

4 BASIC INTERFEROMETRY AND OPTICAL TESTING

4.1 REVIEW OF TWO BEAM INTERFERENCE

4.1.1 PLANE WAVES

$$E_1(\vec{r}, t) = A_1 \hat{p}_1 e^{i(\vec{k}_1 \cdot \vec{r} - \omega_1 t)}$$

$$E_2(\vec{r}, t) = A_2 \hat{p}_2 e^{i(\vec{k}_2 \cdot \vec{r} - \omega_2 t)}$$

$$A_{1,2} = a_{1,2} e^{i\phi_{1,2}} \quad \left\{ \begin{array}{l} a_{1,2} = \text{amplitude} \\ \phi_{1,2} = \text{phase} \end{array} \right.$$

~~Wave vector~~

\vec{k}_1 and \vec{k}_2 are propagation directions of the plane waves

$$|\vec{k}| = \frac{2\pi N}{\lambda} \quad \text{where } N \text{ is index of medium, } \lambda \text{ is wavelength}$$

$$\hat{p}_1 \cdot \vec{k}_1 = 0 \quad \text{for transverse EM wave}$$

similar for \vec{k}_2, \hat{p}_2

Combine these fields to find irradiance

$$I(\vec{r}, t) = |E_1(\vec{r}, t) + E_2(\vec{r}, t)|^2$$

$$I(\vec{r}, t) = |A_1|^2 + |A_2|^2 + A_1 A_2^* (\hat{p}_1 \cdot \hat{p}_2) e^{i[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} - (\omega_1 - \omega_2)t]} + A_1^* A_2 (\hat{p}_1 \cdot \hat{p}_2) e^{-i[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} - (\omega_1 - \omega_2)t]}$$

$$I(\vec{r}, t) = \frac{1}{2} a_1^2 + \frac{1}{2} a_2^2 + a_1 a_2 (\hat{p}_1 \cdot \hat{p}_2) e^{i[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi_1 - \phi_2]} + a_1 a_2 (\hat{p}_1 \cdot \hat{p}_2) e^{-i[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi_1 - \phi_2]}$$

$$\text{define } I_1 = \frac{1}{2} a_1^2 \quad I_2 = \frac{1}{2} a_2^2 \quad \phi_1 - \phi_2 = \phi$$

$$I(\vec{r}, t) = I_1 + I_2 + 2\sqrt{I_1 I_2} (\hat{p}_1 \cdot \hat{p}_2) \cos[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} - (\omega_1 - \omega_2)t + \phi]$$

Usually we assume $\hat{p}_1 \cdot \hat{p}_2 = 1$ i.e. electric field is polarized in same direction
 $\omega_1 - \omega_2 = 0$ i.e. same frequency

Under these assumptions

$$I(\vec{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos[(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + \phi]$$

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2}$$

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2}$$

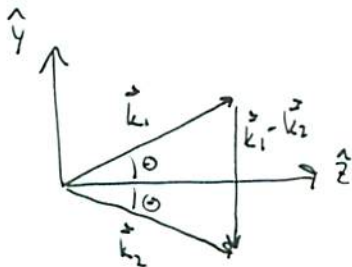
$$\text{Contrast} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}$$

What does this pattern look like?

Bright fringe when $(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + \phi = 2m\pi$ m integer

Dark fringe when $(\vec{k}_1 - \vec{k}_2) \cdot \vec{r} + \phi = (2m+1)\pi$

Let's align our coordinate system so that \vec{k}_1 and \vec{k}_2 are in the yz plane and orient the z axis so it bisects \vec{k}_1 and \vec{k}_2



$$\vec{k}_1 = \frac{2\pi N}{\lambda} [\sin\theta \hat{y} + \cos\theta \hat{z}]$$

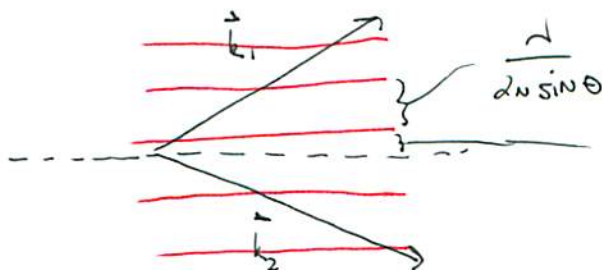
$$\vec{k}_2 = \frac{2\pi N}{\lambda} [-\sin\theta \hat{y} + \cos\theta \hat{z}]$$

$$\vec{k}_1 - \vec{k}_2 = \frac{2\pi N}{\lambda} [2\sin\theta \hat{y}]$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\text{So } (\vec{k}_1 - \vec{k}_2) \cdot \vec{r} = \frac{2\pi N}{\lambda} [2y\sin\theta]$$

Bright fringe when $\frac{2\pi N}{\lambda} [2y\sin\theta] + \phi = 2m\pi$



ϕ controls this offset

4.1.3 | General Wavefront Shapes

$$E_1(\vec{r}, t) = A_1 e^{i \frac{2\pi}{\lambda} \omega_1(\vec{r}, t)}$$

$$E_2(\vec{r}, t) = A_2 e^{i \frac{2\pi}{\lambda} \omega_2(\vec{r}, t)}$$

Again

$$I(\vec{r}) = |E_1(\vec{r}, t) + E_2(\vec{r}, t)|^2$$

$$I(\vec{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \left[\frac{2\pi}{\lambda} (\omega_1 - \omega_2) + \phi \right]$$

Bright fringes when $\frac{2\pi}{\lambda} (\omega_1 - \omega_2) + \phi = 2m\pi$

Interferometry basically takes an unknown wavefront ω_1 (test wavefront) and a known wavefront ω_2 (reference wavefront) and combines them as above. The fringe pattern created by the cosine term is analyzed to recover $\omega_1 - \omega_2$. Furthermore, we can control I_1, I_2 ~~and~~ to maximize fringe contrast. Finally, we can take multiple measurements with different values of ϕ to aid in recovering $\omega_1 - \omega_2$

Why is this useful?

Example 1

Suppose ω_1 is the wavefront emerging from the exit pupil of an optical system and ω_2 is a spherical wave corresponding to the reference sphere, the $\omega_1 - \omega_2$ equals the wavefront error as we have previously defined. Furthermore, various metrics such as wavefront variance and Strehl ratios are related to this value

Example 2

Suppose ω_1 is the wavefront reflected from some spherical surface under test and ω_2 is a spherical wavefront corresponding to the ideal shape of this surface. $\omega_1 - \omega_2$ represents the ^{surface} ~~surface~~ difference between the surfaces. This may tell you ~~to~~ where to grind/polish more or if the part meets spec.

4.1.4 Visibility

Visibility is another name for contrast as we defined previously

Visibility = (I_max - I_min) / (I_max + I_min) = 2*sqrt(I_1*I_2) / (I_1 + I_2)

Note that when I_1 = I_2

Visibility = 2*sqrt(I_1^2) / 2*I_1 = 1

High contrast when individual beams have equal intensities.

4.1.5 COHERENCE AND POLARIZATION

We have ignored these effects by assuming w_1 = w_2 and p_1 dot p_2 = 1. Let's look at the effects of these two. From page 138

I(r, t) = I_1 + I_2 + 2*sqrt(I_1*I_2) (p_1 dot p_2) cos[(k_1 - k_2) dot r + (w_1 - w_2)t + phi]

The location of the fringes doesn't just depend on r, but also on t. In other words, the fringes move with time. If w_1 - w_2 is too big, then we can't measure the fringe pattern since it is moving too fast.

The other issue is random fluctuations in phi with time. To get interference, we need the light to come from the same source. The light is split into a test and reference portion and then recombined to interfere. By coming from the same source, phi_1 and phi_2 correlate making phi essentially constant. We also need the times it takes to traverse the two paths to be similar otherwise the correlation between phi_1 and phi_2 breaks down.

