Special Functions - There are a series of special functions that show up over and over in Fourier optics. Because of their ubiquity, we'll give them names and formal definitions so we can refer to them in short hand notation.

One Dimensional Functions

The arguments for the functions are standardized to $\frac{x-x_0}{b}$.

x is the variable

x_0 is the center of the function

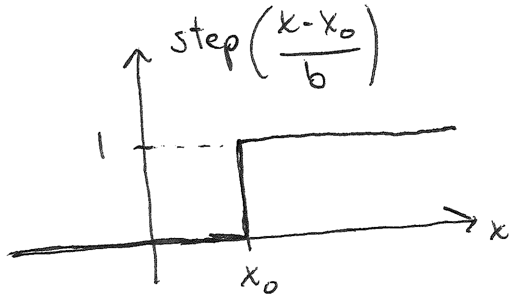
$|b|$ often refers to the area under the function (finite functions)

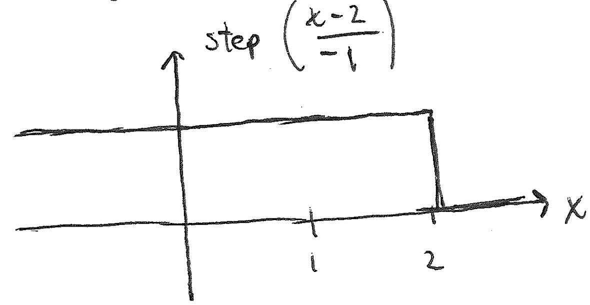
The sign of b reverses the direction of the function. See ⑥ and ⑦.

Step Function

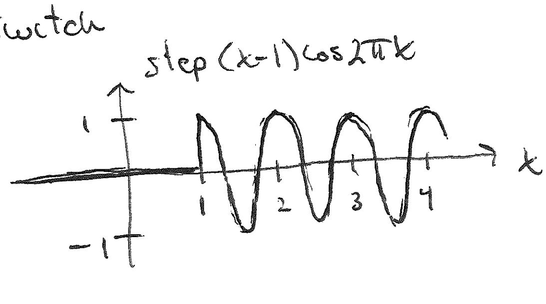
$$\text{step}\left(\frac{x-x_0}{b}\right) = \begin{cases} 0 & \frac{x}{b} < \frac{x_0}{b} \\ \frac{1}{2} & \frac{x}{b} = \frac{x_0}{b} \\ 1 & \frac{x}{b} > \frac{x_0}{b} \end{cases}$$

This function has a discontinuity at $x = x_0$.





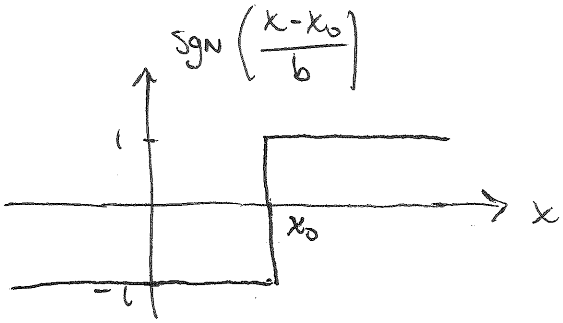
Switch

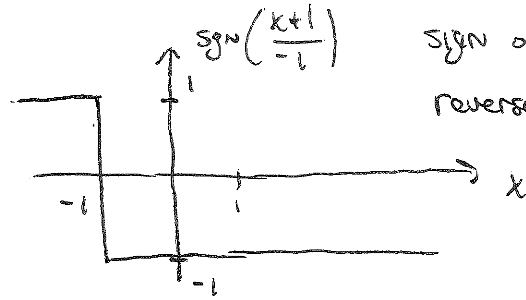


The only purpose for b in the step function is to enable the function to be reflected about x_0 depending on the sign of b.

Sign Function (pronounced "signum")

$$\mathrm{sgn}\left(\frac{x-x_0}{b}\right) = \begin{cases} -1 & \frac{x}{b} < \frac{x_0}{b} \\ 0 & \frac{x}{b} = \frac{x_0}{b} \\ 1 & \frac{x}{b} > \frac{x_0}{b} \end{cases}$$

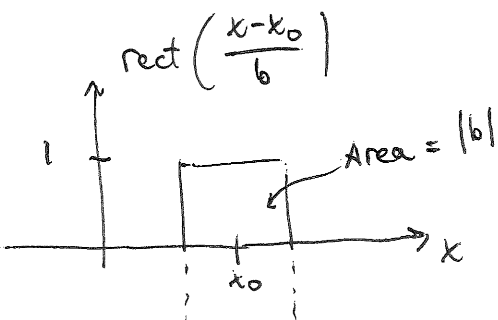


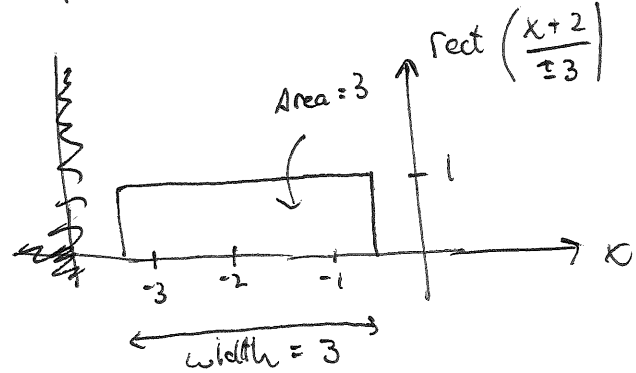


like step in that sign of b just reverses sgn about x_0.

