FOURIER TRANSFORMS 1-D

Previously, we defined the Fourier transform and illustrated a few examples (see pages 35-45). Let's explore some more properties of Fourier transforms

$$F(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi\xi x}\, dx \qquad \text{1-D Fourier transform of } f(x)$$

This function can be inverted so that

$$f(x) = \int_{-\infty}^{\infty} F(\xi)\, e^{i2\pi\xi x}\, d\xi \qquad \text{1-D Inverse Fourier transform of } F(\xi)$$

$f(x)$ and $F(\xi)$ are said to be "Fourier transform pairs".

There are several short hand notations that are common for the preceding relationship's.

Fourier Operator

$$F(\xi) = \mathcal{F}\{f(x)\} \qquad f(x) = \mathcal{F}^{-1}\{F(\xi)\}$$

$$F(\xi) \xleftarrow{\mathcal{F}} f(x) \qquad f(x) \xrightarrow{\mathcal{F}} F(\xi)$$

$$F(\xi) \xrightarrow{\mathcal{F}^{-1}} f(x) \qquad f(x) \xleftarrow{\mathcal{F}^{-1}} F(\xi)$$

Inversion Formula

Let's prove $\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} = f(x)$

Basically, Fourier transforming a function and then inverse transforming it gets you back to the original function

First $\quad \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(\alpha)\, e^{-i2\pi\xi\alpha}\, d\alpha$

Use dummy variable α here to avoid confusion with the x in the inverse Fourier transform. The definite integral gets rid of the dummy variable and simply leaves a function of ξ.

So,

$$\underbrace{\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\}}_{\text{function of } x} = \underbrace{\int_{-\infty}^{\infty} \Big[\underbrace{\int_{-\infty}^{\infty} f(\alpha)\, e^{-i2\pi\xi\alpha}\, d\alpha}_{\text{function of } \xi} \Big] e^{i2\pi\xi x}\, d\xi}_{\text{Definite integral over } \xi \text{ leaves function of } x.}$$

Rearrange order of integration

$$\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} = \int_{-\infty}^{\infty} f(\alpha) \left[\int_{-\infty}^{\infty} e^{-i2\pi\xi(\alpha - x)}\, d\xi \right] d\alpha$$

Recall $\displaystyle\int_{-\infty}^{\infty} e^{-i2\pi\xi(\alpha - x)}\, d\xi = \delta(\alpha - x)$ $\quad$ Compare to formula on page 36 of notes

$$\boxed{\mathcal{F}^{-1}\{\mathcal{F}\{f(x)\}\} = \int_{-\infty}^{\infty} f(\alpha)\, \delta(\alpha - x)\, d\alpha = f(x)}$$

This is sifting property of delta functions from page 18

Many other properties of Fourier transform are derived in similar fashion. Often need the dummy variables to help keep the integration clear.

Can go in other direction too $\quad \mathcal{F}\{\mathcal{F}^{-1}\{F(\xi)\}\} = F(\xi)$

OTHER PROPERTIES OF FOURIER TRANSFORMS

Linearity

If $F(\xi) = \mathcal{F}\{f(x)\}$ and $H(\xi) = \mathcal{F}\{h(x)\}$ and A_1 and A_2 are arbitrary constants then

$$\mathcal{F}\{A_1 f(x) + A_2 h(x)\} = A_1 \mathcal{F}\{f(x)\} + A_2 \mathcal{F}\{h(x)\} = A_1 F(\xi) + A_2 H(\xi)$$

Central Ordinate

$$F(0) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi(0)x}\,dx$$

So central value of $F(\xi)$ is just equal to the area under $f(x)$.

This works in reverse too.

$$f(0) = \int_{-\infty}^{\infty} F(\xi)\,d\xi$$

So central value of $f(x)$ is just equal to area under Fourier transform $F(\xi)$.

