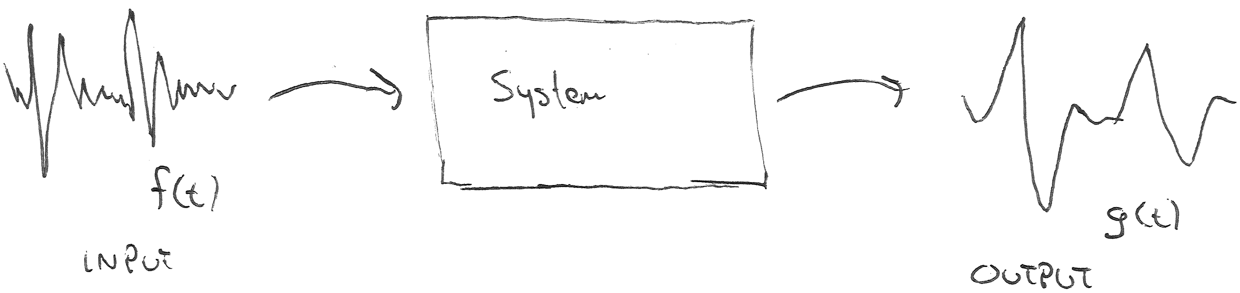
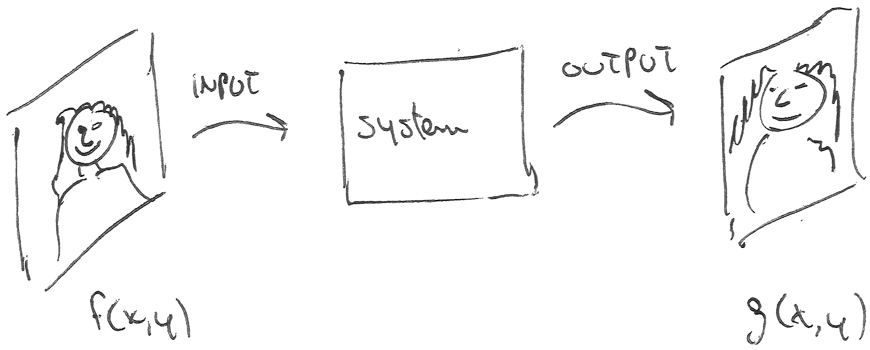
a system is any type of process in which an output signal is generated in response an input signal. For those familiar with electrical engineering, systems are usually handling 1D ~~fo~~ signals that vary in time



For example, an audio signal from a person speaking is an example of a time varying signal f(t). A system might be a microphone that takes in this signal and electronically relay it to a speaker. The output of the speaker g(t) is now the time-vary amplified version of the input. In real systems, g(t) is <u>not</u> a perfect scaled reproduction of the input f(t), but instead some of the information contain in f(t) has been lost or modified by the system. In systems theory, we are often trying to understand how the system modifies various inputs. If we can assume or approximate the system as being <u>linear</u>, we can in general break a complex input signal f(t) into a series of simpler signals. The simple signals are then passed through the system and are modified accordingly. The simple output signals are then reassembled to create the complex output g(t). Thus, the response of the system to the simple signals gives us an idea of what can be passed through the system.

In optics, we often deal with systems where the input and output signals are 2D (or more) representing spatial coordinates instead of temporal coordinates as in the audio example.



For example, $f(x,y)$ might be the irradiance distribution of your face and the system is the camera in your cell phone. The output signal $g(x,y)$ in this case is the "selfie" image captured. Again, the output is not a perfect reproduction of the input, and we would like to understand the performance of the system by splitting the input into a series of simpler signals, passing these simple signals through the system to see how they are modified, and then create the complex output $g(x,y)$ by assembling the simple output signals.

Now, Fourier transforms provide an ideal framework for analyzing linear systems. Fourier transforms are a mathematical technique that breaks complex signals into a series of sinusoidal patterns. Sinusoidal patterns passing through a linear system remain sinusoidal, but the system may modify their amplitude and phase. These shifted and squished sinusoids are then reassembled to give the output.

This course is half math class and half applications. The math fundamentals are important for success. Fourier theory shows up everywhere in optics and having the tools developed in this class will greatly ease the material covered in future classes. I strongly recommend Gaskill's "Linear Systems, Fourier Transforms and Optics". We will be covering Chs. 2-7 and Ch. 9 in this book to develop the necessary math foundations for the class. I also highly recommend Goodman's "Fourier Optics" 4th edition which we'll use for the applications portion of the class.