Polarization affects the visibility

Coherence time delta t approx 1/delta nu

where delta nu = bandwidth of source

Coherence length delta l = c*delta t = c/delta nu where c is speed of light

delta l approx lambda^2 / delta lambda = (mean lambda)^2 / (wavelength band)

4.1.2 Spherical wave

$$E_1(\vec{r}, t) = \frac{A_1}{|\vec{r}-\vec{r}_1|} e^{i(k|\vec{r}-\vec{r}_1| - \omega t)}$$

again $A_1 = a_1 e^{i\phi_1}$

$$E_2(\vec{r}, t) = \frac{A_2}{|\vec{r}-\vec{r}_2|} e^{i(k|\vec{r}-\vec{r}_2| - \omega t)}$$

$$A_2 = a_2 e^{i\phi_2}$$

assume polarized in same direction and have same frequency

$$I(\vec{r}, t) = |E_1(\vec{r}, t) + E_2(\vec{r}, t)|^2$$

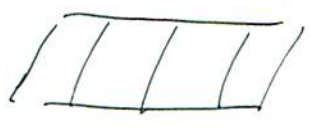
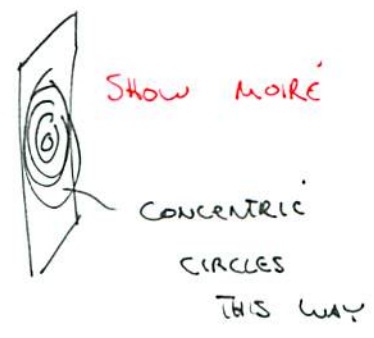
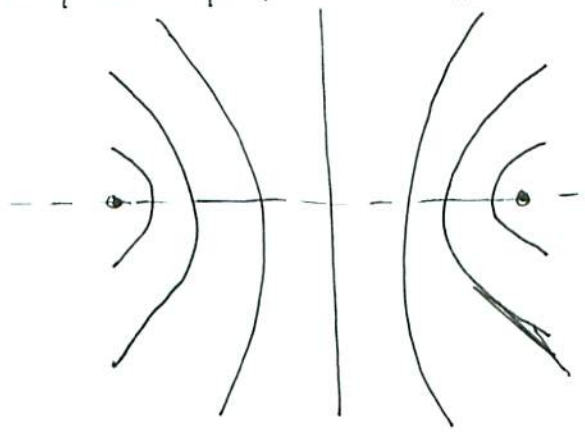
Following same procedure as before

$$I(\vec{r}, t) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left[\frac{2\pi\nu}{\lambda} (|\vec{r}-\vec{r}_1| - |\vec{r}-\vec{r}_2|) + \phi\right]$$

Bright fringes occur when

$$\frac{2\pi\nu}{\lambda} (|\vec{r}-\vec{r}_1| - |\vec{r}-\vec{r}_2|) + \phi = 2m\pi$$

In 3D, the equiphase surfaces are hyperboloids



STRAIGHT LINES THIS WAY

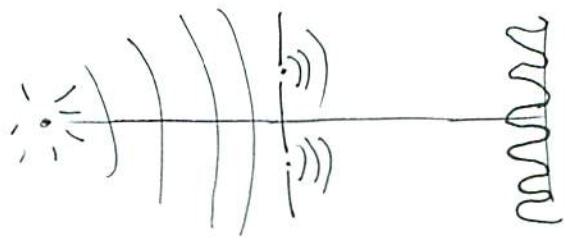
Polarization effects to Visibility

$$\text{Visibility} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} (\hat{p}_1 \cdot \hat{p}_2)$$

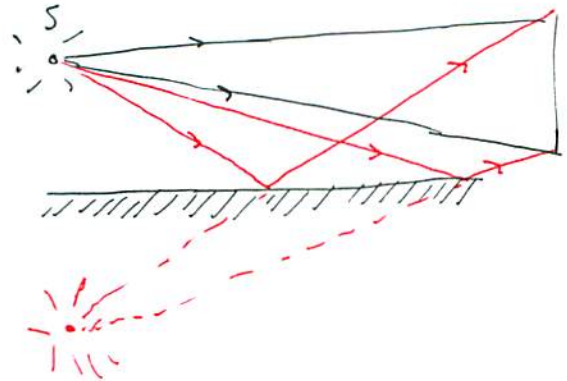
this is just cosine of θ between the two vectors. When $\hat{p}_1 \perp \hat{p}_2$, the fringes have zero visibility

[4.1.6] DIVISION OF WAVEFRONT

Spatially DIVIDE THE wavefront from a single source into two wavefronts and then recombine to get interference.

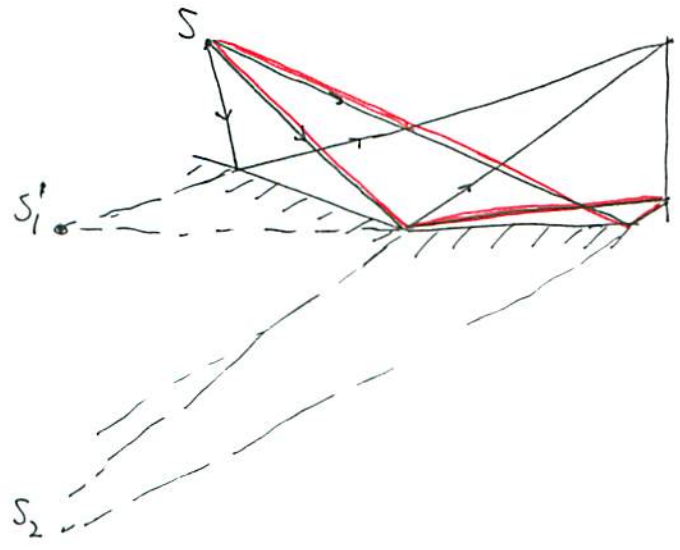


Young's Double Slit



Lloyd's Mirror

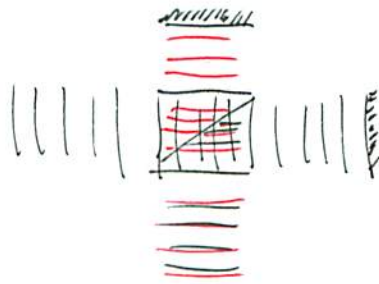
INTERFERENCE OF TWO SPHERICAL WAVES



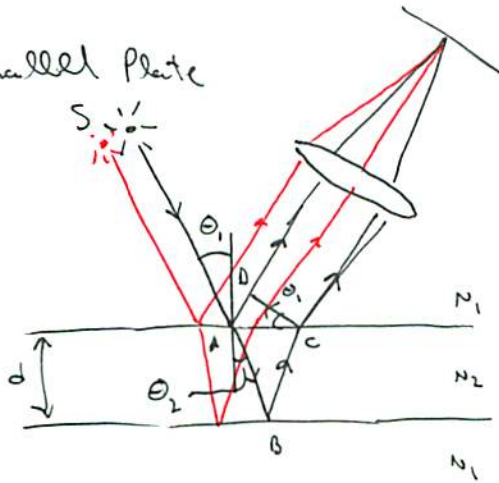
FRESNEL'S MIRRORS

4.1.7 | DIVISION OF AMPLITUDE

USE A BEAMSPLITTER TO DIVIDE THE BEAM AND THE RECOMBINE



Plane Parallel Plate



From geometry

$$AB = BC = \frac{d}{\cos \theta_2}$$

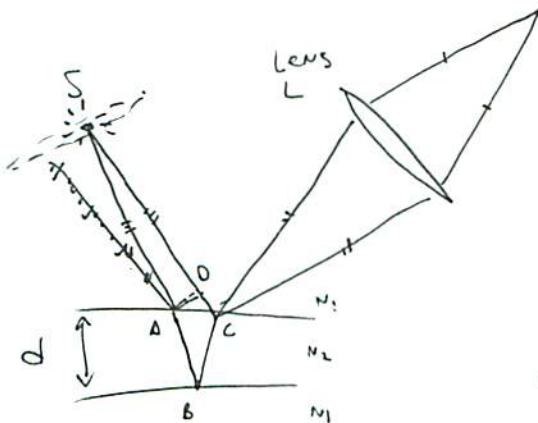
$$AC = 2d \tan \theta_2$$

$$\Delta D = AC \sin \theta_1 = 2d \tan \theta_2 \sin \theta_1$$

$$OPD = n_2 (AB + BC) - n_1 \Delta D + \frac{\lambda}{2}$$

$$OPD = 2n_2 d \cos \theta_2 + \frac{\lambda}{2} \quad \text{Use } n_1 \sin \theta_1 = n_2 \sin \theta_2$$

The $\frac{\lambda}{2}$ occurs because there is a π phase change upon reflection from only one of the interfaces (whenever we go from lower to higher n index). These are known as fringes of equal inclination since extended source gives fringes at same location.