Scaling Property

$$\mathcal{F}\{f(\tfrac{x}{b})\} = \int_{-\infty}^{\infty} f\left(\frac{\alpha}{b}\right) e^{-i2\pi\xi\alpha}\,d\alpha$$

$F(\xi) = \mathcal{F}\{f(x)\}$

Substitute $\beta = \dfrac{\alpha}{b}$

$d\alpha = |b|\,d\beta$

$$= \int_{-\infty}^{\infty} f(\beta)\, e^{-i2\pi\xi b\beta}\,|b|\,d\beta$$

$$= |b| \int_{-\infty}^{\infty} f(\beta)\, e^{-i2\pi(b\xi)\beta}\,d\beta$$

$$= |b|\, F(b\xi)$$

SHIFTING PROPERTY

$$\mathcal{F}\{f(x-x_0)\} = \int_{-\infty}^{\infty} f(\alpha - x_0)\, e^{-i2\pi\xi\alpha}\, d\alpha$$

x_0 real constant

$F(\xi) = \mathcal{F}\{f(x)\}$

substitute $\beta = \alpha - x_0$

$d\beta = d\alpha$

$$= \int_{-\infty}^{\infty} f(\beta)\, e^{-i2\pi\xi(\beta + x_0)}\, d\beta$$

$$= e^{-i2\pi\xi x_0} \int_{-\infty}^{\infty} f(\beta)\, e^{-i2\pi\xi\beta}\, d\beta$$

$$= e^{-i2\pi\xi x_0}\, F(\xi)$$

Transform OF A CONJUGATE

$$\mathcal{F}\{f^*(x)\} = \int_{-\infty}^{\infty} f^*(\alpha)\, e^{-i2\pi\xi\alpha}\, d\alpha$$

$F(\xi) = \mathcal{F}\{f(x)\}$

$$= \left[\int_{-\infty}^{\infty} f(\alpha)\, e^{-i2\pi(-\xi)\alpha}\, d\alpha\right]^*$$

$$= F^*(-\xi)$$

Transform of a TRANSFORM

Suppose we know $F(\xi) = \mathcal{F}\{f(x)\}$ and we want to find $G(\xi) = \mathcal{F}\{F(x)\}$

Remember ξ and x are just variables and $F(\,)$ defines the function

$$G(\xi) = \int_{-\infty}^{\infty} F(\alpha)\, e^{-i2\pi\xi\alpha}\, d\alpha$$

$$= \int_{-\infty}^{\infty} F(\alpha)\, e^{i2\pi(-\xi)\alpha}\, d\alpha$$

This is just inverse transform

$$= f(-\xi)$$

TRANSFORM OF A CONVOLUTION

Suppose $g(x) = f(x) * h(x)$ and $F(\xi) = \mathcal{F}\{f(x)\}$, $H(\xi) = \mathcal{F}\{h(x)\}$

Find $G(\xi) = \mathcal{F}\{g(x)\}$

$$G\{\xi\} = \mathcal{F}\Big\{ \underbrace{\int_{-\infty}^{\infty} f(\beta) h(x-\beta)\, d\beta}_{\text{FUNCTION OF } x} \Big\}$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\beta) h(x-\beta)\, d\beta \right] e^{-i2\pi\xi x}\, dx$$

$$= \int_{-\infty}^{\infty} f(\beta) \left[\int_{-\infty}^{\infty} h(x-\beta) e^{-i2\pi\xi x}\, dx \right] d\beta$$ Reorder integration

$$= \int_{-\infty}^{\infty} f(\beta)\, H(\xi)\, e^{-i2\pi\xi\beta}\, d\beta$$ FROM SHIFTING PROPERTY

$$= H(\xi) \int_{-\infty}^{\infty} f(\beta)\, e^{-i2\pi\xi\beta}\, d\beta$$

$$\boxed{G\{\xi\} = H(\xi)\, F(\xi)}$$

THE FOURIER TRANSFORM OF A CONVOLUTION IS THE PRODUCT OF THE FOURIER TRANSFORM OF THE INDIVIDUAL FUNCTIONS

TRANSFORM OF A PRODUCT

$$\boxed{\mathcal{F}\{ f(x) h(x) \} \Rightarrow F(\xi) * H(\xi)}$$

This can be proved by doing a similar derivation as above.