Rectangle Function "rect"

$$\mathrm{rect}\left(\frac{x-x_0}{b}\right) = \begin{cases} 0 & \left|\frac{x-x_0}{b}\right| > \frac{1}{2} \\ \frac{1}{2} & \left|\frac{x-x_0}{b}\right| = \frac{1}{2} \\ 1 & \left|\frac{x-x_0}{b}\right| < \frac{1}{2} \end{cases}$$

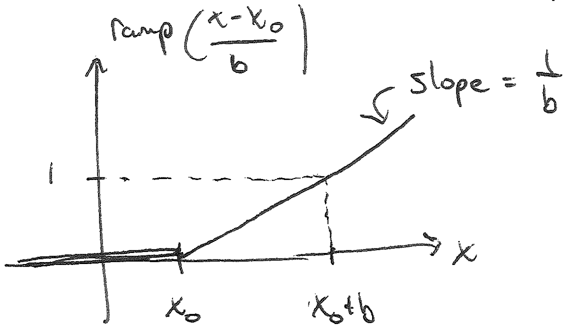


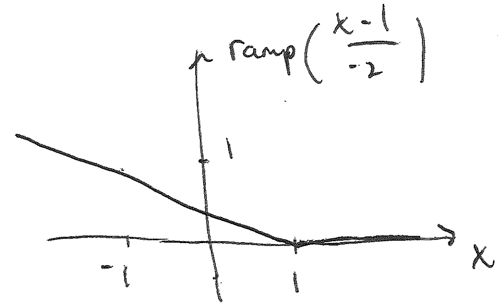


Area under rect is $|b|$. The sign of b reverses the rect about x_0, but this has the same shape.

Ramp Function

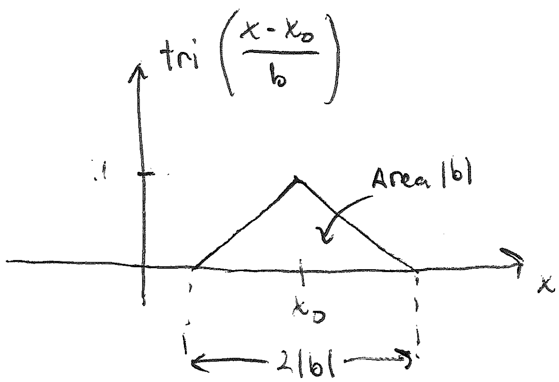
$$\text{ramp}\left(\frac{x-x_0}{b}\right) = \begin{cases} 0 & \frac{x}{b} \le \frac{x_0}{b} \\ \left|\frac{x-x_0}{b}\right| & \frac{x}{b} > \frac{x_0}{b} \end{cases}$$

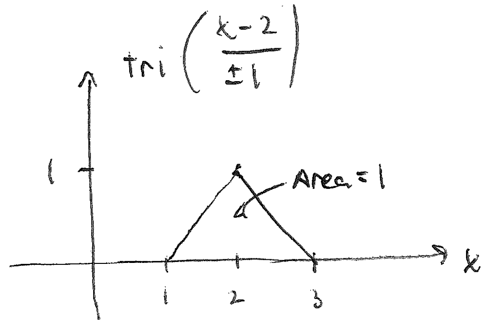




Again, sign of b reverses about x_0.

Triangle Function (tri)





$$\text{tri}\left(\frac{x-x_0}{b}\right) = \begin{cases} 0 & \left|\frac{x-x_0}{b}\right| \ge 1 \\ 1-\left|\frac{x-x_0}{b}\right| & \left|\frac{x-x_0}{b}\right| < 1 \end{cases}$$

Base of triangle is $2|b|$ to give area $\frac{1}{2} \cdot 2|b| \cdot 1 = |b|$.
Sign of b reverses triangle, but has no effect because of symmetry

Combining different definitions

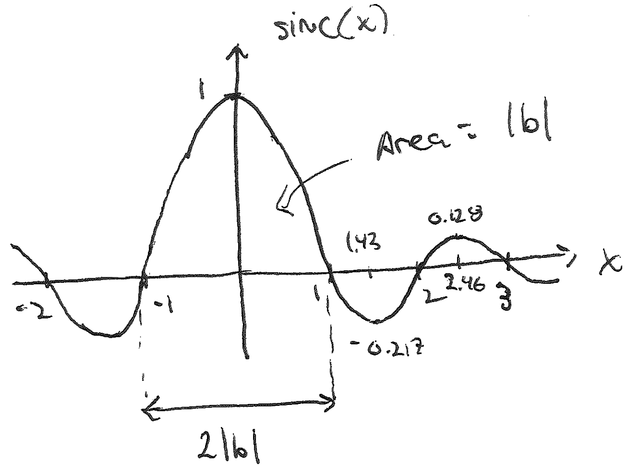
$$\text{sgn}\left(\frac{x-x_0}{b}\right) = 2\,\text{step}\left(\frac{x-x_0}{b}\right) - 1$$

$$\text{tri}\left(\frac{x-1}{2}\right) = \text{ramp}\left(\frac{x+1}{2}\right)\text{step}\left(\frac{x-1}{-1}\right) + \text{ramp}\left(\frac{x-3}{-2}\right)\text{step}\left(\frac{x-1}{1}\right)$$

Sinc function pronounced "sink"

