Mathematical Operators and Linear Shift-Invariant Systems

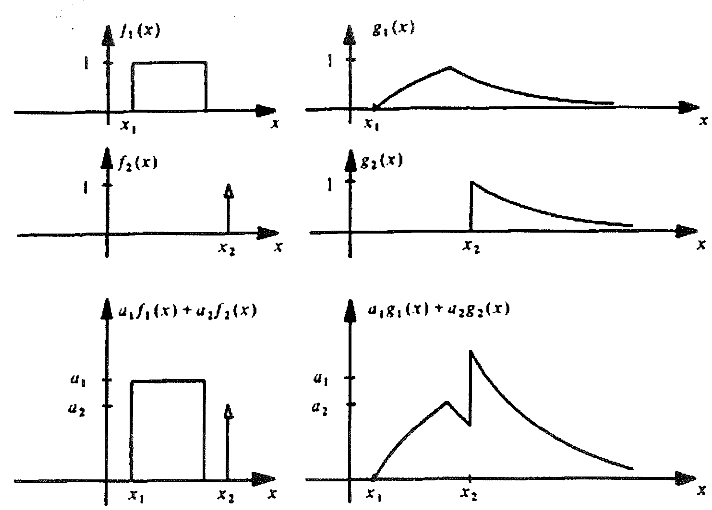
In our initial discussion of systems, we envisioned a system with an input $f_i(x)$ and an output $g_i(x)$. The system transforms the input to the output. We can capture this with operator notation.

$$f_i(x) \xrightarrow{\text{input}} \boxed{\begin{matrix}\text{System}\\ S\{\,\}\end{matrix}} \xrightarrow{\text{output}} g_i(x)$$

For now, we'll just use $S\{\,\}$ as the operator notation

$$g_i(x) = S\{f_i(x)\} \qquad i = 1, 2 \ldots$$

Basically $S\{\,\}$ is just some set of rules for converting the input to the output. In general, $S\{\,\}$ can be very complicated and we can do much besides the conceptual picture above. However, if $S\{\,\}$ has some nice properties, we can make some additional statements about the system.



Linearity - Nice property #1

If

$$S\{f_1(x)\} = g_1(x)$$

$$S\{f_2(x)\} = g_2(x)$$

then the system is linear if

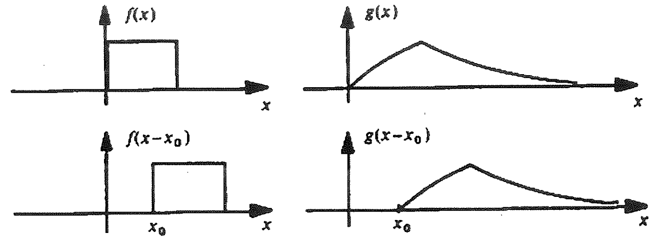
$$S\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 S\{f_1(x)\} + a_2 S\{f_2(x)\}$$

$$= a_1 g_1(x) + a_2 g_2(x)$$

where a_1 and a_2 are arbitrary complex constants.

Linearity basically says that if I know the outputs for a given set of inputs, then I know the net output for any combination of the inputs.

<u>Shift-Invariance</u> - Nice property #2



If $\quad S\{f(x)\} = g(x)$

then the system is shift invariant if

$$S\{f(x-x_0)\} = g(x-x_0)$$

Basically says the system doesn't care about where (or when) $f()$ is located, it just cares about the shape of $f()$

We'll use a special operator notation $\mathcal{L}\{\ \}$ for systems that are <u>both</u> linear and shift invariant. (LSI)

$$\mathcal{L}\{a_1 f_1(x-x_1) + a_2 f_2(x-x_2)\} = a_1 g_1(x-x_1) + a_2 g(x-x_2)$$

<u>Impulse Response</u> - The impulse response is the output of the system when the input is a delta function.