Often convolution integral is difficult to solve, but product of Fourier transforms is easy.

Properties of Fourier Transforms

A_1 and A_2 arbitrary constants
b and d real nonzero constants
x_0 and ξ_0 real constants
k a positive integer

$g(x)=\int_{-\infty}^{\infty} G(\beta)e^{j2\pi\beta x}d\beta$	$G(\xi)=\int_{-\infty}^{\infty} g(\alpha)e^{-j2\pi\alpha\xi}d\alpha$
$f(\pm x)$	$F(\pm\xi)$
$f^*(\pm x)$	$F^*(\mp\xi)$
$F(\pm x)$	$f(\mp\xi)$
$F^*(\pm x)$	$f^*(\pm\xi)$
$f\left(\frac{x}{b}\right)$	$\lvert b\rvert F(b\xi)$
$\lvert d\rvert f(dx)$	$F\left(\frac{\xi}{d}\right)$
$f(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}F(\xi)$
$e^{\pm j2\pi\xi_0 x}f(x)$	$F(\xi\mp\xi_0)$
$f\left(\frac{x\pm x_0}{b}\right)$	$\lvert b\rvert e^{\pm j2\pi x_0\xi}F(b\xi)$
$\lvert d\rvert e^{\pm j2\pi\xi_0 x}f(dx)$	$F\left(\frac{\xi\mp\xi_0}{d}\right)$
$h(x)$	$H(\xi)$
$A_1f(x)+A_2h(x)$	$A_1F(\xi)+A_2H(\xi)$
$f(x)\ast h(x)$	$F(\xi)H(\xi)$
$f(x)h(x)$	$F(\xi)\ast H(\xi)$
$f(x)\star h(x)$	$F(\xi)H(-\xi)$
$f(x)h(-x)$	$F(\xi)\star H(\xi)$
$\gamma_{fh}(x)=f(x)\star h^*(x)$	$F(\xi)H^*(\xi)$
$f(x)h^*(x)$	$\gamma_{FH}(\xi)=F(\xi)\star H^*(\xi)$
$\gamma_f(x)=f(x)\star f^*(x)$	$\lvert F(\xi)\rvert^2$
$\lvert f(x)\rvert^2$	$\gamma_F(\xi)=F(\xi)\star F^*(\xi)$

Elementary Fourier Transform Pairs

x_0 and ξ_0 real constants
a and c real constants
k a nonnegative integer
x and ξ real variables

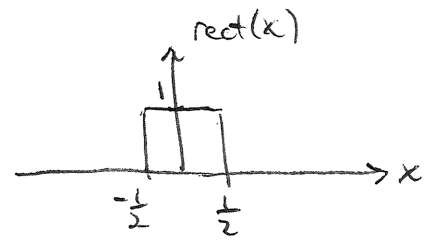
$g(x)=\int_{-\infty}^{\infty} G(\beta)e^{j2\pi x\beta}d\beta$	$G(\xi)=\int_{-\infty}^{\infty} g(\alpha)e^{-j2\pi\xi\alpha}d\alpha$
1	$\delta(\xi)$
$\delta(x)$	1
$\delta(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}$
$e^{\pm j2\pi\xi_0 x}$	$\delta(\xi\mp\xi_0)$
$\cos(2\pi\xi_0 x)$	$\frac{1}{2\lvert\xi_0\rvert}\delta\delta\left(\frac{\xi}{\xi_0}\right)$
$\frac{1}{2\lvert x_0\rvert}\delta\delta\left(\frac{x}{x_0}\right)$	$\cos(2\pi x_0\xi)$
$\sin(2\pi\xi_0 x)$	$\frac{j}{2\lvert\xi_0\rvert}\delta\delta\left(\frac{\xi}{\xi_0}\right)$
$\frac{j}{2\lvert x_0\rvert}\delta\delta\left(\frac{x}{x_0}\right)$	$-\sin(2\pi x_0\xi)$
$\mathrm{rect}(x)$	$\mathrm{sinc}(\xi)$
$\mathrm{sinc}(x)$	$\mathrm{rect}(\xi)$
$\mathrm{tri}(x)$	$\mathrm{sinc}^2(\xi)$
$\mathrm{sinc}^2(x)$	$\mathrm{tri}(\xi)$
$\mathrm{comb}(x)$	$\mathrm{comb}(\xi)$
$\mathrm{Gaus}(x)$	$\mathrm{Gaus}(\xi)$