REVIEW OF FUNCTIONS AND THEIR PROPERTIES

Functions can be thought of as a "rule" which takes some input value(s) and creates an output value. In 1D, we can have

$$y = f(x)$$

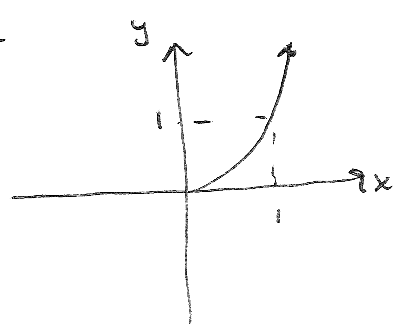
x is the input value or independent variable

y is the output value or dependent variable

$f(x)$ is the "rule" that converts x into y.

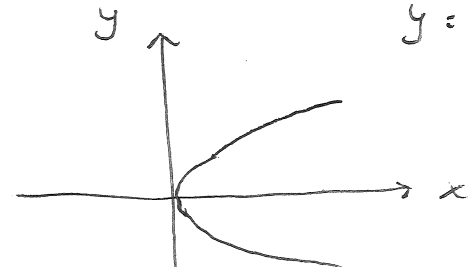
Examples x real and NON-negative (i.e. $x \geq 0$)

$y = x^2$



SINGLE VALUED FUNCTION
ONE y for ONE x

$y = \pm\sqrt{x}$



MULTI-VALUED FUNCTION
MORE THAN ONE y for ONE x

Note: For the previous examples if x is real and $-\infty < x < \infty$

then $y = x^2$ only has real values of y, so this is a real-valued function.

then $y = \pm\sqrt{x}$ has imaginary values when $x < 0$, so this is a complex-valued function. More on these later

Periodic Functions

Periodic functions satisfy the following relationship

$$f(x) = f(x + nX) \quad \text{where } n \text{ is an integer}$$
X is real positive constant

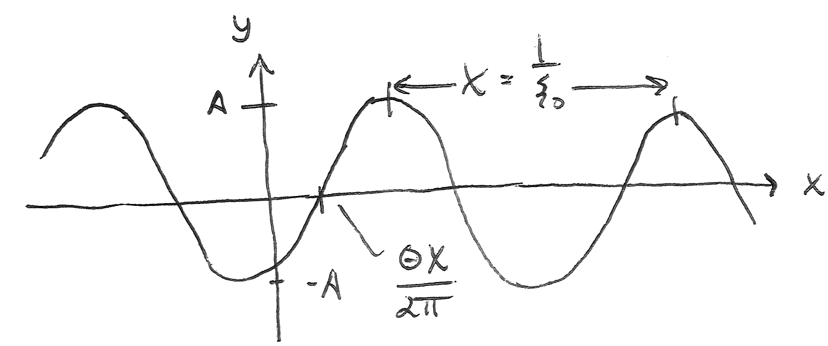
X is known as the period. The shape of the function repeats itself every integer multiple of X. If x is a spatial coordinate, then the reciprocal of the period is called the fundamental spatial frequency ξ_0

$$\xi_0 = \frac{1}{X}$$

so if X in units of mm then ξ_0 is in units of cycles/mm

One cycle is the fundamental shape in $f(x)$ that gets repeated. The period X tells the length of that shape. The fundamental spatial frequency ξ_0 tells how many times the shape repeats in a given distance.

$$f(x) = A \sin(2\pi \xi_0 x - \theta)$$



Sinusoidal patterns are common examples of periodic functions

$$f(x) = A \sin(2\pi \xi_0 x - \Theta)$$

↑ Amplitude ↑ spatial frequency ↑ Phase Shift

Zeroes occur when $2\pi \xi_0 x - \Theta = n\pi$ n integer

or $$x = \frac{\Theta + n\pi}{2\pi \xi_0} = \frac{(\Theta + n\pi) X}{2\pi}$$

For periodic we need $f(x) = f(x + nX)$

$$\cancel{A} \sin(2\pi \xi_0 x - \Theta) = \cancel{A} \sin(2\pi \xi_0 (x + nX) - \Theta)$$

$$\sin(2\pi \xi_0 x - \Theta) = \sin(2\pi \xi_0 x - \Theta + 2n\pi(\overset{1}{\cancel{\xi_0 X}}))$$

$$\sin(2\pi \xi_0 x - \Theta) = \sin(2\pi \xi_0 x - \Theta) \overset{1}{\cancel{\cos(2n\pi)}} + \cos(2\pi \xi_0 x - \Theta) \overset{0}{\cancel{\sin(2n\pi)}}$$

$$\sin(2\pi \xi_0 x - \Theta) = \sin(2\pi \xi_0 x - \Theta) \quad \text{TRUE}$$

We can also talk about functions that are periodic in time.