For a general system

$$S\{f(x)\} = g(x)$$

when $f(x) = \delta(x-x_0)$, the impulse response will be denoted as $h(x;x_0)$

$$S\{\delta(x-x_0)\} = h(x;x_0)$$

in general the impulse response depends on the location of x_0

For a linear shift invariant (LSI)

$$\mathcal{L}\{\delta(x)\} = h(x;0) \quad \text{delta function at origin}$$

$$\mathcal{L}\{\delta(x-x_0)\} = h(x;x_0) \quad \text{delta function at } x_0$$

But shift invariance says

$$\mathcal{L}\{\delta(x-x_0)\} = h(x-x_0;0)$$

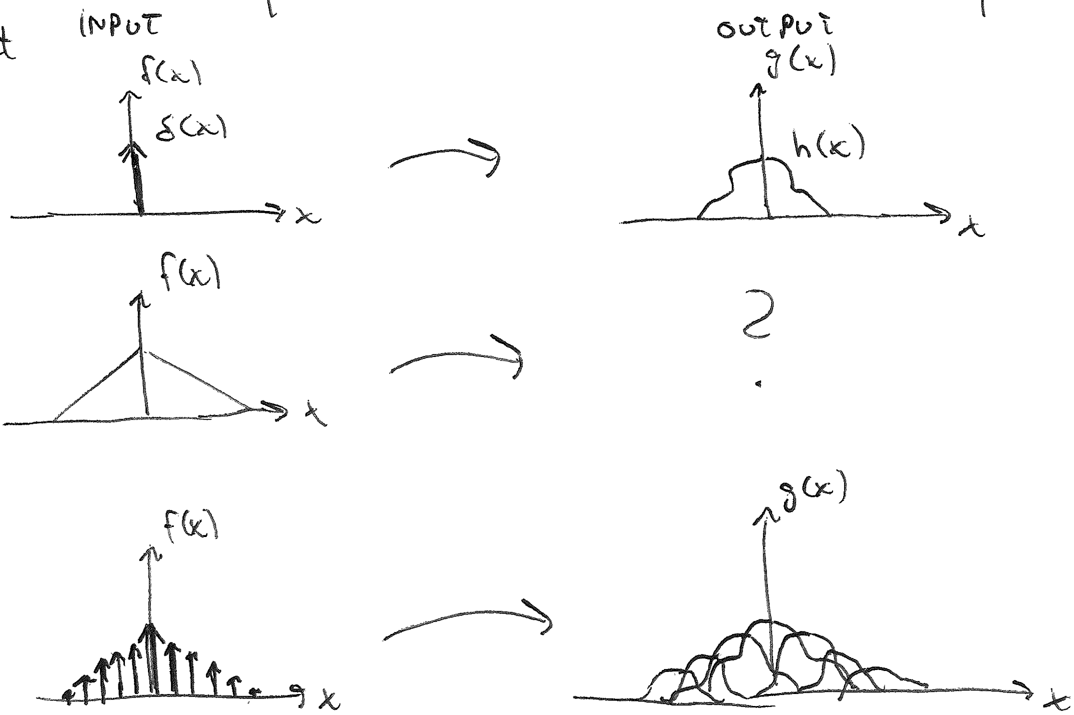
So the impulse response

$$h(x;x_0) = h(x-x_0;0) = h(x;0)$$

(i.e. the impulse response is the same no matter where the δ-function is located) The extra 0 is also superfluous for LSI systems

$$\mathcal{L}\{\delta(x-x_0)\} = h(x-x_0)$$

This is nice because if we know $h(x)$, the we can represent any input as a bunch of delta functions and use linearity to get the output



EIGENFUNCTIONS OF LSI SYSTEMS

Eigenfunctions of a systems are specific functions that when input into the system give an output that is the same function scale by a complex constant. These complex constants are called <u>eigenvalues</u>.

complex constant

same function as input

$$\mathcal{L}\{\psi(x;\xi)\} = H(\xi)\,\psi(x;\xi)$$

we'll allow $\psi(x;\xi)$ to depend on ξ as well

Let's represent $H(\xi) = A(\xi)e^{-i\Phi(\xi)}$ since it is complex.

$$\mathcal{L}\{\psi(x;\xi)\} = A(\xi)e^{-i\Phi(\xi)}\,\psi(x;\xi)$$

would like to find $\psi(x;\xi)$ for LSI systems.

