A system is any type of process in which an output signal is generated in response to an input signal. For those familiar with electrical engineering, systems are usually handling 1D signals that vary in time.

For example, an audio signal from a person speaking is an example of a time-varying signal $f(t)$. A system might be a microphone that takes in this signal and electronically relay it to a speaker. The output of the speaker is now the time-varying amplified version of the input. In real systems, $g(t)$ is not a perfect scalar reproduction of the input $f(t)$, but instead some of the information content in $f(t)$ has been lost or modified by the system. In systems theory, we are often trying to understand how the system modifies various inputs. If we can assume or approximate the system as being linear, we can in general break a complex input signal into a series of simpler signals. The simpler signals are then passed through the system and are modified accordingly. The simple output signals are then reassembled to create the complex output $g(t)$. Thus, the response of the system to the simple signals gives us an idea of what can be passed through the system.
In optics, we often deal with systems where the input and output signals are 2D representing spatial coordinates instead of temporal coordinates as in the audio example.

\[ f(x,y) \quad \rightarrow \quad \text{system} \quad \rightarrow \quad g(x,y) \]

For example, \( f(x,y) \) might be the incidence distribution of your face and the system is the camera in your cell phone. The output signal \( g(x,y) \) in this case is the "selfie" image captured. Again, the output is not a perfect reproduction of the input, and we would like to understand the performance of the system by splitting the input into a series of simpler signals, passing these simple signals through the system to see how they are modified, and then create the complex output \( g(x,y) \) by assembling the simple output signals.

Now, Fourier transforms provide an ideal framework for analyzing linear systems. Fourier transforms are a mathematical technique that breaks complex signals into a series of sinusoidal patterns. Sinusoidal patterns passing through a linear system remain sinusoidal, but the system may modify their amplitude and phase. These shifted and squashed sinusoids are then reassembled to give the output.
This course is half math class and half applications. The math fundamentals are important for success. Fourier theory shows up everywhere in optics and having the tools developed in this class will greatly ease the material covered in future classes. I strongly recommend Gaskells “Linear Systems, Fourier Transforms and Optics.” We will be covering Ch. 2-7 and Ch. 9 in this book to develop the necessary foundations for the class. I also highly recommend Goodman’s “ Fourier Optics” 4th edition which will use for the applications portion of the class.

**Review of Functions and Their Properties**

Functions can be thought of as a “rule” which takes some input value and creates an output value. In this, we can have

\[ y: f(x) \]

*\( x \) is the input value or independent variable

*\( y \) is the output value or dependent variable

*\( f(x) \) is the “rule” that converts \( x \) into \( y \).

**Examples**  \( x \) real and non-negative (i.e. \( x \geq 0 \))

- \( y = x^2 \)
- \( y = \pm \sqrt{x} \)

**Single-Valued Function**

One \( y \) for one \( x \)

**Multi-Valued Function**

More than one \( y \) for one \( x \)
Note: For the previous examples if \( x \) is real and \(-\infty < x < \infty\) then \( y = x^2 \) only has real values of \( y \), so this is a real-valued function.

Then \( y = \pm \sqrt{x} \) has imaginary values when \( x < 0 \), so this is a complex-valued function. More on these later.

### Periodic Functions

Periodic functions satisfy the following relationship:

\[
f(x) = f(x + nx) \quad \text{where } n \text{ is an integer}
\]

\( X \) is real positive constant.

\( X \) is known as the period. The shape of the function repeats itself every integer multiple of \( X \). If \( x \) is a spatial coordinate, then the reciprocal of the period is called the fundamental spatial frequency \( f_0 \):

\[
f_0 = \frac{1}{X}
\]

so if \( x \) is units of \( \text{m} \)

then \( f_0 \) is in units of \( \text{cycles/m} \)

One cycle is the fundamental shape of \( f(x) \) that gets repeated. The period \( X \) tells the length of that shape. The fundamental spatial frequency \( f_0 \) tells how many times the shape repeats in a given distance.

\[
f(x) = A \sin(2\pi f_0 x - \Theta)
\]
Sinusoidal patterns are common examples of periodic functions.

\[ f(x) = A \sin \left( 2\pi \frac{x}{T} + \Theta \right) \]

Amplitude spatial Phase shift frequency

Zeros occur when \( 2\pi \frac{x}{T} + \Theta = n\pi \) \( n \) integer

\[ x = \frac{\Theta + n\pi}{2\pi \frac{A}{T}} = \left( \frac{\Theta + n\pi}{2\pi} \right) \]

For periodic we need \( f(x) = f(x + nX) \)

\[ A \sin \left( 2\pi \frac{x}{T} + \Theta \right) = A \sin \left( 2\pi \frac{x + nX}{T} + \Theta \right) \]

\[ \sin \left( 2\pi \frac{x}{T} + \Theta \right) = \sin \left( 2\pi \frac{x}{T} + \Theta + 2\pi n \frac{X}{T} \right) \]

\[ \sin \left( 2\pi \frac{x}{T} + \Theta \right) = \sin \left( 2\pi \frac{x}{T} + \Theta \right) \cos \left( 2\pi n \frac{X}{T} \right) + \cos \left( 2\pi \frac{x}{T} + \Theta \right) \sin \left( 2\pi n \frac{X}{T} \right) \]

\[ \sin \left( 2\pi \frac{x}{T} + \Theta \right) = \sin \left( 2\pi \frac{x}{T} + \Theta \right) \text{ TRUE} \]

We can also talk about functions that are periodic in time.