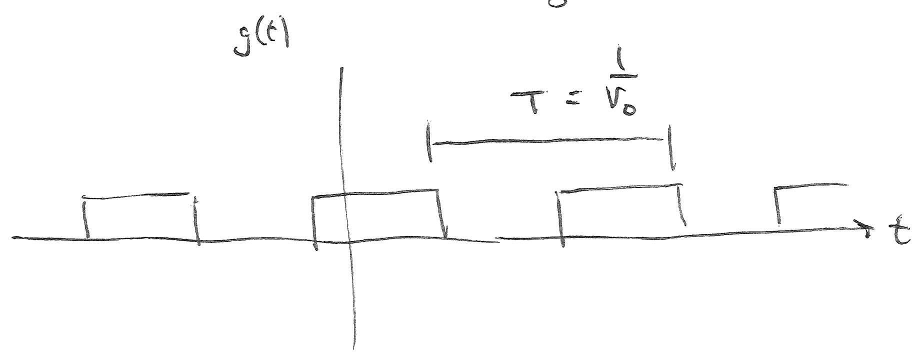
They satisfy $$g(t) = g(t + nT)$$ n integer

T is period

The reciprocal of T is called the fundamental temporal frequency, ν_0

$$\nu_0 = \frac{1}{T}$$ for T in seconds

ν_0 is in Hz $\left(\frac{1}{\text{sec}}\right)$



Periodic functions don't necessarily need to be sinusoidal

Adding Periodic Functions

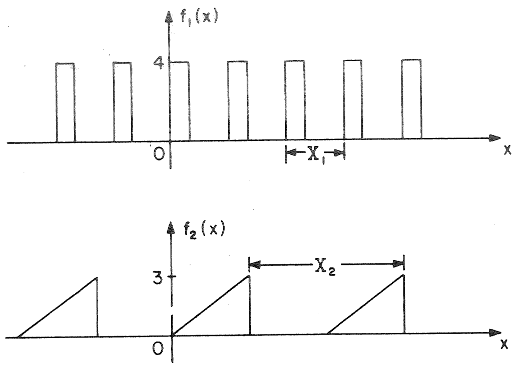


Figure 2-5 Periodic functions of period X_1 and X_2.

Adding two periodic functions will result in a periodic function if the ratio of the periods of the two functions is rational. Figure 2.5 shows two functions with periods X_1 and X_2. The sum of these functions for $\frac{X_2}{X_1} = 2$ is shown in Fig. 2-6 (periodic)

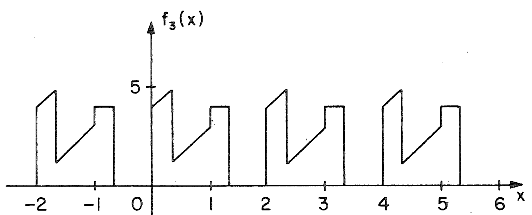


Figure 2-6 Sum of $f_1(x)$ and $f_2(x)$ of Fig. 2-5 when $X_2/X_1=2$.

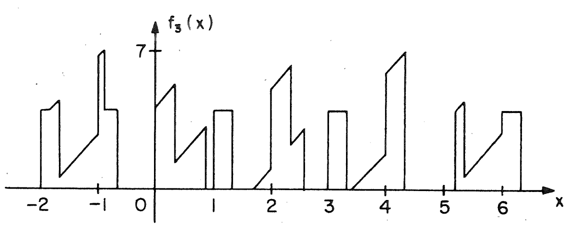
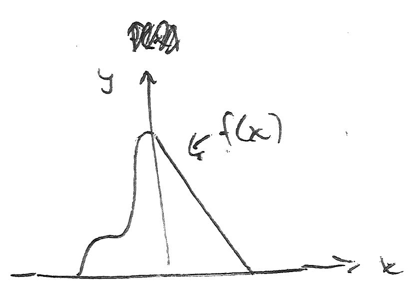


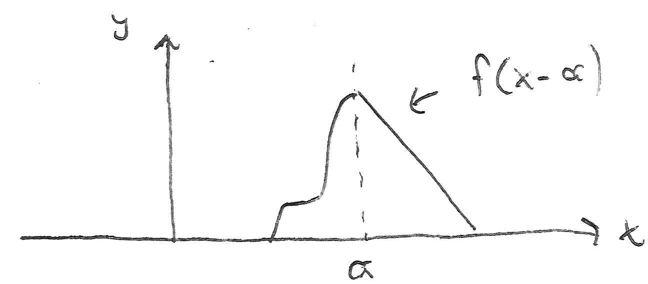
Figure 2-7 An almost-periodic function.