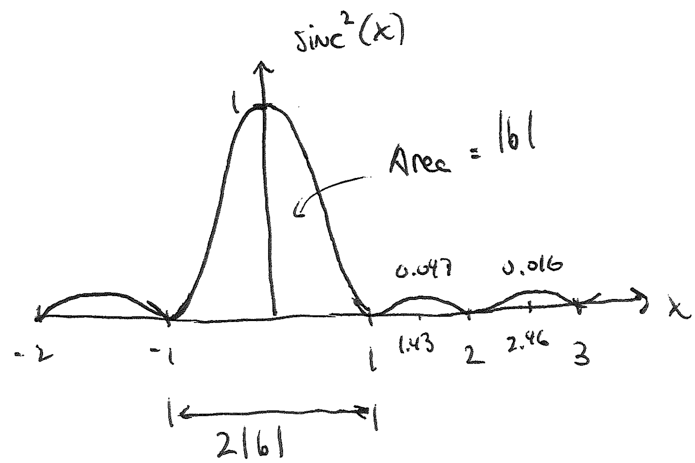
$$\operatorname{sinc}\left(\frac{x-x_0}{b}\right) = \frac{\sin \pi\left(\frac{x-x_0}{b}\right)}{\pi\left(\frac{x-x_0}{b}\right)}$$

Note: not everybody defines the sinc function with the π (e.g. Mathematica). Be careful!



Including the π in the definition makes the zeros occur at $x_0 \pm Nb$ where $N = 0, 1, 2 \ldots$ and the area $= |b|$. Furthermore, at $x = x_0$ the function is 1, much like the other function we defined. Sign of b, reverses function about x_0, but has no effect because of symmetry.

Sinc^2 function - The square of the sinc( ) function



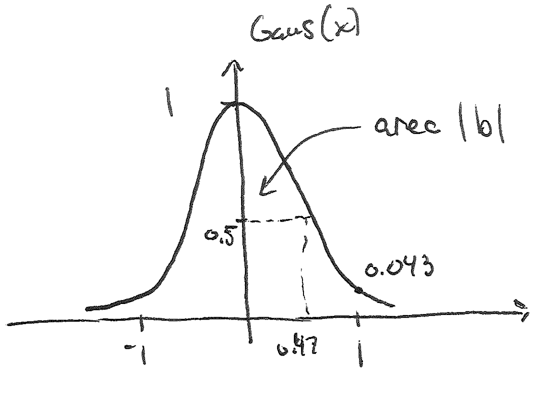
It may seem counterintuitive that sinc( ) and sinc^2( ) both have an area of $|b|$, but realize that sinc( ) has positive and negative lobes, whereas sinc^2( ) is strictly positive.

Sign of b reverses function about x_0, but has no effect because of symmetry.

Gaussian Function

$$\mathrm{Gaus}\left(\frac{x-x_0}{b}\right) = \exp\left[-\pi\left(\frac{x-x_0}{b}\right)^2\right]$$

Again we'll incorporate the π here to keep consistent with our previous definitions.



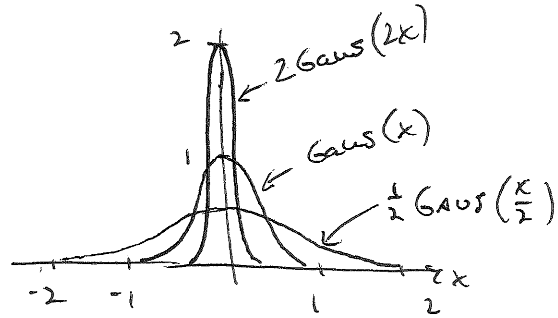
So Gaus( ) peaks at 1 when $x = x_0$ and an area $= |b|$. Again, sign of b has no effect since symmetric about x_0.

Note how this is different from the normal distribution in probability has an area $= 1$.

DIRAC'S DELTA FUNCTION OR IMPULSE FUNCTION

Dirac's delta function, symbolized by $\delta(x)$ is one of those concepts that gives mathematicians fits, but is readily used by engineers. In optics, we are often interested in point sources of light or the value of a temporal function at a point in time. The delta function can be used to represent such point objects. Defining $\delta(x)$ is a bit of a challenge though. One way to define $\delta(x)$ is as a limit such as

$$\delta(x) = \lim_{b \to 0} \frac{1}{|b|}\mathrm{Gaus}\left(\frac{x}{b}\right)$$



All have area = 1

As $|b| \to 0$ the Gaussian function becomes infinitely high and infinitely narrow. The area of the function however remains at 1. $\delta(x)$ is therefore a spike with area 1.

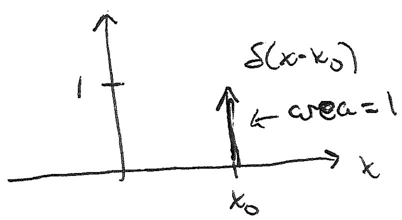
This definition doesn't really provide any insight into the properties of $\delta(x)$.

It is far better to define $\delta(x)$ in terms of two mathematical properties

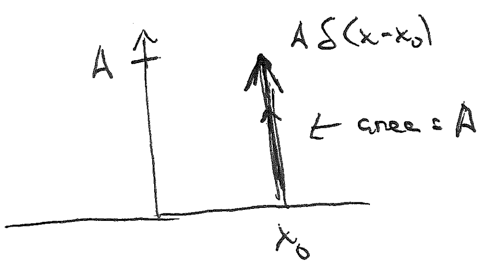
$$\boxed{\begin{aligned} &\delta(x-x_0) = 0 \qquad \text{for } x \neq x_0 \\ &\int_{x_1}^{x_2} f(\alpha)\,\delta(\alpha - x_0)\,d\alpha = f(x_0) \qquad x_1 < x_0 < x_2 \end{aligned}}$$ "sifting"

In words, the delta function is zero everywhere where its argument is non-zero (i.e. $x - x_0 \neq 0$). If we multiply a function $f(x)$ by a delta function and integrate, the result is just the value of the function at the point x_0. Note, if $f(x)$ is discontinuous at x_0, then the integral gives the average value of $f(x_0)$

Plotting delta functions



The delta function is drawn as an arrow centered on the point $x = x_0$. The height of the arrow is the area of the delta function, not the height of the function.