They satisfy \( g(t) = g(t + nT) \) \( n \) integer \( T \) is period

The reciprocal of \( T \) is called the fundamental temporal frequency, \( V_0 \)

\[ V_0 = \frac{1}{T} \]

for \( T \) in seconds

\( V_0 \) is in Hz (s^{-1})

\[ g(t) \]

\[ T = \frac{1}{V_0} \]

Periodic functions do not necessarily need to be sinusoidal
Adding periodic functions will result in a periodic function if the ratios of the periods of the two functions is rational. Figure 2-5 shows two functions with periods \( X_1 \) and \( X_2 \). The sum of these functions for \( \frac{X_2}{X_1} = 2 \) is shown in Fig. 2-6 (periodic).

This is what happens when \( \frac{X_2}{X_1} = \sqrt{3} \) is irrational \( \Rightarrow \) non-periodic.

**Adding Periodic Functions**

**Figure 2-5** Periodic functions of period \( X_1 \) and \( X_2 \).

**Figure 2-6** Sum of \( f_1(x) \) and \( f_2(x) \) of Fig. 2-5 when \( X_2/X_1 = 2 \).

**Figure 2-7** An almost-periodic function.

**SHIFTING AND REVERSING FUNCTIONS**

\[ f(x - \alpha) \quad (\alpha > 0) \] means function shifted in \( +x \) direction.

\[ f(x + \alpha) \quad (\alpha > 0) \] means function shifted in \( -x \) direction.
Even and Odd Functions

Functions with even and odd properties are useful because often calculation of their Fourier transform is simplified by using their symmetry properties.

Even Function

\[ e(x) = e(-x) \]

Symmetric about y-axis

Odd Function

\[ o(x) = -o(-x) \]

Symmetric about origin for 180° rotation

An arbitrary function \( f(x) \) can always be represented as a combination of an even function \( f_e(x) \) and an odd function \( f_o(x) \)

\[ f(x) = f_e(x) + f_o(x) \]

where

\[ f_e(x) = \frac{1}{2} [ f(x) + f(-x) ] \]

\[ f_o(x) = \frac{1}{2} [ f(x) - f(-x) ] \]
Integrals of even and odd functions

\[ \int_{-a}^{a} e(x) \, dx = 2 \int_{0}^{a} e(x) \, dx \]

\[ \int_{-a}^{a} o(x) \, dx = 0 \]

Products of even and odd functions lead to even and odd functions.

\[ \int_{-a}^{a} e_1(x) e_2(x) \, dx = 2 \int_{0}^{a} e_1(x) e_2(x) \, dx \]

\[ \int_{-a}^{a} o_1(x) o_2(x) \, dx = 2 \int_{0}^{a} o_1(x) o_2(x) \, dx \]

\[ \int_{-a}^{a} e(x) o(x) \, dx = 0 \]

Finally, combining various results gives

\[ \int_{-a}^{a} f(x) \, dx = 2 \int_{0}^{a} f_e(x) \, dx \]
Two-Dimensional Functions

\[ z = f(x, y) \]  Two inputs \( x, y \) give one output \( z \) that follows the "rule" \( f() \)

For this cause, \( x, y \) are usually the spatial coordinates on a plane.

- \( x, y \) Cartesian coordinates \( r = \sqrt{x^2 + y^2} \) \( x = r \cos \theta \)
- \( r, \theta \) Polar coordinates \( \theta = \tan^{-1} \left( \frac{y}{x} \right) \) \( y = r \sin \theta \)

Example: Gaussian function

\[ f(x, y) = A \exp \left[ -\pi \left( \frac{x^2 + y^2}{a^2} \right) \right] \]

Conversion to polar coordinates

\[ f(r, \theta) = A \exp \left[ -\pi \frac{r^2}{a^2} \right] \]

This function also illustrates an example of rotational symmetry since \( f(r, \theta) \) doesn't depend on \( \theta \).