Try complex exponentials (think sines + cosines)

$$\psi(x;\xi) = \exp(i2\pi\xi x)$$

$$\mathcal{L}\{\exp(i2\pi\xi x)\} = g(x;\xi)$$

$$\mathcal{L}\{\exp(i2\pi\xi(x-x_0))\} = \exp(-i2\pi\xi x_0)\,\mathcal{L}\{\exp(i2\pi\xi x)\}$$

$$= \exp(-i2\pi\xi x_0)\,g(x;\xi) \qquad \text{By linearity}$$

But from shift invariance

$$\mathcal{L}\{\exp(i2\pi\xi(x-x_0))\} = g(x-x_0;\xi)$$

Equating the two

$$g(x-x_0;\xi) = \exp(-i2\pi\xi x_0)\,g(x;\xi)$$

This equality is satisfied when

$$g(x;\xi) = H(\xi)\exp(i2\pi\xi x)$$ where $H(\xi)$ is arbitrary complex constant

So putting it altogether

$H(\xi)$ is called the <u>Transfer Function</u>

$$\mathcal{L}\{\exp(i2\pi\xi x)\} = \underbrace{H(\xi)}_{\text{eigenvalue}}\,\underbrace{\exp(i2\pi\xi x)}_{\text{eigenfunction}}$$

Basically, if we represent the system input as a bunch of sines and cosines, then the output is a bunch of sines and cosines, but they may be scaled by $A(\xi)$ and shifted by $\Phi(\xi)$.

Spoiler Alert: The impulse response $h(x)$ and the Transfer Function $H(\xi)$ are related by the Fourier transform

Convolution

Convolution is an important operation for analyzing LSI systems. If the impulse response of the system is known, then convolving it with the system input gives the system output. Conceptually, this is just replacing each point in the input with its weighted impulse response. Mathematically, convolution is described as

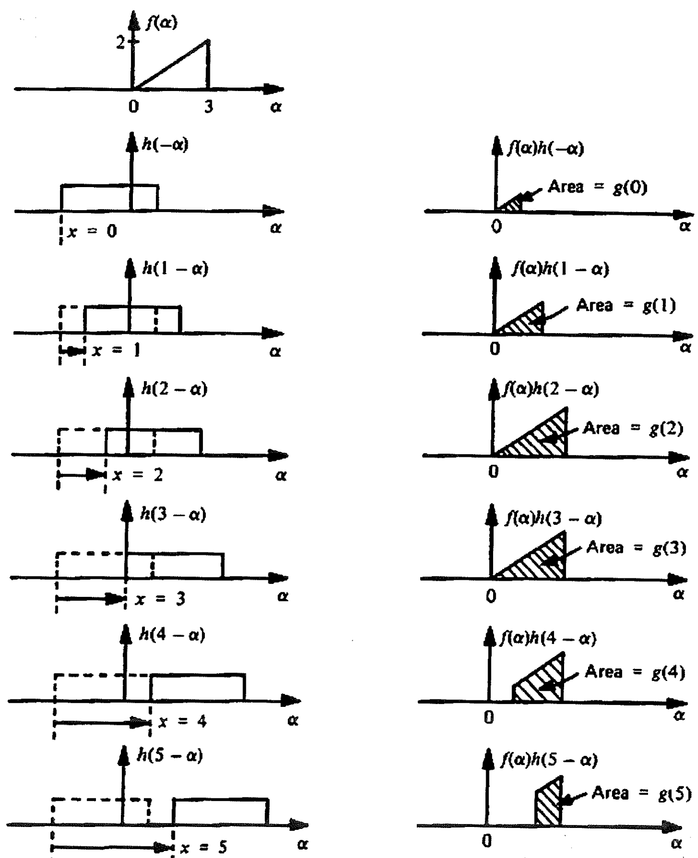
$$\underbrace{f(x) * h(x)}_{\text{shorthand notation}} = \int_{-\infty}^{\infty} \underbrace{f(\alpha)\, h(x-\alpha)\, d\alpha}_{\text{formal definition}}$$

Since α is the integration variable, it should be obvious that

$$f(x) * h(x) = g(x)$$

The convolution of two functions of x gives a function of x.