Fringes of equal thickness

$$OPD \approx 2n_2 d \cos \theta_2 + \frac{\lambda}{2}$$

Here the key is that we are imaging the surface boundary so the fringes are localized at the interface. In general, the fringe pattern changes with position on an extended source. For small d (i.e. a few waves) a large source can be used. However, as the thickness d increases the source must approach a point. The lens L can be the eye.

4.2 Newton's Rings

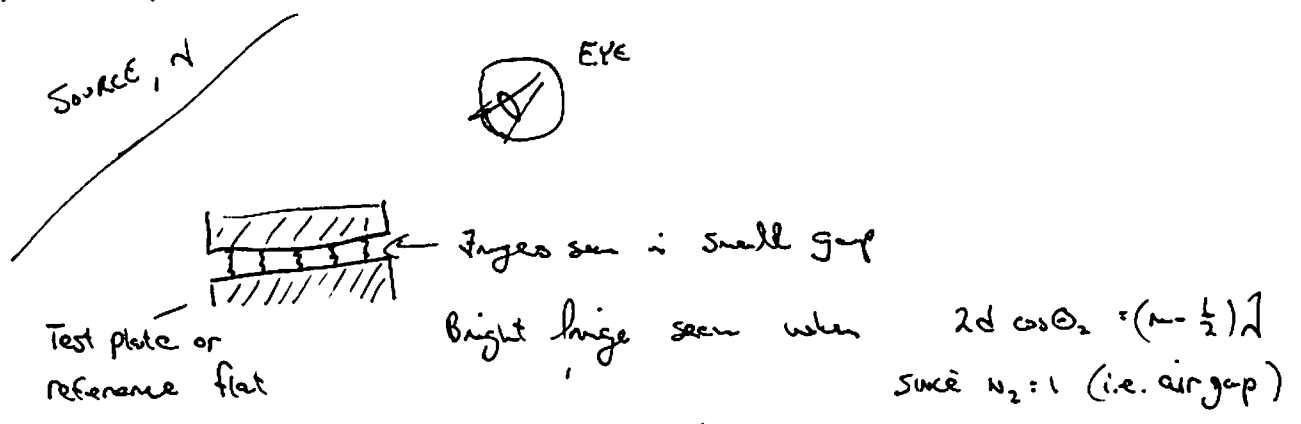
Newton's rings are examples of fringes of equal thickness. One example is the color fringe pattern seen from a thin oil film or bubble surface. Since the film is thin, a large source (sun, sky) can be used. ~~For~~ Bright fringes occur when

$$2n_2 d \cos \theta_2 + \frac{\lambda}{2} = m \lambda \quad m \text{ Integer}$$

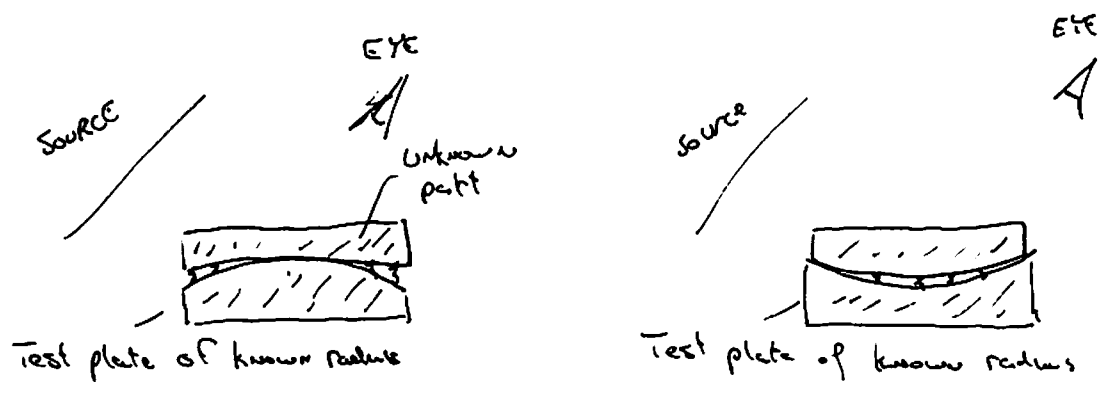
$$2n_2 d \cos \theta_2 = (m - \frac{1}{2}) \lambda$$

Different wavelengths correspond to different values of d , hence the color fringing effect.

We can use this for optical testing as well. Consider ~~testing~~ testing how flat a surface is. If we place the test surface in contact with a known flat surface and the spacing is small, then fringes can be seen with an eye at the gap between the two surfaces. If we use a monochromatic source, the color fringe effect will be removed and the fringes will have higher contrast.