The area of the

Scaling Properties – The definitions above can also be used to see what happens for scaled and shifted versions of delta function $\delta\left(\frac{x-x_0}{b}\right)$. We know that this function will be zero everywhere except when $x = x_0$.

Now integration $I_1 = \int_{-\infty}^{\infty} f(\alpha)\,\delta\left(\frac{\alpha - x_0}{b}\right) d\alpha = |b| \int_{-\infty}^{\infty} f(b\beta)\,\delta\left(\beta - \frac{x_0}{b}\right) d\beta$

substitute $\alpha = b\beta$

Now from sifting property

$$I_1 = |b|\, f(b\beta)\Big|_{\beta = \frac{x_o}{b}} = |b|\, f(x_o)$$

This is exactly the same result as

$$I_2 = \int_{-\infty}^{\infty} f(\alpha)\, |b|\, \delta(\alpha - x_o)\, d\alpha = |b|\, f(x_o)$$

So we can equate the two delta functions

$$\boxed{\delta\left(\frac{x - x_o}{b}\right) = |b|\, \delta(x - x_o)}$$

other variations on this scaling result

$$\delta(ax - x_o) = \frac{1}{|a|}\, \delta\left(x - \frac{x_o}{a}\right)$$

$$\delta(-x + x_o) = \delta(x - x_o)$$

$$\delta(-x) = \delta(x) \longleftarrow$$ $\delta(x)$ acts like an even function

Multiplication Properties

$$f(x)\,\delta(x - x_o) = f(x_o)\,\delta(x - x_o)$$

$$x\,\delta(x - x_o) = x_o\,\delta(x - x_o)$$

$$\delta(x)\,\delta(x - x_o) = 0 \qquad x_o \neq 0$$

$\delta(x - x_o)\,\delta(x - x_o)$ undefined

Integral Properties

$$\boxed{\begin{aligned} &\int_{-\infty}^{\infty} A\,\delta(\alpha - x_0)\, d\alpha = A \\ &\int_{-\infty}^{\infty} \delta(\alpha - x_0)\, d\alpha = 1 \\ &\int_{-\infty}^{\infty} \delta(\alpha - x_0)\,\delta(x - \alpha)\, d\alpha = \delta(x - x_0) \end{aligned}}$$

For this last result, use the following

$$\beta = \alpha - x_0 \Rightarrow \alpha = \beta + x_0$$

$$d\alpha = d\beta$$

$$\int_{-\infty}^{\infty} \delta(\alpha - x_0)\,\delta(x - \alpha)\, d\alpha = \int_{-\infty}^{\infty} \delta(\beta)\,\delta(x - \beta - x_0)\, d\beta$$

$$= \int_{-\infty}^{\infty} \delta(\beta)\,\delta\big(-\beta + (x - x_0)\big)\, d\beta$$ reorder argument

$$= \int_{-\infty}^{\infty} \delta(\beta)\,\delta\big(\beta - (x - x_0)\big)\, d\beta$$ from multiplication properties

$$= \delta(x - x_0)$$ from sifting property. The first $\delta(\beta)$ is acting like our function $f(\,)$

Relatives to the Dirac Delta Function

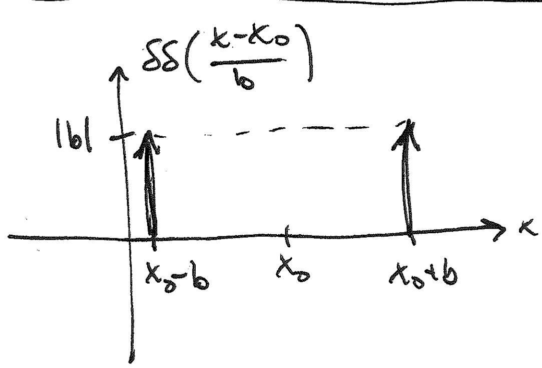
Even and Odd Impulse Functions - Pairs of delta functions

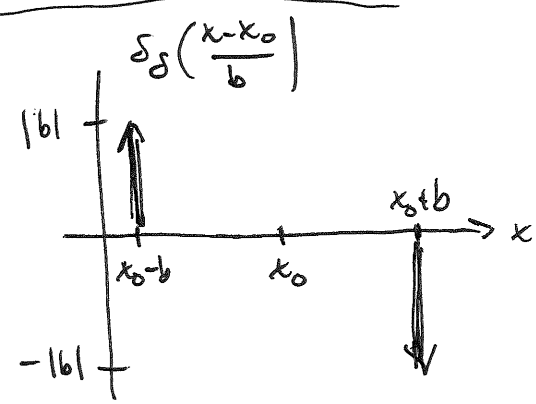
even pair
$$\delta\delta(x) = \left[\delta(x+1) + \delta(x-1)\right]$$

odd pair
$$\delta_\delta(x) = \left[\delta(x+1) - \delta(x-1)\right]$$

I'm not a huge fan of this notation with the dropped delta function in the odd pair. Using the scaling properties of delta functions

$$\boxed{\begin{aligned} \delta\delta\left(\frac{x-x_0}{b}\right) &= |b|\left[\delta(x-x_0+b) + \delta(x-x_0-b)\right] \\ \delta_\delta\left(\frac{x-x_0}{b}\right) &= |b|\left[\delta(x-x_0+b) - \delta(x-x_0-b)\right] \end{aligned}}$$