Let's use ~~one~~ our properties to prove some of these.

Fourier Transform of rect(x)

$$G(\xi) = \mathcal{F}\{\mathrm{rect}(x)\}$$

$$= \int_{-\infty}^{\infty} \mathrm{rect}(x)\, e^{-i2\pi\xi x}\, dx$$



$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-i2\pi\xi x}\, dx = \left.\frac{e^{-i2\pi\xi x}}{-i2\pi\xi}\right|_{x=-\frac{1}{2}}^{x=\frac{1}{2}} = \frac{e^{-i\pi\xi} - e^{i\pi\xi}}{-i2\pi\xi} = \frac{\sin(\pi\xi)}{\pi\xi}$$

$$G(\xi) = \mathrm{sinc}(\xi)$$

FOURIER TRANSFORM OF sinc(x)

$$G(\xi) = \mathcal{F}\{\mathrm{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\pi x} e^{-i2\pi\xi x}\, dx$$

This integral is particularly nasty and solving it involves a contour integral and the residue theorem. However, we know the transform of a transform property.

Given $F(\xi) = \mathcal{F}\{f(x)\}$

then $\mathcal{F}\{F(x)\} = f(-\xi)$

Given $\mathrm{sinc}(\xi) = \mathcal{F}\{\mathrm{rect}(x)\}$

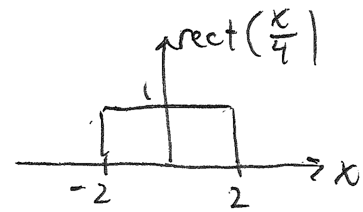
then $\mathcal{F}\{\mathrm{sinc}(x)\} = \mathrm{rect}(-\xi)$

$\mathcal{F}\{\mathrm{sinc}(x)\} = \mathrm{rect}(\xi)$ since rect( ) is even function

FOURIER TRANSFORM OF $\mathrm{rect}\left(\frac{x}{4}\right)$

$G(\xi) = \mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{4}\right)\right\}$ we know $\mathrm{sinc}(\xi) = \mathcal{F}\{\mathrm{rect}(x)\}$

The scaling property says

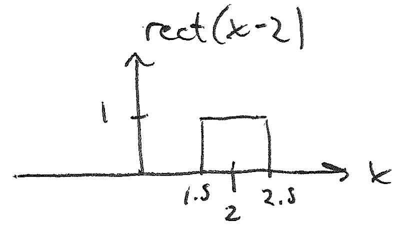


$$\mathcal{F}\left\{f\left(\frac{x}{b}\right)\right\} = |b|\, F(b\xi)$$

So $\mathcal{F}\left\{\mathrm{rect}\left(\frac{x}{4}\right)\right\} = 4\,\mathrm{sinc}(4\xi)$ ← Compare this to page 42 with $X = 8$