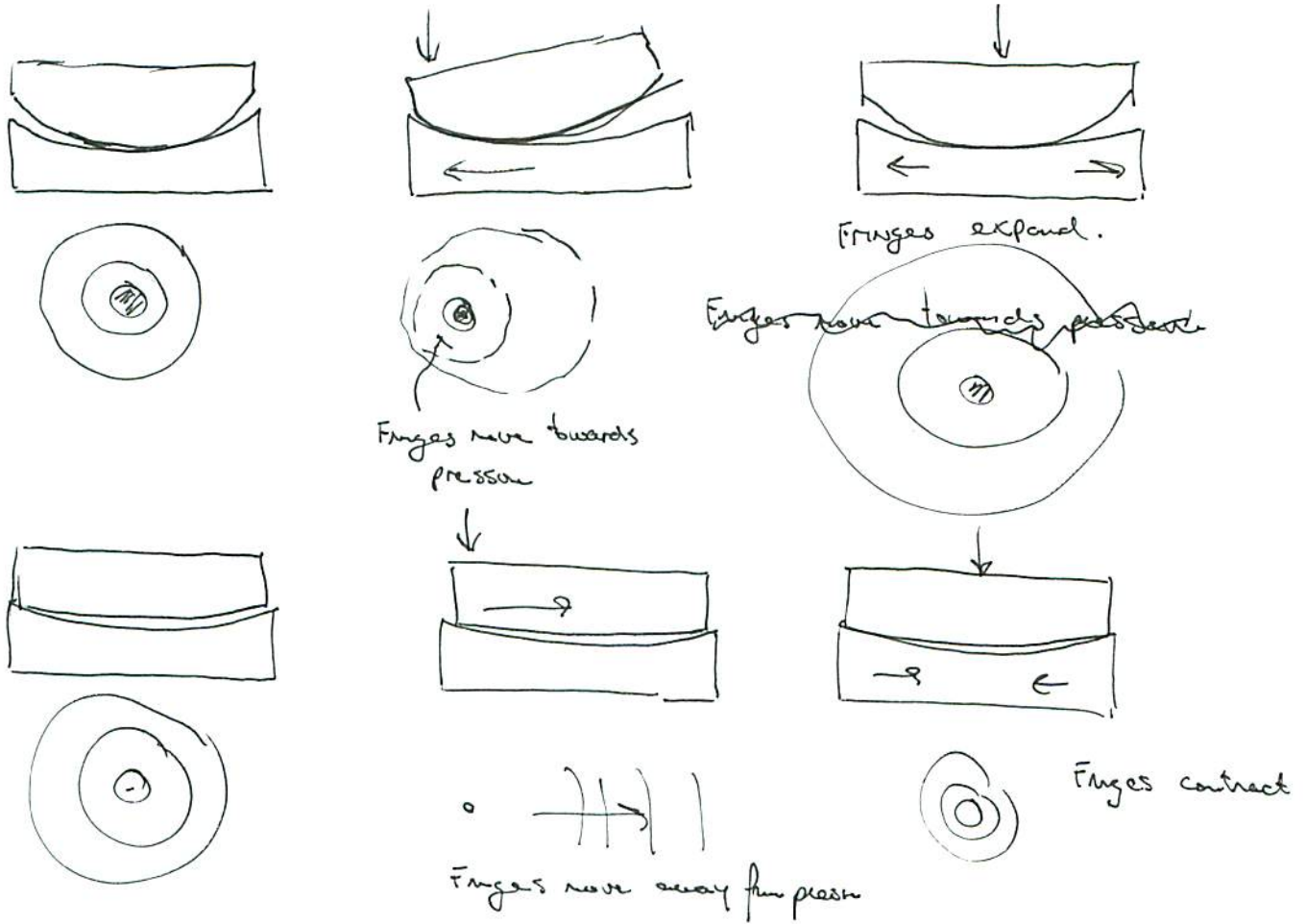


This technique can be extended to curved surfaces as well



In general, the test plate must be well characterized and at least as big as the unknown part. The parts must match pretty closely to see the fringes.

4.2.2 Determining if radius is shorter or longer than test plate



Show ~~FILE~~ PARK'S VIDEO

Show DEMO

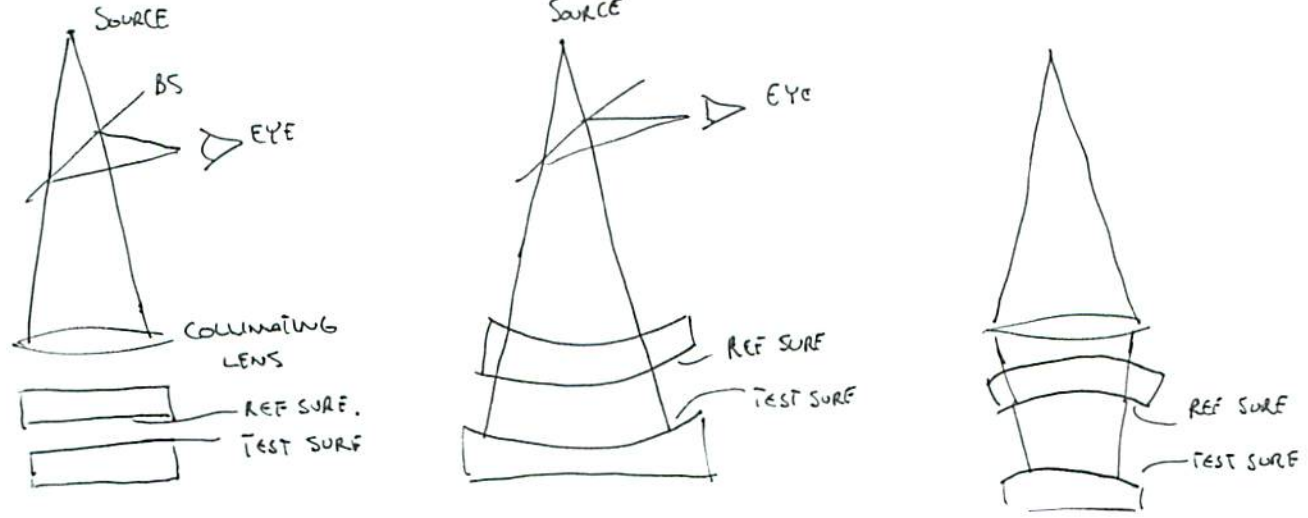
4.2.3 TEST PLATES ~~FILE~~ Show opcolab.com/page200.html

Most manufacturers have sets of test plates with well characterized radii. When ordering lenses from such a company you may specify test plate fits, but your design must be close to available test plates. Alternatively, test plates can be made for your part, but this is expensive and time consuming. Going this route is dictated by budget, timeframe and # of parts to be made/tested. It may be worthwhile to modify your design to better match available test plates.

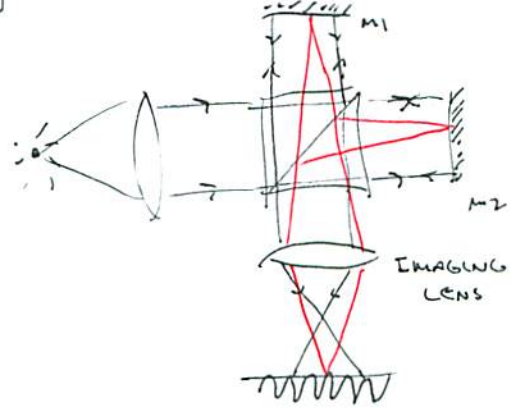
4.3 FIZEAU INTERFEROMETER

If gap between the reference and test surfaces becomes too large, or concerns regarding putting surfaces in contact then a ~~Fizeau~~ Fizeau interferometer can be used. A small monochromatic source is needed in this case.

4.3.1 Common Configurations

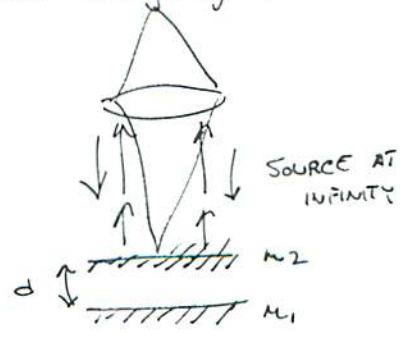


4.4 TWYMAN - GREEN INTERFEROMETER



The Twyman - GREEN uses a collimated monochromatic source. The imaging lens relays mirrors M1 and M2 to a common plane where the interference pattern is observed.

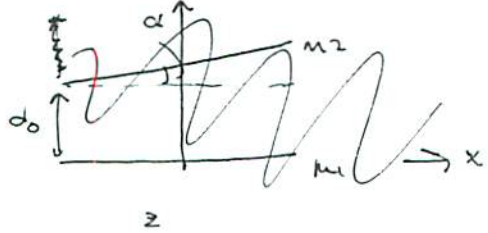
If we unfold the optical paths, we essentially have our plane parallel plate system but everything is coaxial



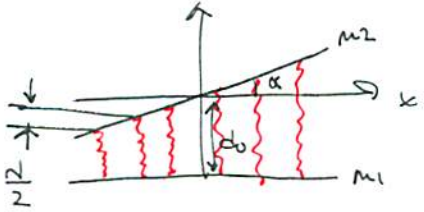
Fringes of equal thickness located in "gap" between M1 and M2. Main difference is that there isn't a π phase shift at one of the surfaces.