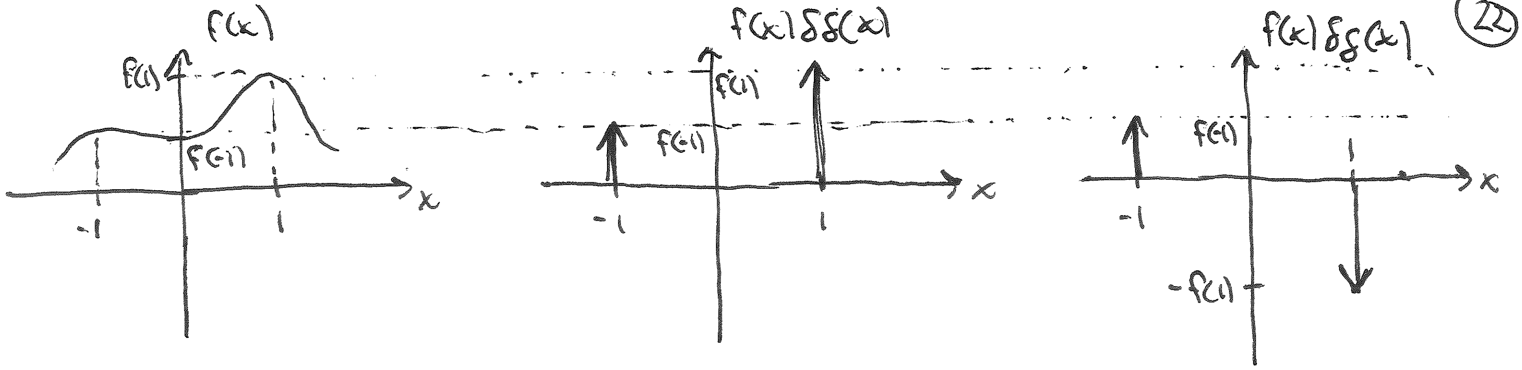




Using the multiplication properties of delta functions

$$f(x)\,\delta\delta\left(\frac{x-x_0}{b}\right) = |b|\left[f(x_0-b)\,\delta(x-x_0+b) + f(x_0+b)\,\delta(x-x_0-b)\right]$$

$$f(x)\,\delta_\delta\left(\frac{x-x_0}{b}\right) = |b|\left[f(x_0-b)\,\delta(x-x_0+b) - f(x_0+b)\,\delta(x-x_0-b)\right]$$



using the sifting properties

$$\int_{-\infty}^{\infty} f(\alpha)\,\delta\delta\left(\frac{\alpha - x_0}{b}\right) d\alpha = |b|\left[f(x_0 - b) + f(x_0 + b)\right]$$

$$\int_{-\infty}^{\infty} f(\alpha)\,\delta_\delta\left(\frac{\alpha - x_0}{b}\right) d\alpha = |b|\left[f(x_0 - b) - f(x_0 + b)\right]$$

From all of these

$\delta\delta(-x) = \delta\delta(x)$ even function $\Rightarrow \int_{-\infty}^{\infty} \delta\delta\left(\frac{\alpha}{b}\right) d\alpha = 2|b|$

$\delta_\delta(-x) = -\delta_\delta(x)$ odd function $\Rightarrow \int_{-\infty}^{\infty} \delta_\delta\left(\frac{\alpha}{b}\right) d\alpha = 0$

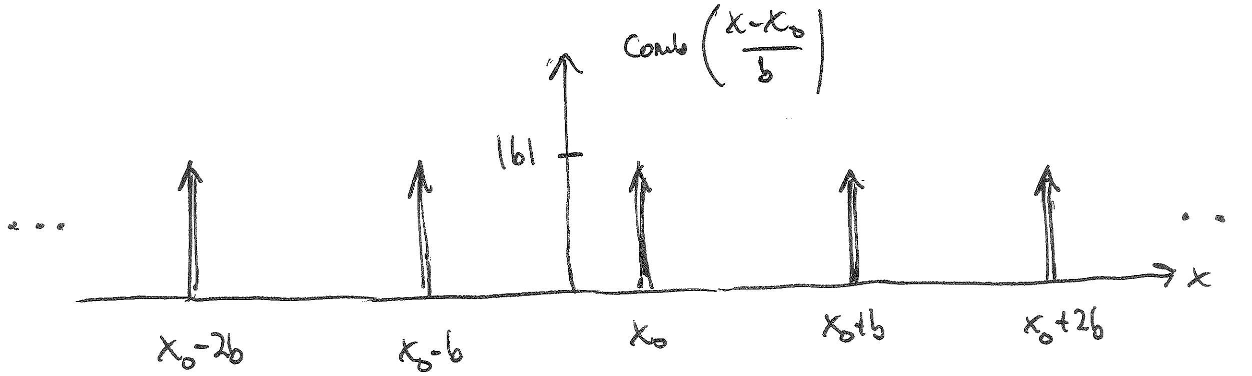
COMB FUNCTION

The comb function is a periodic array of delta functions.

$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$ where n is an integer.