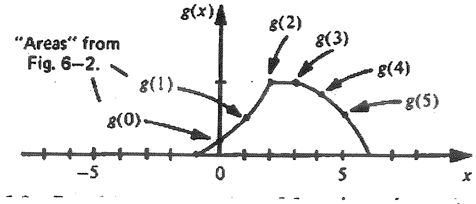
Changing the value of x has the effect of shifting $h(x-\alpha)$ relative to $f(\alpha)$



The convolution integral records the area of overlap of $f(\alpha)h(x-\alpha)$ for each value of x.

$g(x)$ is just a description of all these areas as a function of x.

Plotting $g(x)$ gives the convolution.



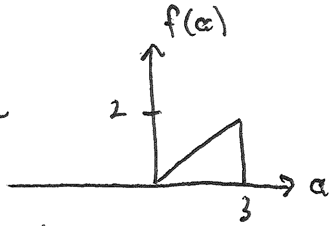
To do this integral mathematically, the convolution needs to be broken up into regions,

The functions are

$f(\alpha) = \frac{2}{3}\alpha$

when $0 \le \alpha \le 3$

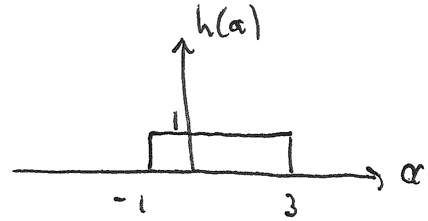
$f(\alpha) = 0$ otherwise



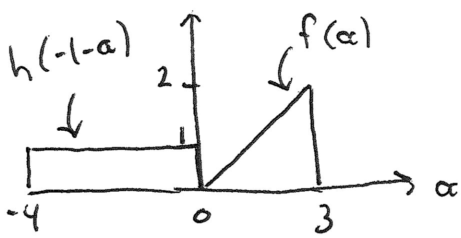
$h(\alpha) = 1$

when $-1 \le \alpha \le 3$

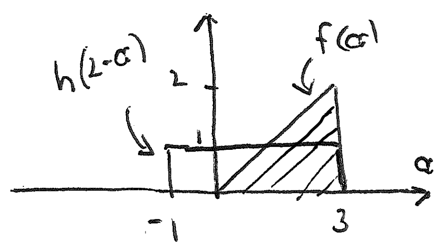
$h(\alpha) = 0$ otherwise



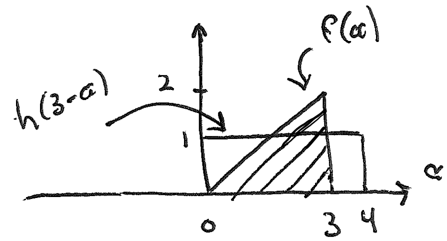
we're interested in $h(x-\alpha) = 1$ when $-1 \le x - \alpha \le 3$ or $\boxed{x-3 \le \alpha \le x+1}$



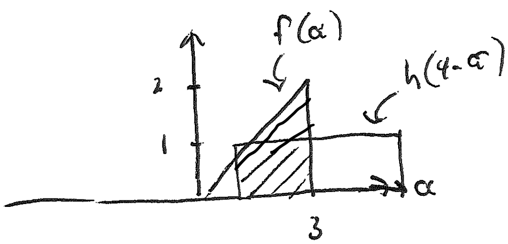
$x = -1$



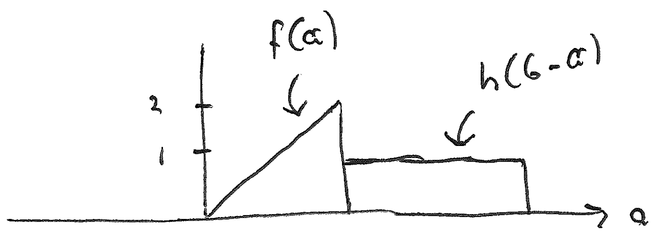
$x = 2$



$x = 3$



$x = 4$



$x = 6$

Let's look at the integral in these different regions

FOR $x \le -1$ $\quad f(\alpha)h(x-\alpha) = 0$, so $g(x) = 0$

FOR $-1 < x \le 2$

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)\,d\alpha$$

$$= \int_{0}^{3} \frac{2\alpha}{3} h(x-\alpha)\,d\alpha \qquad \text{region where } f(\alpha) \text{ is NON-zero}$$