OPD = $2d \Rightarrow$ bright fringe when $2d = m\lambda$
 $d = m\frac{\lambda}{2}$

If we tip M_1 with respect to M_2 , we get straight line fringes



$$d(x) = d_0 +$$

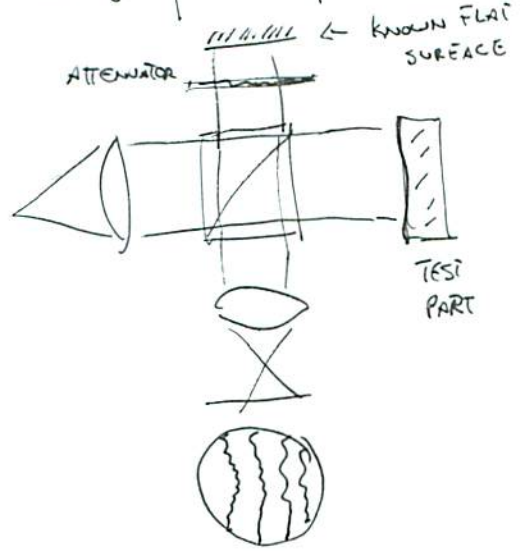


$$d(x) : d_0 + x \tan \alpha$$

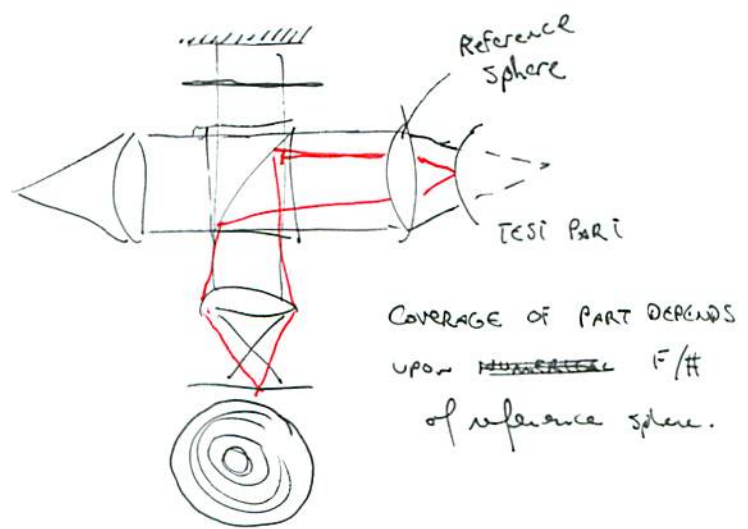
So fringes when $d_0 + x \tan \alpha = \frac{n\lambda}{2}$

If the fringes are not straight lines then one (or both) mirrors are not flat

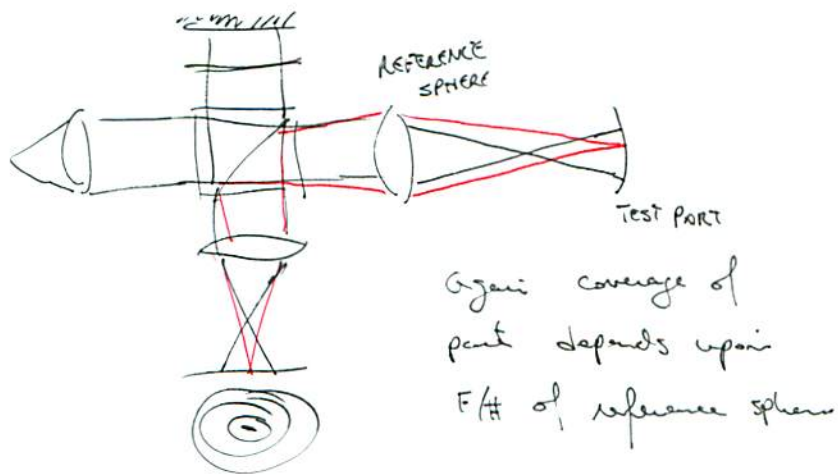
Testing flat surfaces



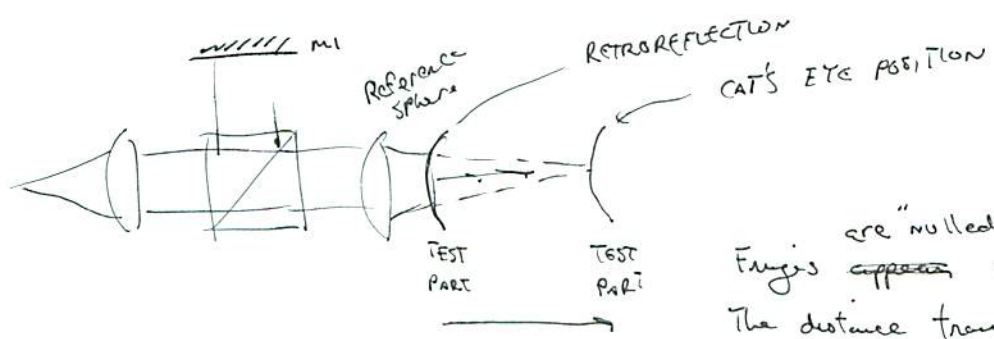
Testing Convex Part



Testing Concave Part



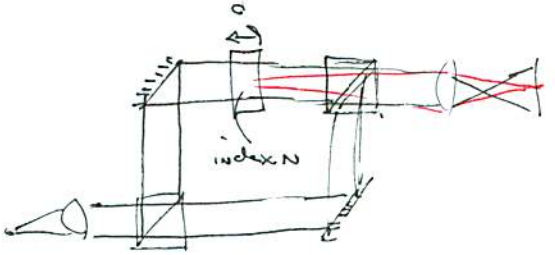
Measuring Radius of curvature



are "nulled"
Fringes appear at the two positions.
The distance traveled is the radius of the test part.

Fringes are "nulled" when plane wave from MI and recollimated reflection from test part are parallel.

4.5 | MACH-ZEHNDER INTERFEROMETER



measures paths in transmission
OPD = ~~nd~~ nd
BRIGHT FRINGE WHEN $nd = m\lambda$

EXAMPLE OF USE: SUPPOSE WANT TO MEASURE UNIFORMITY OF REFRACTIVE INDEX. CONSTANT THICKNESS SLAB OF GLASS, THEN FRINGE PATTERN DEPENDS ON INDEX vs. position

EXAMPLE: MEASURING A WINDOW

4.6 | LATERAL SHEARING INTERFEROMETERS

Split wavefront $w(x,y)$ and displace it by Δx . Combine displaced and original wavefront to get interference pattern

4.6.1 Common Configs

Shear plate

OPD = $w_1(x,y) - w_2(x+\Delta x, y)$ proportional to $\frac{dw}{dx}$

Defocus
 $w = w_{020} (x^2 + y^2) + w_{111} y$
OPD = $2w_{020} x \Delta x + w_{111} y$
lens introduced by wedge α

Sketch VIDEO

In the case of a shear plate

$$w_1(x, y) = w_{020} (x^2 + y^2)$$

$$w_2(x + \Delta x, y) = w_{020} ((x + \Delta x)^2 + y^2) + w_{111} y$$

So

$$OPD = w_1(x, y) - w_2(x + \Delta x, y) = w_{020} (x^2 + y^2) - w_{020} (x^2 + 2\Delta x \cdot x + \Delta x^2 + y^2) - w_{111} y$$

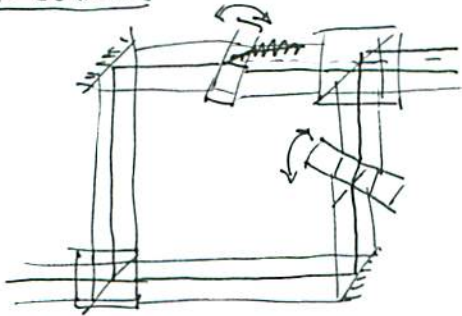
$$OPD = -2w_{020} \Delta x \cdot x - w_{020} \Delta x^2 - w_{111} y = m \lambda \text{ for bright fringe}$$

$m = \text{integer}$

$$y = \frac{-2w_{020} \Delta x}{w_{111}} x - \frac{w_{020} \Delta x^2 - m \lambda}{w_{111}} \text{ Eq. of a line}$$

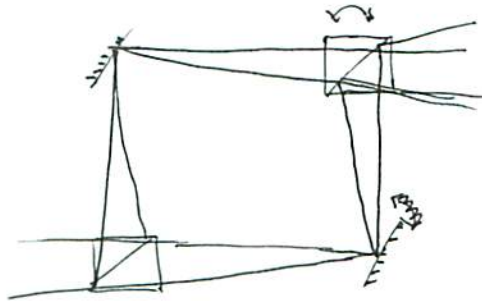
$$\text{slope} = \frac{-2w_{020} \Delta x}{w_{111}}$$