This is what happens when

$\frac{X_2}{X_1} = \sqrt{3}$ irrational $\Rightarrow$ non-periodic

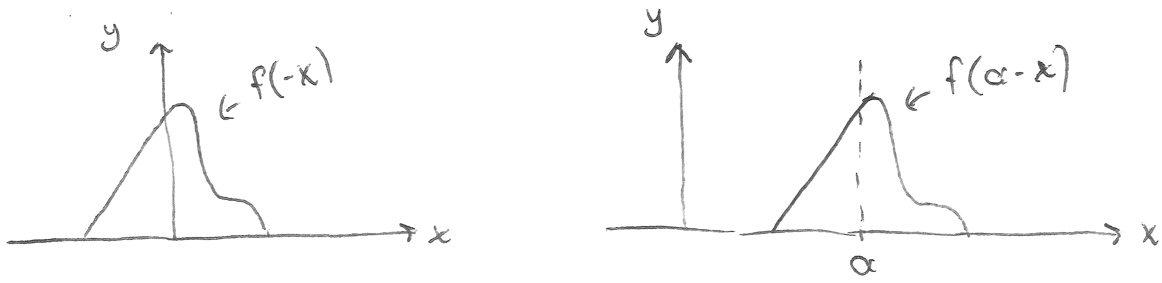
SHIFTING AND REVERSING FUNCTIONS





$f(x-a)$ $(a>0)$ means function shifted in $+x$ direction

$f(x+a)$ $(a>0)$ $\Rightarrow$ $f(x-(-a))$ means function shifted in $-x$ direction



$f(\alpha - x)$ means the function is reflected about the y axis and shifted to be centered on α.

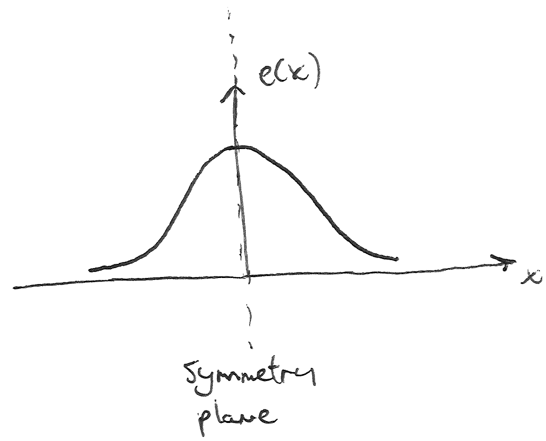
EVEN AND ODD FUNCTIONS

Functions with even and odd ~~symmetry~~ properties are useful because often calculation of their Fourier transform is simplified by using their symmetry properties.

Even Function

$$e(x) = e(-x)$$

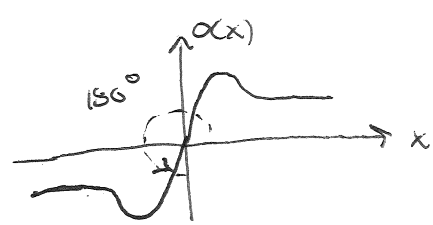
Symmetric about y-axis



Odd Function

$$o(x) = -o(-x)$$

symmetric about origin for 180° rotation



An arbitrary function $f(x)$ can always be represented as a combination of an even function $f_e(x)$ and an odd function $f_o(x)$

$$f(x) = f_e(x) + f_o(x)$$

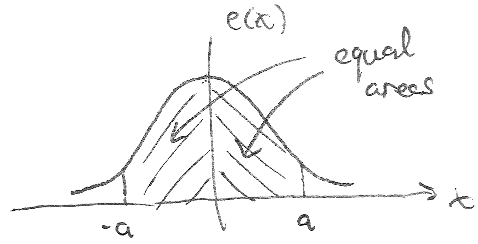
where

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x) \right]$$

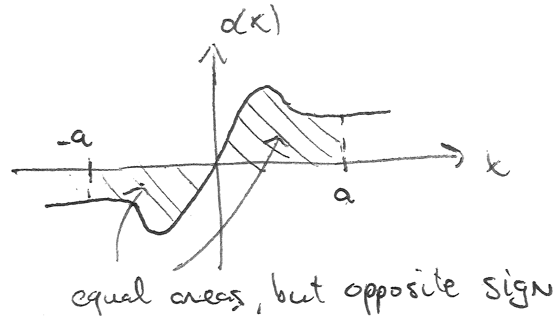
$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x) \right]$$

Integrals of even and odd functions

$$\int_{-a}^{a} e(x)\,dx = 2\int_{0}^{a} e(x)\,dx$$



$$\int_{-a}^{a} o(x)\,dx = 0$$



Products of even and odd functions lead to even and odd functions.