This function is also an example of a separable function. In general, 2D separable functions can be split into a product of 1D functions of the variables.

\[ f(x, y) = g(x) h(y) \] For Gaussian \( f(x, y) = A \exp \left[ -\frac{\pi x^2}{a^2} \right] \exp \left[ -\frac{\pi y^2}{a^2} \right] \)

Functions may be separable in one coordinate system, but not in another.
Infinite extent

Consider the function \( f(x,y) = \begin{cases} 1 & |x| \leq b, |y| \leq d \\ 0 & \text{otherwise} \end{cases} \)

No \( y \) dependence, so \( y \) can take on any value be \(-\infty, \infty\), but \( x \) is constrained to \([-b, b]\).

Now consider \( f_2(x,y) = \begin{cases} 1 & |x| \leq b, |y| \leq d \\ 0 & \text{otherwise} \end{cases} \)

\( f_2(x,y) \) is a separable function in \( x \) and \( y \) since we can write this as

\[ f_2(x,y) = g(x)h(y) \]

where

\[ g(x) = \begin{cases} 1 & |x| \leq b \\ 0 & \text{otherwise} \end{cases} \]

\[ h(y) = \begin{cases} 1 & |y| \leq d \\ 0 & \text{otherwise} \end{cases} \]

\( f_2(x,y) \) is not separable in polar coordinates.

\( f_2(x,y) \) would not be separable if the pattern was rotated \( 45^\circ \) around the \( z \)-axis.

We like separable functions because integration often simplifies.

\[ \iint f(x,y) \, dx \, dy = \left( \int g(x) \, dx \right) \left( \int h(y) \, dy \right) \]

2D integral \( \Rightarrow \) product of two 1D integrals.
Complex Numbers Review

A complex number \( u \) can be written as

\[
u = v + i\omega \quad \text{Cartesian Coordinates}
\]

where \( v \) and \( \omega \) are real and \( i = \sqrt{-1} \). Note, many engineering bodies including Gaskill and Goodman use \( j = \sqrt{-1} \). However, \( u \)-optics "\( i \)" is more common, so we'll stick with that notation.

Complex numbers can also be written in polar coordinates with

\[
u = r \exp(i\phi)
\]

where \( r = |u| = \sqrt{v^2 + \omega^2} \) and \( \phi = \arg(u) = \tan^{-1}\left( \frac{\omega}{v} \right) \)

Polar form is often much easier to use for algebraic manipulations.

Operations

Complex conjugate: \( u^* = v - i\omega = \bar{u} = r \exp(-i\phi) \)

Addition

\[
u_1 = v_1 + i\omega_1 \\
u_2 = v_2 + i\omega_2
\]

useful results

\[
u_1 + u_2 = v_1 + v_2 + i(\omega_1 + \omega_2)
\]

add real and imaginary parts separately.

Multiplication

\[
u_1u_2 = (v_1 + i\omega_1)(v_2 + i\omega_2)
\]

useful results

\[
u_1u_2 = |u_1|^2 = v_1^2 + \omega_1^2 = r_1^2
\]

\[
u_1u_2^* = |u_1|^2 = v_1^2 + \omega_2^2 = r_1^2
\]

\[
u_1u_2^* = v_1v_2 - i(\omega_1\omega_2) + i(v_1\omega_2 + v_2\omega_1) = r_1r_2 \exp[i(\phi_1 + \phi_2)]
\]
Division - Division is much more complicated in Cartesian than in polar.

\[ \frac{u_1}{u_2} = \frac{u_1 u_2^*}{u_2 u_1^*} = \frac{u_1 u_2^*}{|u_1|^2} = \frac{(V_1 V_2 + \omega_1 \omega_2) + i (\omega_1 V_1 - V_1 \omega_2)}{V_2^2 + \omega_2^2} \]

In polar,

\[ \frac{u_1}{u_2} = \frac{r_1 \exp[i \phi_1]}{r_2 \exp[i \phi_2]} = \frac{r_1}{r_2} \exp[i (\phi_1 - \phi_2)] \]

Other useful results

\[ |u_1 u_2| = |u_1| \cdot |u_2| \]
\[ |u_1 u_1^*| = |u_1|^2 \]
\[ (u_1 + u_2)^* = u_1^* + u_2^* \]
\[ (u_1 u_2)^* = u_1^* u_2^* \]
\[ \left( \frac{u_1}{u_2} \right)^* = \frac{u_1^*}{u_2^*} \]
\[ \frac{1}{u_1} = \frac{u_1^*}{|u_1|^2} \]

Euler's Formula

\[ \exp[i \pi \xi_0 x] = \cos(2\pi \xi_0 x) + i \sin(2\pi \xi_0 x) \]
\[ \cos(2\pi \xi_0 x) = \frac{1}{2} \left[ \exp(i 2\pi \xi_0 x) + \exp(-i 2\pi \xi_0 x) \right] \]
\[ \sin(2\pi \xi_0 x) = \frac{1}{2i} \left[ \exp(i 2\pi \xi_0 x) - \exp(-i 2\pi \xi_0 x) \right] \]