FOURIER TRANSFORM OF $\mathrm{rect}(x-2)$

$G(\xi) = \mathcal{F}\{\mathrm{rect}(x-2)\}$



The shifting property says

$$\mathcal{F}\{f(x-x_0)\} = e^{-i2\pi\xi x_0} F(\xi)$$

So

$$\mathcal{F}\{\mathrm{rect}(x-2)\} = e^{-i4\pi\xi}\, \mathrm{sinc}(\xi)$$

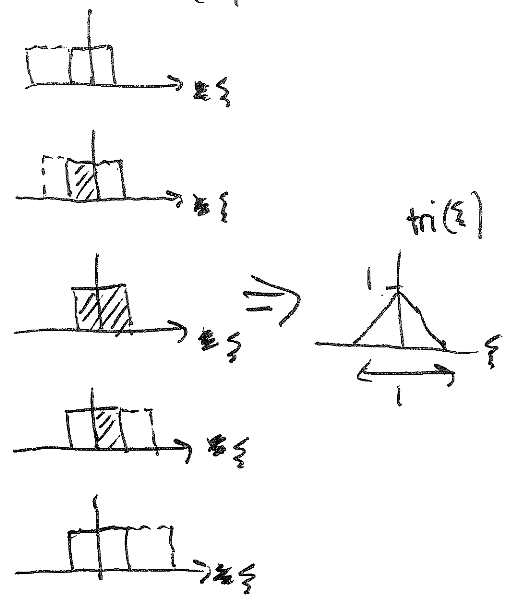
FOURIER TRANSFORM OF $\text{sinc}^2(x)$

$$G(\xi) = \mathcal{F}\{\text{sinc}^2(x)\}$$

This is another particularly nasty problem to do by direct integration.

Let's use transform of a product instead

rect(ξ) * rect(ξ)



$$\mathcal{F}\{f(x)h(x)\} = F(\xi) * H(\xi)$$

$$\mathcal{F}\{\text{sinc}(x)\cdot\text{sinc}(x)\} = \text{rect}(\xi) * \text{rect}(\xi)$$

equal

$$\mathcal{F}\{\text{sinc}^2(x)\} = \text{tri}(\xi)$$

Convince yourself of the last step where

$$\text{tri}(\xi) = \text{rect}(\xi) * \text{rect}(\xi)$$

FOURIER TRANSFORM OF $\delta(x)$

$$G(\xi) = \mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x) e^{-i2\pi\xi x}\, dx$$

Sifting property of delta functions say

$$\int_{-\infty}^{\infty} f(\alpha)\,\delta(\alpha - x_0)\, d\alpha = f(x_0)$$

Replace dummy variable α with x and let $x_0 = 0$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx = f(0)$$

Comparing to above $f(x) = e^{-i2\pi\xi x}$

So

$$\mathcal{F}\{\delta(x)\} = e^{-i2\pi\xi(0)} = 1$$

FOURIER TRANSFORM OF $e^{i2\pi\xi_0 x}$

$$G(\xi) = \mathcal{F}\{e^{i2\pi\xi_0 x}\} = \int_{-\infty}^{\infty} e^{i2\pi\xi_0 x} e^{-i2\pi\xi x}\,dx = \int_{-\infty}^{\infty} e^{-i2\pi(\xi-\xi_0)x}\,dx$$

$$\mathcal{F}\{e^{i2\pi\xi_0 x}\} = \delta(\xi-\xi_0)$$

see page 36

FOURIER TRANSFORM OF $\cos(2\pi\xi_0 x)$

$$G(\xi) = \mathcal{F}\{\cos(2\pi\xi_0 x)\} = \mathcal{F}\{\tfrac{1}{2}(e^{i2\pi\xi_0 x} + e^{-i2\pi\xi_0 x})\}$$

use linearity here

$$\mathcal{F}\{\cos(2\pi\xi_0 x)\} = \frac{1}{2}\mathcal{F}\{e^{i2\pi\xi_0 x}\} + \frac{1}{2}\mathcal{F}\{e^{i2\pi(-\xi_0)x}\}$$

From the previous result

$$\mathcal{F}\{\cos(2\pi\xi_0 x)\} = \frac{1}{2}\left[\delta(\xi-\xi_0) + \delta(\xi+\xi_0)\right]$$