MACH-ZEHNDER



Rotate plane parallel plates to give shear

ROTATE FOR SHEAR



4.7 Interferograms

SHOW SLIDES

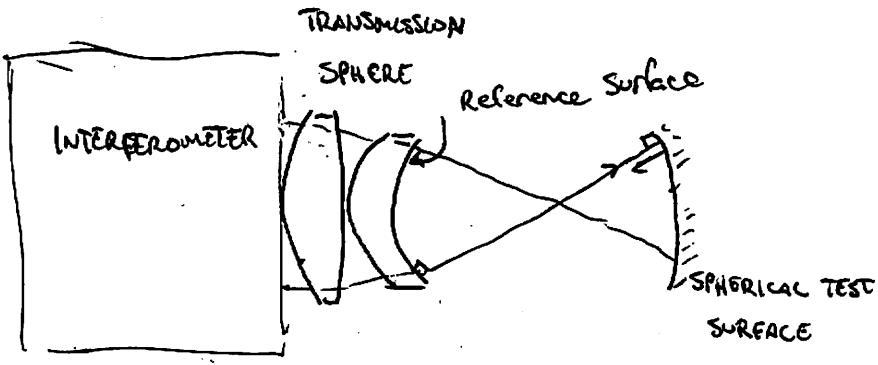
Visibility decrease with differences in beam intensities, but not that fast

Suppose $I_2 = 0.04 I_1$, roughly glass/mirror comparison

$$\text{Visibility} = \frac{2\sqrt{0.04} I_1}{I_1 (1.04)} = 0.385$$

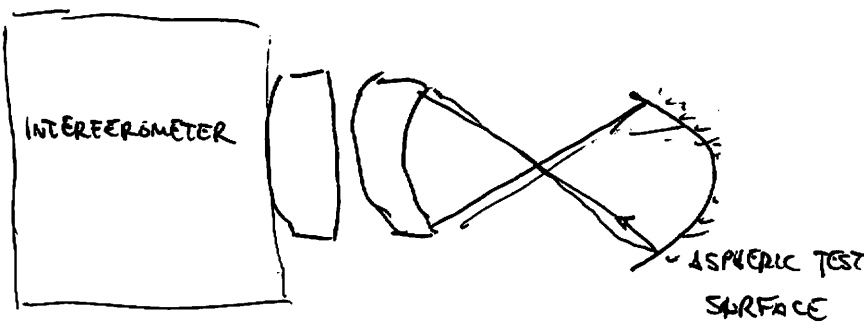
Aliasing occurs when fringes get too close together

Testing aspheres are especially prone to aliasing due to the inability to easily match the aspheric wavefront with the reference wavefront.



If everything is aligned then a null pattern would result. Typically put in slight tilt to get straight line fringes

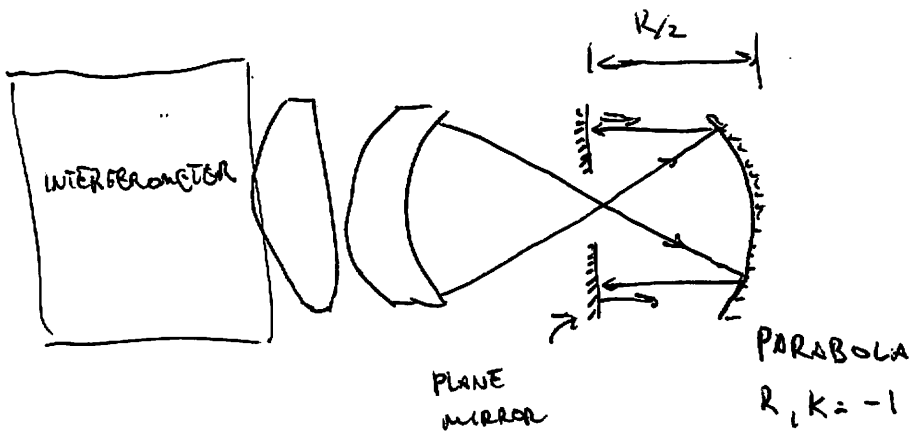
Rays pass through same regions of the transmission sphere on the way in and on way out. Aberrations from transmission sphere cancel.



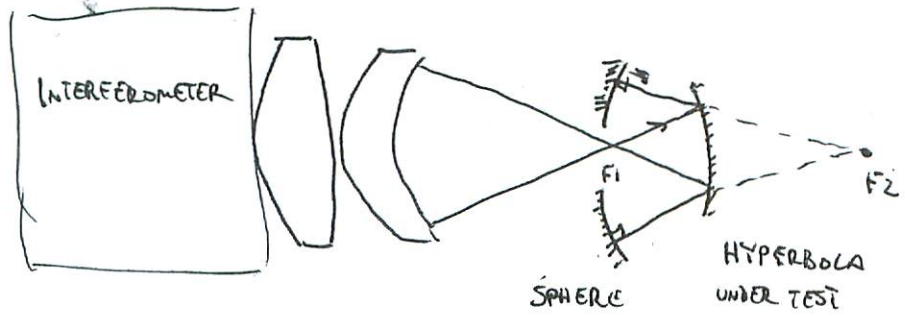
Return beam has high spherical aberration which can quickly cause aliasing. Also return rays pass through different parts of transmission sphere so retro errors appear (i.e. aberrations for transmission sphere enter into wavefront)

NULL Tests for ~~Conic~~ Aspheres

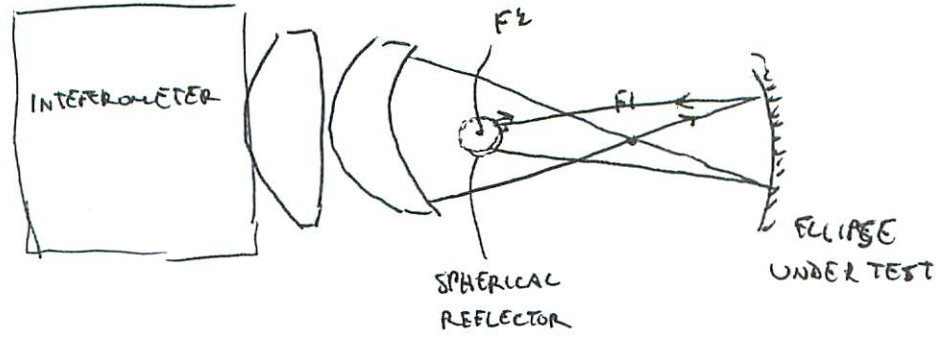
For aspheres we can sometimes get creative with setup to cancel the spherical aberration and still do null testing



Light passing through focal point of parabola will be collimated. Flat mirror retroreflects collimated beam



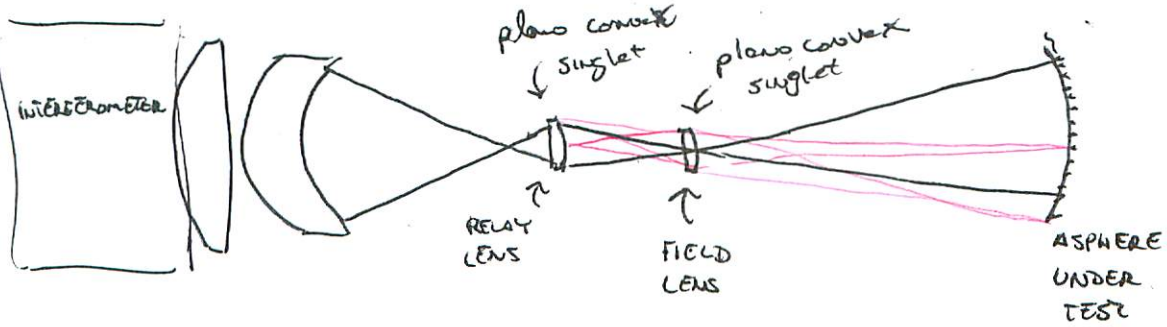
Center of curvature of spherical mirror is placed at F_2 of hyperbola to get retroreflection



Spherical reflector placed at F_2 of ellipse under test to get retroreflection

Problems with these setups are that the auxiliary mirror needs to be as large as the test mirror (parabola, hyperbola), center region of the test surface not tested, different test mirrors might require different auxiliary mirrors.

