rayleigh's Theorem

Given two complex functions $f(x)$ and $g(x)$ the following holds

$$\int_{-\infty}^{\infty} f(x)\,g^*(x)\,dx = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} F(\beta)\,e^{i2\pi x\beta}\,d\beta\right]\left(\int_{-\infty}^{\infty} G(\beta')\,e^{i2\pi x\beta'}\,d\beta'\right)^*}_{\text{REPLACE FUNCTIONS WITH THEIR FOURIER TRANSFORMS}} dx$$

Reorder the integration

$$= \iint_{-\infty}^{\infty} F(\beta)\,G^*(\beta')\left[\underbrace{\int_{-\infty}^{\infty} e^{i2\pi(\beta-\beta')x}\,dx}_{\text{delta function}}\right] d\beta\,d\beta'$$

$$= \iint_{-\infty}^{\infty} F(\beta)\,G^*(\beta')\,\delta(\beta-\beta')\,d\beta\,d\beta'$$

$$= \int_{-\infty}^{\infty} F(\beta)\,G^*(\beta)\,d\beta \quad \text{from sifting}$$

A special case of this result occurs when $g(x) = f(x)$

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(\beta)|^2\,d\beta} \quad \text{Rayleigh's Theorem}$$

Basically $|f(x)|^2$ describes the energy within $f(x)$ and this is a statement of conservation of energy since $F(\beta)$ contains all the information about $f(x)$

Let's go one step further and assume $f(x)$ is a periodic function. We know that the Fourier transform of a periodic function is given by

$$F(\xi) = \sum_{N=-\infty}^{\infty} c_N \, \delta(\xi - N\xi_0)$$ see pages 36 - 41

Rayleigh's Theorem reduces to

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \sum_{N=-\infty}^{\infty} |c_N|^2}$$

Parseval's Theorem for Fourier Series

The heights (or more exactly their squared magnitudes) of the delta functions tell how much energy is contributed by that frequency ξ_0.

<u>Wiener - Khinchine Theorem</u>

$$\mathcal{F}\{ f(x) \star f^*(x) \} = |F(\xi)|^2$$

The Fourier transform of an autocorrelation is just the magnitude squared (power spectrum) of the Fourier transform of $f(x)$

similar ~~definition of~~ derivation to that of a transform of a convolution from page 63

The power spectrum (also called power spectral density) is a measure of the energy per unit time in a temporal signal and is related to something like contrast for a given spatial frequency in 2D signals. It can be useful for recognizing signals due to its independence on the phase of $f(x)$, and for filtering noisy images.