Same pattern as integer addition

$$\int_{-a}^{a} e_1(x)\,e_2(x)\,dx = 2\int_{0}^{a} e_1(x)\,e_2(x)\,dx$$

even x even = even

2 + 4 = 6

$$\int_{-a}^{a} o_1(x)\,o_2(x)\,dx = 2\int_{0}^{a} o_1(x)\,o_2(x)\,dx$$

odd x odd = ~~odd~~ even

3 + 5 = 8

$$\int_{-a}^{a} e(x)\,o(x)\,dx = 0$$

even x odd = odd

4 + 5 = 9

Finally, combining various results gives

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f_e(x)\,dx$$

Two-Dimensional Functions

$z = f(x,y)$ Two inputs x,y give one output z that follows the "rule" $f(\ )$

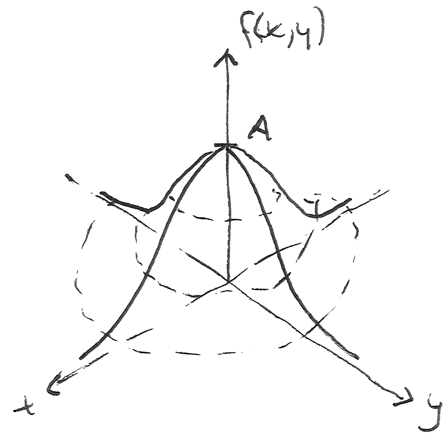
For this course, x,y are usually the spatial coordinates on a plane.

x,y Cartesian coordinates $r = \sqrt{x^2+y^2}$ $x = r\cos\theta$

r,θ Polar coordinates $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $y = r\sin\theta$

Example: Gaussian Function

$$f(x,y) = A\exp\left[-\pi\left(\frac{x^2+y^2}{d^2}\right)\right]$$



Conversion to polar coordinates

$$f(r,\theta) = A\exp\left[-\pi\frac{r^2}{d^2}\right]$$

This function also illustrates an example of rotational symmetry since $f(r,\theta)$ doesn't depend on θ.

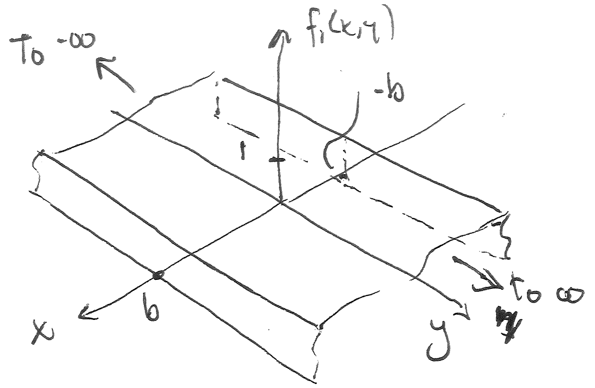
This function is also an example of a separable function. In general, 2D separable functions can be split into a product of 1D functions of the variables.

e.g. $f(x,y) = g(x)h(y)$ For Gaussian $f(x,y) = A\exp\left[\frac{-\pi x^2}{d^2}\right]\exp\left[\frac{-\pi y^2}{d^2}\right]$

Functions may be separable in one coordinate system, but not in another.

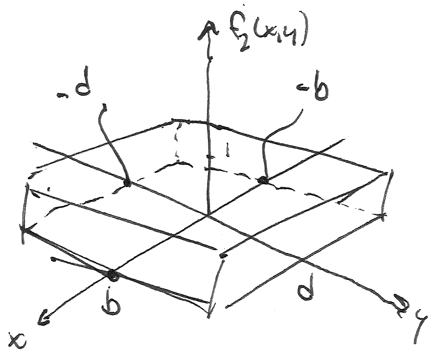
Infinite extent

Consider the function $f_1(x,y) = \begin{cases} 1 & |x| \le b \\ 0 & \text{otherwise} \end{cases}$



No y dependence, so y can take on any value be $-\infty, \infty$, but x is constrained to $\pm b$.