FOURIER TRANSFORM OF $\text{comb}(x)$

Recall $\text{comb}(x) = \sum_{N=-\infty}^{\infty} \delta(x-N)$ where N is an integer (Page 22)

$$G(\xi) = \mathcal{F}\{\text{comb}(x)\} = \sum_{N=-\infty}^{\infty} \mathcal{F}\{\delta(x-N)\}$$ linearity lets us pull $\mathcal{F}\{\}$ inside of sum

We know $\mathcal{F}\{\delta(x)\} = 1$ from previous page

we know $\mathcal{F}\{f(x-x_0)\} = e^{-i2\pi\xi x_0} F(\xi)$ from Shifting property

So $\mathcal{F}\{\delta(x-N)\} = e^{-i2\pi\xi N}\cdot 1$

$$\mathcal{F}\{\text{comb}(x)\} = \sum_{N=-\infty}^{\infty} e^{-i2\pi\xi N}$$

Now what?

$$\sum_{m=-\infty}^{\infty} e^{i2\pi\xi m}$$ substitute $m=-N$

Let's try a different strategy with the comb function. Recall that comb is a periodic function so we can write it as a Fourier series.

$$f(x) = \sum_m a_m e^{i2\pi m \xi_0 x}$$

$$a_m = \frac{1}{X}\int_{-\frac{X}{2}}^{\frac{X}{2}} f(x)\exp\left[-i2\pi m \xi_0 x\right]dx$$

From page 34

$$\mathrm{comb}(x) = \sum_{m=-\infty}^{\infty} a_m e^{i2\pi m \xi_0 x}$$

$$a_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{comb}(x)\exp\left(-i2\pi m \xi_0 x\right)dx$$

$$a_m = \int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(x)\exp\left(-i2\pi m \xi_0 x\right)dx$$

$$a_m = 1 = \exp\left(-i2\pi m \xi_0 \cdot 0\right) \quad \text{sifting}$$

$$\mathrm{comb}(x) = \sum_{m=-\infty}^{\infty} e^{i2\pi m x}$$

Remember $\xi_0 = \frac{1}{X}$ but $X = 1$

$$\mathcal{F}\{\mathrm{comb}(x)\} = \sum_{m=-\infty}^{\infty} \mathcal{F}\{e^{i2\pi m x}\}$$

From top of previous page

$$\mathcal{F}\{\mathrm{comb}(x)\} = \sum_{m=-\infty}^{\infty} \delta(\xi - m)$$

$$\mathcal{F}\{\mathrm{comb}(x)\} = \mathrm{comb}(\xi)$$

That's better!

FOURIER TRANSFORM AND LSI SYSTEMS

Suppose $f(x)$ is the input to an LSI system with impulse response $h(x)$. We know the output $g(x)$ is given by

$$g(x) = f(x) * h(x)$$ see page 56

If we know $F(\xi) = \mathcal{F}\{f(x)\}$ and $H(\xi) = \mathcal{F}\{h(x)\}$ then

$$G(\xi) = \mathcal{F}\{g(x)\} = F(\xi)\,H(\xi)$$

from Fourier transform of convolution property

The output is then given by

$$g(x) = \mathcal{F}^{-1}\{F(\xi)\,H(\xi)\}$$

For LSI systems, there are two routes to get the output

$$\begin{tikzcd}
\text{input} & \text{convolution} & \text{impulse response} & & \text{output} \\
f(x) \arrow[rrrr, bend left] \arrow[d, "\mathcal{F}\{\}"'] & * & h(x) \arrow[d, "\mathcal{F}\{\}"] & = & g(x) \\
F(\xi) & \cdot & H(\xi) & = & G(\xi) \arrow[u, "\mathcal{F}^{-1}\{\}"'] \\
\text{input spectrum} & \text{product} & \text{transfer function} & & \text{output spectrum}
\end{tikzcd}$$

Fourier transform route is often computationally faster!