$$= \frac{2}{3}\int_{0}^{x+1} \alpha\,d\alpha \qquad \text{region where both } f(\alpha) \text{ and } h(x-\alpha) \text{ NON-zero}$$

$$= \frac{(x+1)^2}{3}$$

FOR $2 < x \le 3$

$g(x)$ doesn't change and keeps the same value as $g(2)$

$$g(x) = \frac{(2+1)^2}{3} = 3$$

FOR $3 < x \le 6$

$$g(x) = \frac{2}{3}\int_{x-3}^{3} \alpha\,d\alpha$$

$$g(x) = 3 - \frac{(x-3)^2}{3}$$

FOR $x > 6$ NO overlap so $g(x) = 0$

Evaluating the convolution integral is challenging for simple functions and becomes nearly impossible for more complex functions. We'll show how to get the convolution more simply using Fourier Transforms.

PROPERTIES OF CONVOLUTION

Commutative Property

$$f(x) * h(x) = \int_{-\infty}^{\infty} f(\alpha) h(x-\alpha)\, d\alpha$$

change variables $\beta = x - \alpha \qquad d\beta = -d\alpha$

$$f(x) * h(x) = -\int_{\infty}^{-\infty} f(x-\beta)\, h(\beta)\, d\beta$$

$$f(x) * h(x) = \int_{-\infty}^{\infty} h(\beta)\, f(x-\beta)\, d\beta$$

$$\boxed{f(x) * h(x) = h(x) * f(x)}$$

Distributive Property

$$\big(a\, v(x) + b\, w(x)\big) * h(x) = \int_{-\infty}^{\infty} \big(a\, v(\alpha) + b\, w(\alpha)\big)\, h(x-\alpha)\, d\alpha$$

a, b constants

$$= a \int_{-\infty}^{\infty} v(\alpha) h(x-\alpha)\, d\alpha + b \int_{-\infty}^{\infty} w(\alpha) h(x-\alpha)\, d\alpha$$

$$= a\big(v(x) * h(x)\big) + b\big(w(x) * h(x)\big)$$

Shift Invariance

If $\quad f(x) * h(x) = \int_{-\infty}^{\infty} f(\alpha) h(x-\alpha)\, d\alpha = g(x)$

then

$$f(x-x_0) * h(x) = \int_{-\infty}^{\infty} f(\alpha - x_0)\, h(x-\alpha)\, d\alpha$$

substitute $\beta = \alpha - x_0$

$d\beta = d\alpha$

$$= \int_{-\infty}^{\infty} f(\beta)\, h(x - x_0 - \beta)\, d\beta$$

$$= g(x - x_0)$$

Associative Property

$$[v(x) * w(x)] * h(x) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} v(\beta)\, w(\alpha - \beta)\, d\beta \right] h(x - \alpha)\, d\alpha$$

change the order of integration

$$= \int_{-\infty}^{\infty} v(\beta) \underbrace{\left[\int_{-\infty}^{\infty} w(\alpha - \beta)\, h(x - \alpha)\, d\alpha \right]}_{w(x-\beta) * h(x) \,:=\, u(x-\beta)} d\beta$$

$$= \int_{-\infty}^{\infty} v(\beta)\, u(x - \beta)\, d\beta$$

$$= v(x) * u(x)$$

$$= v(x) * [w(x) * h(x)]$$ order of convolution doesn't matter

Convolution with delta functions

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(\alpha)\, \delta(x - \alpha)\, d\alpha$$

$$f(x) * \delta(x) = \int_{-\infty}^{\infty} f(\alpha)\, \delta(\alpha - x)\, d\alpha$$ since δ is even function

$$\boxed{f(x) * \delta(x) = f(x)}$$ from sifting property of delta functions

Scaling Property

$$f\left(\frac{x}{b}\right) * h\left(\frac{x}{b}\right) = |b|\, g\left(\frac{x}{b}\right)$$ where $f(x) * h(x) = g(x)$

Area Under Convolution = product of areas of individual functions

$$\int_{-\infty}^{\infty} g(\beta)\, d\beta = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(\alpha)\, h(\beta - \alpha)\, d\alpha \right] d\beta = \int_{-\infty}^{\infty} f(\alpha) \left[\int_{-\infty}^{\infty} h(\beta - \alpha)\, d\beta \right] d\alpha$$

$$= \left[\int_{-\infty}^{\infty} f(\alpha)\, d\alpha \right] \left[\int_{-\infty}^{\infty} h(\beta)\, d\beta \right]$$