Much like before, a scaled and shifted version can be defined

$$\boxed{\text{comb}\left(\frac{x - x_0}{b}\right) = |b| \sum_{n=-\infty}^{\infty} \delta(x - x_0 - nb)}$$



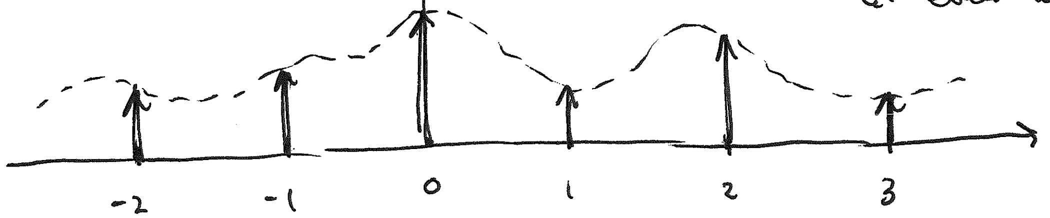
Divide through by $|b|$

$$\frac{1}{|b|}\,\mathrm{comb}\left(\frac{x-x_0}{b}\right) = \sum_{N=-\infty}^{\infty} \delta(x-x_0-Nb)$$

Array of delta functions each with unit area.

Multiplication Properties

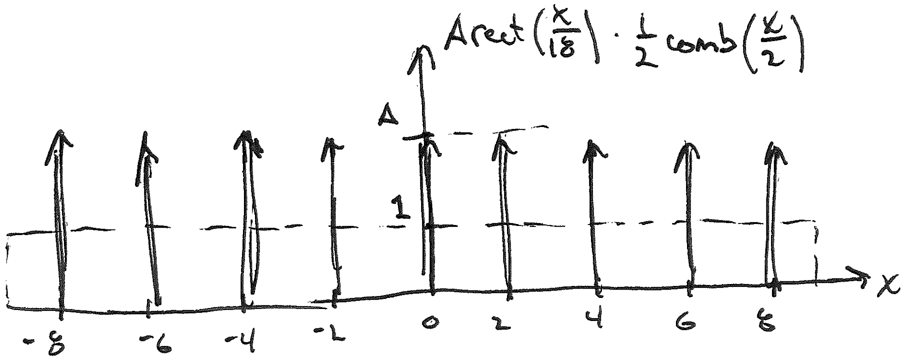
$$f(x)\left[\frac{1}{|b|}\,\mathrm{comb}\left(\frac{x-x_0}{b}\right)\right] = \sum_{N=-\infty}^{\infty} f(x_0+Nb)\,\delta(x-x_0-Nb)$$



comb samples function at even intervals

Often we only want a finite number of delta function. For example consider 9 delta functions with spacing $b=2$ units and $x_0=0$

$$f(x) = A\sum_{N=-4}^{4}\delta(x-2N) = A\big[\delta(x+8)+\delta(x+6)+\delta(x+4)+\delta(x+2)+\delta(x)$$
$$+\,\delta(x-2)+\delta(x-4)+\delta(x-6)+\delta(x-8)\big]$$



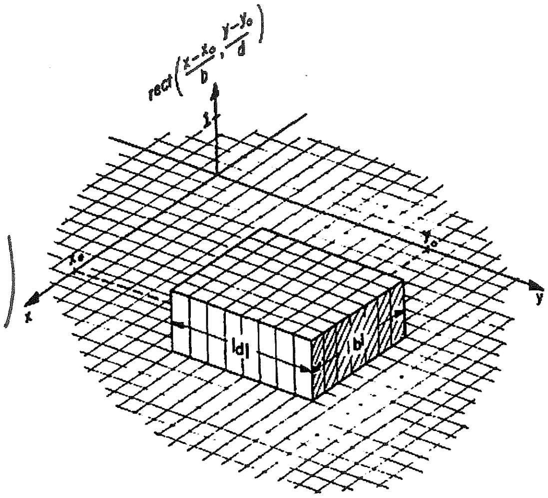
More convenient to write as

$$f(x) = A\,\mathrm{rect}\left(\frac{x}{18}\right)\cdot\frac{1}{2}\,\mathrm{comb}\left(\frac{x}{2}\right)$$

Two-Dimensional Functions

Functions separable in Cartesian coordinates - These functions are just products of their 1D counterparts in x and y.

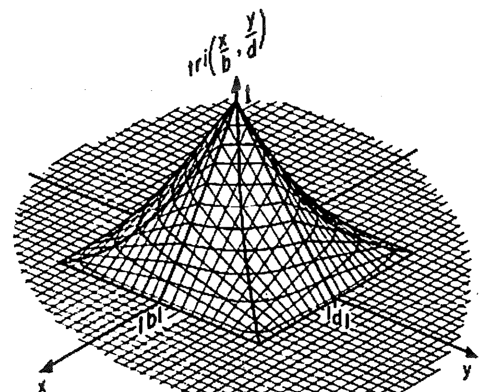
Rectangle Function



$$\mathrm{rect}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right) = \mathrm{rect}\left(\frac{x-x_0}{b}\right)\mathrm{rect}\left(\frac{y-y_0}{d}\right)$$

Rectangular region centered on point (x_0, y_0) with dimensions b and d in the x and y directions, respectively. The height is 1. Volume $= |bd|$.

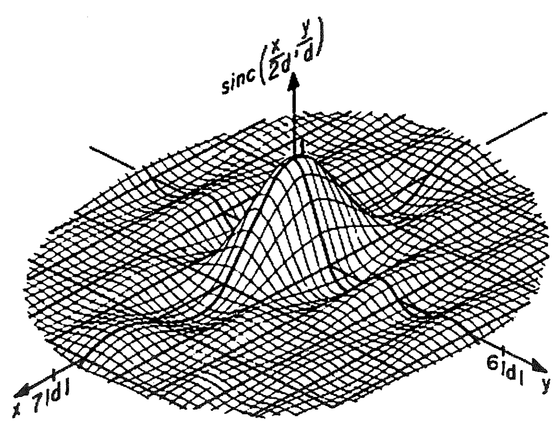
Triangle Function



$$\mathrm{tri}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right) = \mathrm{tri}\left(\frac{x-x_0}{b}\right)\mathrm{tri}\left(\frac{y-y_0}{d}\right)$$

The cross-sections look like triangles with widths $2|b|$ and $2|d|$ as expected. However, the slopes are non-linear in other directions so this is Not a pyramid. The peak height is 1. The volume is $|bd|$.