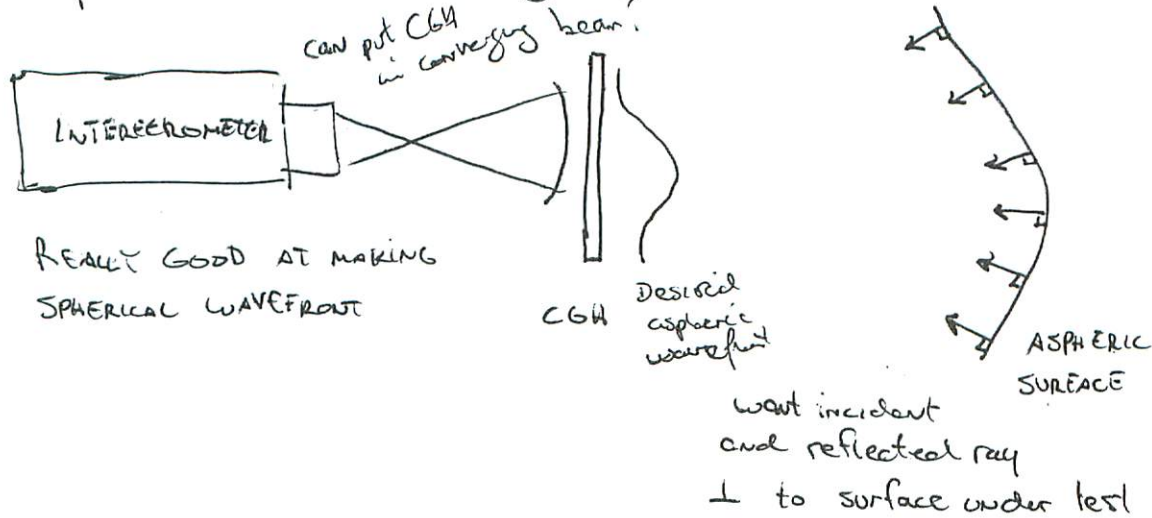
OFFNER RELAY



RELAY LENS IS INTENTIONALLY CHOSEN TO INTRODUCE SPHERICAL ABERRATION. THE FIELD LENS IMAGES THE RELAY LENS ONTO THE TEST SURFACE. CONSEQUENTLY THE SPHERICAL ABERRATION OF THE RELAY LENS IS CHOSEN TO CANCEL THE SPHERICAL ABERRATION INDUCED BY THE TEST SURFACE. THE FIELD LENS INDUCES MINIMAL SPHERICAL ABERRATION SINCE MARGINAL RAYS PASS THROUGH ITS CENTER.

HUBBLE SIDES

Computer Generated Holograms



We pick a plane somewhere between interferometer and test surface. The CGH is designed to convert incident spherical wavefront into an aspheric wavefront. This aspheric wavefront will evolve as it propagates so that it matches the test surface when it gets there.

This offers an advantage over traditional null correctors in that everything happens at the plane of the CGH. You still need to align the CGH properly

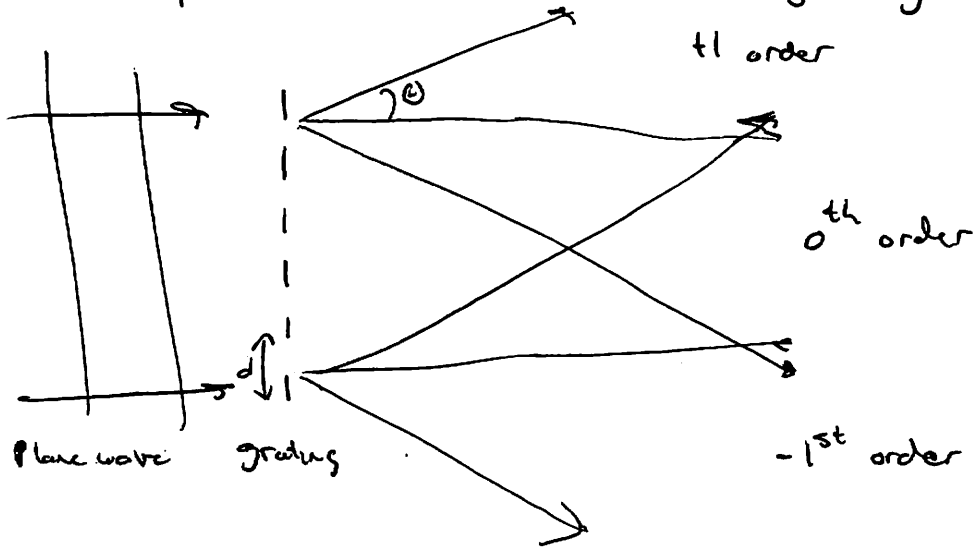
In an ideal world, we would just make a hologram from the interference of the spherical wave and the desired aspheric wavefront. However, creating that desired aspheric wavefront to record the hologram is challenging.

Alternatively, we can use Computer Generated Holograms (CGH) where the interference pattern is calculated numerically and then written to a substrate. To keep fabrication costs low, the CGH typically takes the form of a binary phase pattern for transmission or binary amplitude (e.g. chrome on glass) pattern for reflection. CGH good to roughly 0.5 μ m aspheric departure from best fit sphere, but also slope limited.

CGH Slide

CGH looks like a distorted grating.

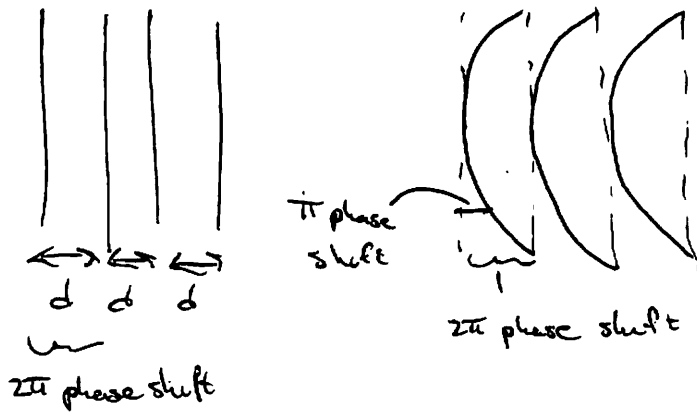
What are the properties of a conventional grating



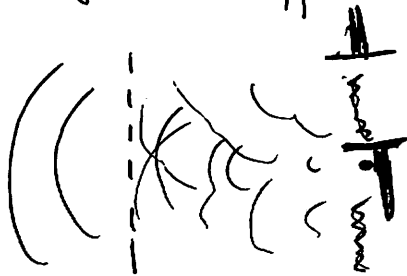
Dictated by grating equation, 0^{th} order beam unaffected, but other orders diffract in different directions.

$$d \sin \theta = m \lambda$$

For CGH, we distort the separate d locally to introduce phase shift



The desired phase pattern is created by forming transitions shifted some fraction of the average spacing d . The average spacing d is chosen so separate diffractive orders don't overlap



d is called a linear carrier

For example in converging beam, 0^{th} order focuses to point, ± 1 order carry distorted wavefront. We can place a stop to block 0^{th} and -1^{st} order and only pass $+1^{\text{st}}$ order which propagates to test surface

Types of CGHs

Amplitude (e.g. Chrome on glass)

25% into 0th order does nothing

10% into $\pm 1^{st}$ orders need to select one of these

Ok for highly reflective surfaces such as metal, but otherwise fringe contrast suffers.