Now consider $f_2(x,y) = \begin{cases} 1 & |x| \le b,\ |y| \le d \\ 0 & \text{otherwise} \end{cases}$



$f_2(x,y)$ is a separable function in x,y since we can write this as

$$f_2(x,y) = g(x)\,h(y) \quad \text{where}$$

$$g(x) = \begin{cases} 1 & |x| \le b \\ 0 & \text{otherwise} \end{cases} \qquad h(y) = \begin{cases} 1 & |y| \le d \\ 0 & \text{otherwise} \end{cases}$$

$f_2(r,\theta)$ is Not separable in polar coordinates.

$f_2(x,y)$ would not be separable if the pattern was rotated 45° around the z-axis.

We like separable functions because integration often simplifies.

$$\iint f(x,y)\,dx\,dy = \left[\int g(x)\,dx\right]\left[\int h(y)\,dy\right]$$

2D integral $\Rightarrow$ product of two 1D integrals.

Complex Numbers Review

A complex number u can be written as

$$u = v + i\omega \qquad \text{Cartesian Coordinates}$$

where v and ω are real and $i = \sqrt{-1}$. Note, many engineering books including Gaskill and Goodman use $j = \sqrt{-1}$. However, in optics "i" is more common, so we'll stick with that notation

Complex numbers can also be written in polar coordinates with

$$u = r \exp[i\phi]$$

where $r = |u| = \sqrt{v^2 + \omega^2}$ and $\phi = \arg(u) = \tan^{-1}\left(\frac{\omega}{u}\right)$

modulus or absolute value — argument or phase

Polar form is often much easier to use for algebraic manipulations.

Operations

Complex conjugate $\quad u^* = v - i\omega = r\exp[-i\phi]$

Addition

$$u_1 = v_1 + i\omega_1$$
$$u_2 = v_2 + i\omega_2$$

useful results

$$u_1 + u_1^* = 2v_1$$
$$u_1 - u_1^* = i2\omega_1$$

$$u_1 + u_2 = v_1 + v_2 + i(\omega_1 + \omega_2)$$

add real and imaginary parts separately.

Multiplication

useful results

$$u_1 u_1^* = |u_1|^2 = v_1^2 + \omega_1^2 = r_1^2$$

$$u_1 u_2 = (v_1 + i\omega_1)(v_2 + i\omega_2)$$
$$= v_1 v_2 + (i)^2 \omega_1 \omega_2 + i v_2 \omega_1 + i v_1 \omega_2$$
$$= (v_1 v_2 - \omega_1 \omega_2) + i(v_1 \omega_2 + v_2 \omega_1) = r_1 r_2 \exp[i(\phi_1 + \phi_2)]$$

Division – Division is much more complicated in Cartesian than in polar.

$$\frac{u_1}{u_2} = \frac{u_1 u_2^*}{u_2 u_2^*} = \frac{u_1 u_2^*}{|u_2|^2} = \frac{(v_1 v_2 + w_1 w_2) + i(v_2 w_1 - v_1 w_2)}{v_2^2 + w_2^2}$$

In polar

$$\frac{u_1}{u_2} = \frac{r_1 \exp[i\phi_1]}{r_2 \exp[i\phi_2]} = \frac{r_1}{r_2} \exp[i(\phi_1 - \phi_2)]$$

Other useful results

$$|u_1 u_2| = |u_1| \cdot |u_2|$$

$$|u_1 u_1^*| = |u_1|^2$$

$$(u_1 + u_2)^* = u_1^* + u_2^*$$

$$(u_1 u_2)^* = u_1^* u_2^*$$

$$\left(\frac{u_1}{u_2}\right)^* = \frac{u_1^*}{u_2^*}$$

$$\frac{1}{u_1} = \frac{u_1^*}{|u_1|^2}$$

Euler's Formula

$$\exp[i2\pi\xi_0 x] = \cos(2\pi\xi_0 x) + i\sin(2\pi\xi_0 x)$$

$$\cos(2\pi\xi_0 x) = \frac{1}{2}\left[\exp(i2\pi\xi_0 x) + \exp(-i2\pi\xi_0 x)\right]$$

$$\sin(2\pi\xi_0 x) = \frac{1}{2i}\left[\exp(i2\pi\xi_0 x) - \exp(-i2\pi\xi_0 x)\right]$$