Sinc Function



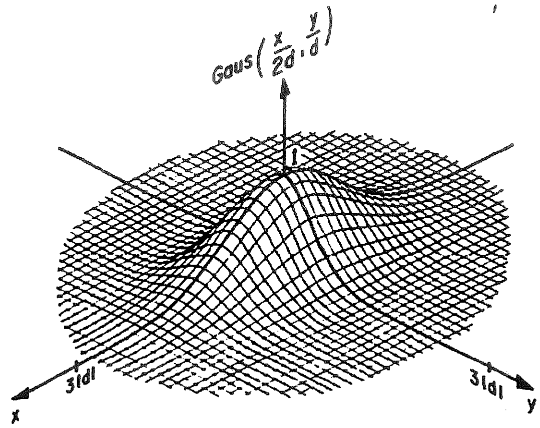
$$\mathrm{sinc}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right) = \mathrm{sinc}\left(\frac{x-x_0}{b}\right)\mathrm{sinc}\left(\frac{y-y_0}{d}\right)$$

This function can be squared as well to give sinc^2 function in 2D. The peaks of both are 1, and the volumes of both are $|bd|$.

Gaussian Function

$$\text{Gaus}\left(\frac{x-x_0}{b}, \frac{y-y_0}{d}\right) = \text{Gaus}\left(\frac{x-x_0}{b}\right)\text{Gaus}\left(\frac{y-y_0}{d}\right)$$

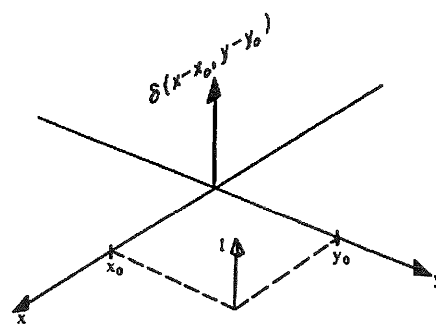
allows an elliptical Gaussian function to be created. The peak is again 1 and the volume is again $|bd|$.



Dirac delta function or impulse function

$$\delta(x-x_0, y-y_0) = \delta(x-x_0)\,\delta(y-y_0)$$

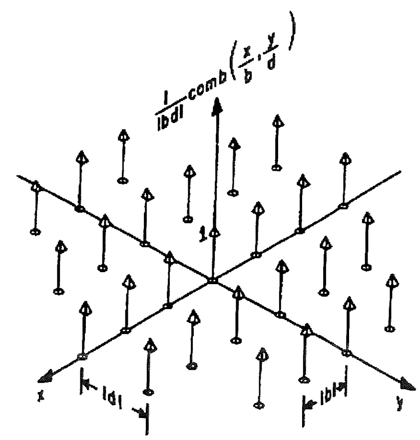
$\delta(x-x_0)$ acts like a line of delta functions along $x=x_0$. Similarly, $\delta(y-y_0)$ acts like a line of delta functions along $y=y_0$. The product of these two only has value at the point (x_0, y_0). Again, the height of the delta function denotes the volume.



Comb Function

$$\text{comb}(x,y) = \text{comb}(x)\,\text{comb}(y)$$

This is a 2D array of unit volume delta functions spaced at integral values of x and y. The spacing can be modified, as well as the center point with



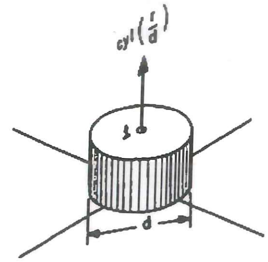
$$\frac{1}{|bd|}\text{comb}\left(\frac{x-x_0}{b}, \frac{y-y_0}{d}\right) = \frac{1}{|b|}\text{comb}\left(\frac{x-x_0}{b}\right)\frac{1}{|d|}\text{comb}\left(\frac{y-y_0}{d}\right)$$

with this normalization, the volumes of each delta function is still 1, but the spacings are now $|b|$ and $|d|$.

POLAR COORDINATES (r,θ) with $0 \le r < \infty$, $0 \le \theta < 2\pi$

Cylinder Function

$$\text{cyl}\left(\frac{r}{d}\right) = \begin{cases} 1 & 0 \le r < \frac{d}{2} \\ \frac{1}{2} & r = \frac{d}{2} \\ 0 & r > \frac{d}{2} \end{cases}$$



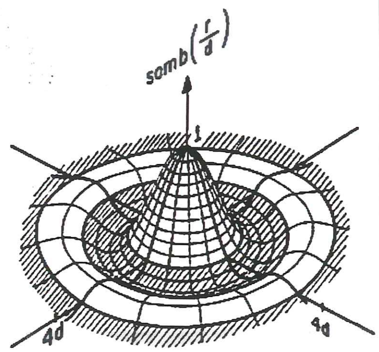
This is a rotationally symmetric function since there is no θ dependence. Cyl() has a height of 1 and a diameter of d. The volume should be $V = \pi\left(\frac{d}{2}\right)^2 h = \frac{\pi d^2}{4}$ where h is the height. Let's prove this with integration.