CONVOLUTION AND LSI SYSTEMS

For a general input function $f(x)$, the output $g(x)$ is

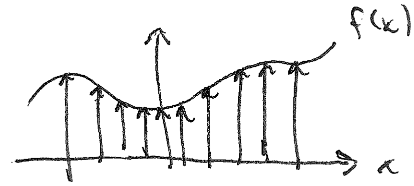
$$g(x) = \mathcal{L}\{f(x)\}$$

When the input is a delta function, the result is the Impulse Response $h(x)$

$$h(x) = \mathcal{L}\{\delta(x)\}$$

Suppose now the function $f(x)$ is represented as a bunch of weighted delta functions

$$f(x) = \int_{-\infty}^{\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha$$



Then

$$g(x) = \mathcal{L}\left\{ \int_{-\infty}^{\infty} f(\alpha)\,\delta(x-\alpha)\,d\alpha \right\}$$

Since the system is linear

$$g(x) = \int_{-\infty}^{\infty} \mathcal{L}\{\delta(x-\alpha)\}\, f(\alpha)\,d\alpha$$

Remember $\mathcal{L}\{\}$ is acting on functions of x. $f(\alpha)\,d\alpha$ is just a constant as far as $\mathcal{L}\{\}$ is concerned.

But we know $\mathcal{L}\{\delta(x)\} = h(x)$ and that $\mathcal{L}\{\}$ is shift-invariant.

So

$$g(x) = \int_{-\infty}^{\infty} f(\alpha)\,h(x-\alpha)\,d\alpha$$

$$\boxed{g(x) = f(x) * h(x)}$$

The output of a LSI system is simply the convolution of the input with the impulse response.

CROSS CORRELATION AND AUTOCORRELATION

Cross correlation is a relative to convolution. It is defined for two complex functions $f(x)$ and $g(x)$ as

$$f(x) \star g(x) = \int_{-\infty}^{\infty} f(\alpha)\, g(\alpha - x)\, d\alpha$$

star instead of asterisk

order is reversed compared to convolution

Unlike convolution, in general

$$f(x) \star g(x) \neq g(x) \star f(x)$$

Cross correlation does not commute

we can write

$$\int_{-\infty}^{\infty} f(\alpha)\, g(\alpha - x)\, d\alpha = \int_{-\infty}^{\infty} f(\alpha)\, g\left(\frac{x-\alpha}{-1}\right) d\alpha = f(x) * g\left(\frac{x}{-1}\right)$$

so $\quad f(x) \star g(x) = f(x) * g(-x)$

If $f(x) = g(x)$, we call this the <u>auto correlation</u> of $f(x)$.

These definitions can be extended the <u>complex cross correlation</u> and <u>complex autocorrelation</u>. In this case

$$\gamma_{fg}(x) = f(x) \star g^*(x) = \int_{-\infty}^{\infty} f(\alpha)\, g^*(\alpha - x)\, d\alpha$$

shorthand notation

Much like before $\quad \gamma_{fg}(x) = f(x) * g^*(-x)$

Other properties

$$\gamma_{gf}(x) = \gamma_{fg}^*(-x)$$

$$\gamma_{fg}(x) = \gamma_{gf}^*(-x)$$

For complex autocorrelation $f(x) = g(x)$, so

$$\gamma_f(x) = f(x) \star f^*(x) = \int_{-\infty}^{\infty} f(\alpha) f^*(\alpha - x)\, d\alpha$$

Second f usually dropped since redundant

other properties

$\gamma_f(x) = \gamma_f^*(-x)$ This is a property called <u>Hermitian</u>

$|\gamma_f(x)| \leq \gamma_f(0)$ The magnitude for any non-zero value of x never exceeds the value at $x=0$.

<u>A general note on convolution and correlation.</u>

Convolution can be thought of as a smoothing operation. In general, the convolution of two functions results in a smoother and broader function than the originals.

Cross Correlation can be thought of as a matching operation. Cross correlation highlights regions in the two functions where their shapes are similar.