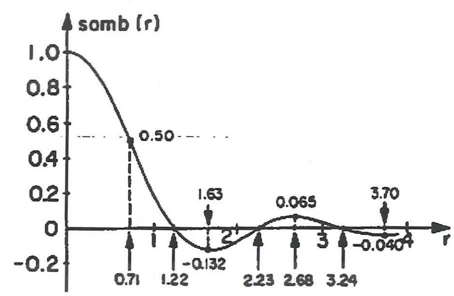
$$\int_0^{2\pi}\int_0^{\infty} \text{cyl}\left(\frac{r}{d}\right) r\,dr\,d\theta = 2\pi\int_0^{d/2} r\,dr = 2\pi\left[\frac{1}{2}r^2\right]_0^{d/2} = \frac{\pi d^2}{4}$$

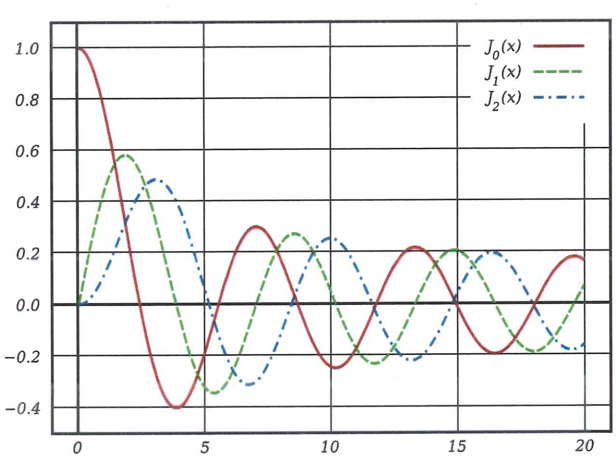
Sombrero Function

$$\text{somb}\left(\frac{r}{d}\right) = \frac{2 J_1\left(\frac{\pi r}{d}\right)}{\frac{\pi r}{d}}$$

where $J_1()$ is the first-order Bessel function of the first kind.







$J_1(x)$ is zero when

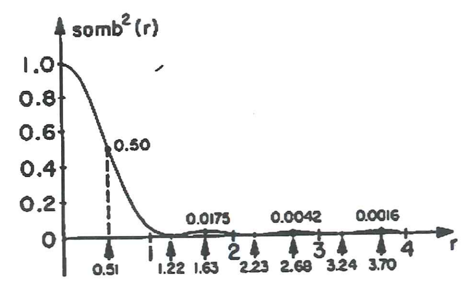
$x = 3.8317$

$x = 7.0156$

$x = 10.1735$

$x = 13.3237$

⋮



$\text{somb}^2()$ shows up in optics quite a bit as well

Volume $= \frac{4d^2}{\pi}$, Rotationally symmetric

Gaussian Function

$$\text{Gaus}\left(\frac{r}{d}\right) = e^{-\pi\left(\frac{r}{d}\right)^2}$$

The Gaussian function in polar coordinates is the rotationally symmetric version of of the 2D Gaussian function given previously in Cartesian coordinates. The peak value is 1 and the volume is d^2.

$$\text{Gaus}\left(\frac{r}{d}\right) = \text{Gaus}\left(\frac{x}{d}\right)\text{Gaus}\left(\frac{y}{d}\right)$$

same

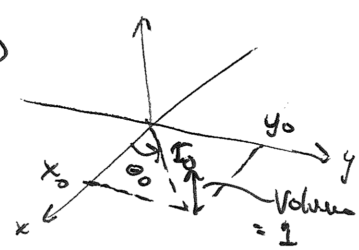
$$\int_0^{2\pi}\int_0^{\infty} \text{Gaus}\left(\frac{r}{d}\right) r\,dr\,d\theta = 2\pi\int_0^{\infty} e^{-\pi\left(\frac{r}{d}\right)^2} r\,dr = -2\pi\cdot\frac{d^2}{2\pi}\int_{-\infty}^{0} e^{u}\,du = d^2$$

Impulse Response or Dirac Delta function

In Cartesian $\delta(x-x_0, y-y_0) = \delta(x-x_0)\,\delta(y-y_0)$ is a delta function located at point (x_0, y_0) with volume 1. We want to determine the equivalent expression in polar coordinates. Let $\vec{r} = \cancel{(x,y)}\ (r,\theta)$ and $\vec{r}_0 = \cancel{(x_0,y_0)}\ (r_0,\theta_0)$ with $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$$\delta(\vec{r}-\vec{r}_0) = \frac{\delta(r-r_0)}{r_0}\,\delta(\theta-\theta_0) \quad \text{for } r_0 > 0$$

$$\delta(\vec{r}) = \frac{\delta(r)}{\pi r} \quad \text{for } r_0 = 0$$



The extra factors appear because we still want the volume to be 1.

$$\int_0^{2\pi}\int_0^{\infty} \frac{\delta(r-r_0)}{r_0}\,\delta(\theta-\theta_0)\,r\,dr\,d\theta = \int_0^{2\pi}\delta(\theta-\theta_0)\,d\theta\int_0^{\infty}\frac{\delta(r-r_0)}{r_0}\,r\,dr = 1\cdot\frac{r_0}{r_0} = 1$$

can change limits if $\theta_0 = 0$ or $\theta_0 = 2\pi$ use sifting

$$\int_0^{2\pi}\int_0^{\infty}\frac{\delta(r)}{\pi r}\,r\,dr\,d\theta = 2\pi\,\frac{1}{\pi}\int_0^{\infty}\delta(r)\,dr = \int_{-\infty}^{\infty}\delta(r)\,dr = 1$$