Phase etch glass

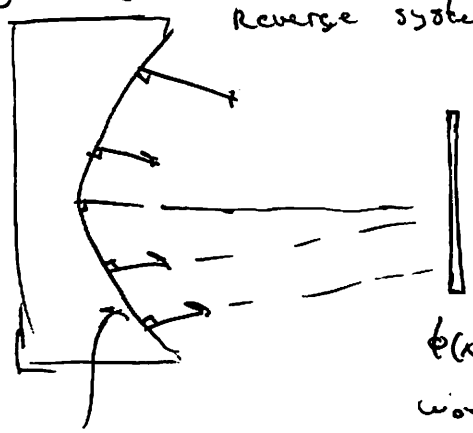
0% into 0th order

40% into $\pm 1^{st}$ orders again select one of these

Good for ~~phase~~ plain glass surfaces since only ~4% reflection from test surface. Need as much light as possible

How to Calculate CGH

Raytracing code can be used to determine phase profile of CGH. Reverse system



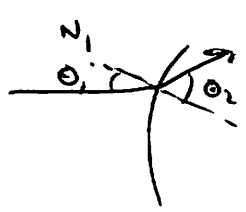
aspheric surface under test

$\phi(x,y)$

would like to figure out required phase at CGH plane

We would like to create rays that leave test surface such that they are \perp to surface.

How can we do this?



$N_2 = 1$

Snell's Law $N_1 \sin \theta_1 = N_2 \sin \theta_2$

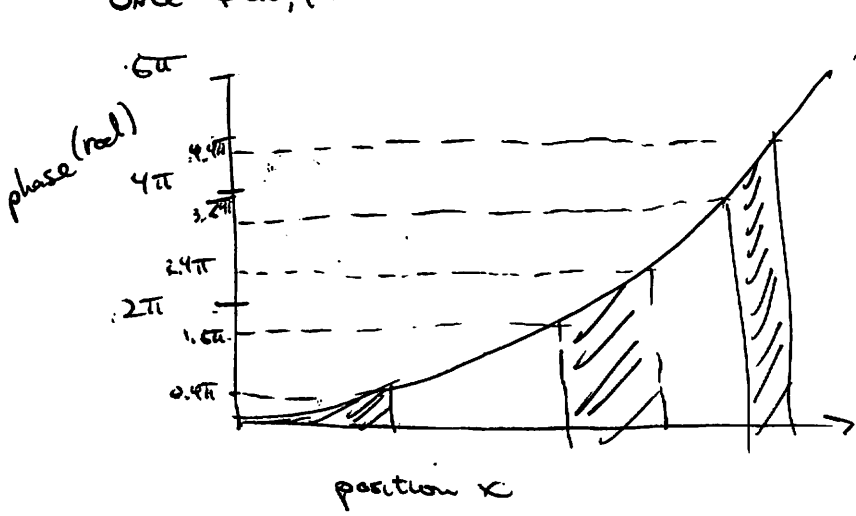
$\theta_2 = \sin^{-1} \left(\frac{N_1}{N_2} \sin \theta_1 \right)$

want $\theta_2 = 0$ irrespective of ~~theta_1~~ θ_1 .

We can do this by hacking raytrace code and setting $N_1 = 0$
Zemax demo

Fabrication of CGH

Once $\phi(x, y)$ is known at CGH plane

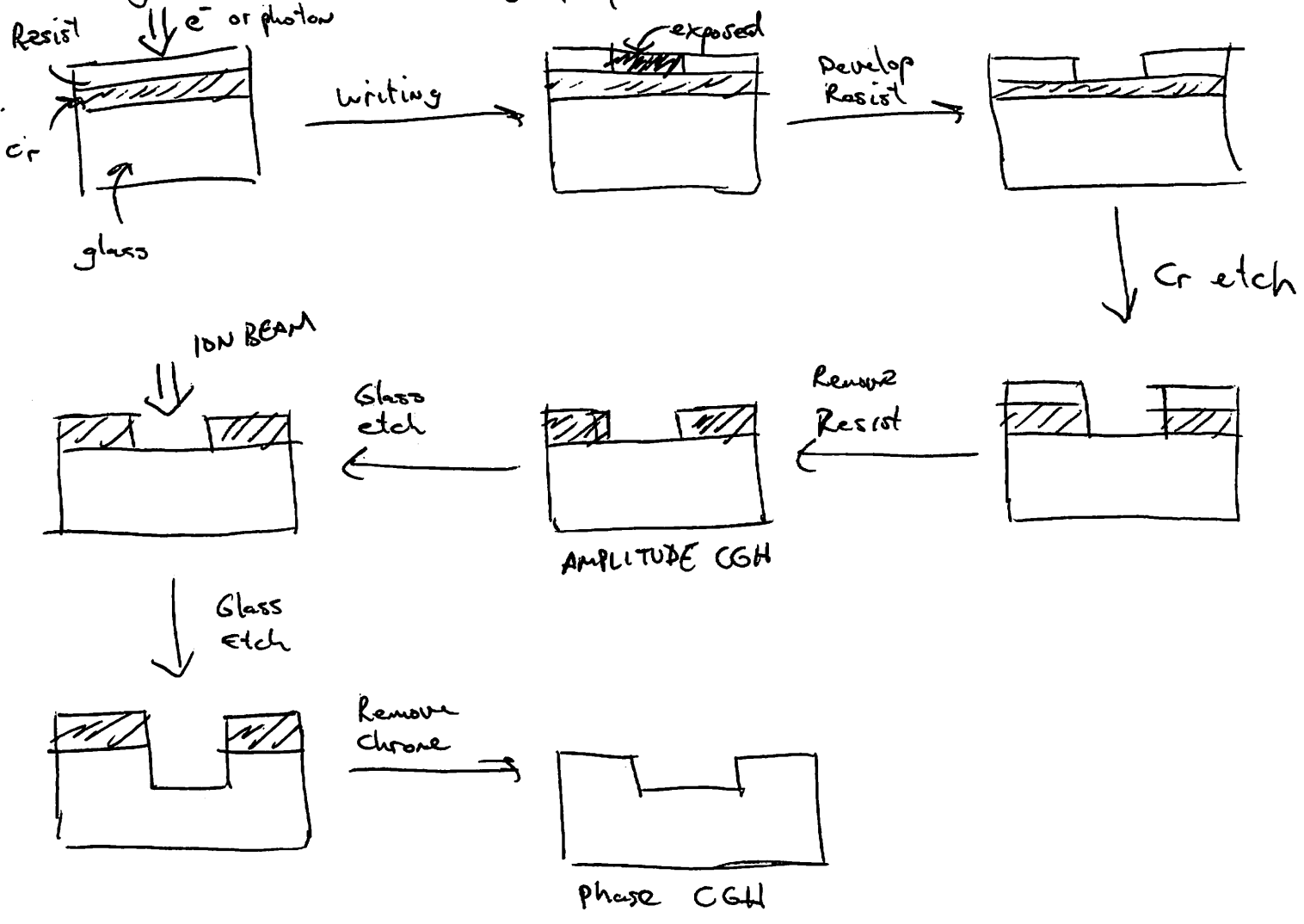


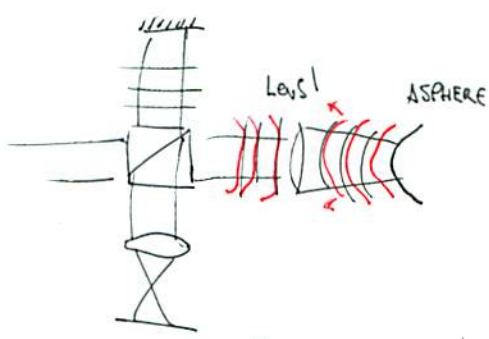
desired phase profile $\phi(x, y)$

For $\pm 1^{st}$ orders, create transitions on either side of multiples of 2π .
 Here, we want $\pm 0.4\pi$. Shaded regions would be chrome covered for amplitude CGHs or π phase shifted for phase gratings.

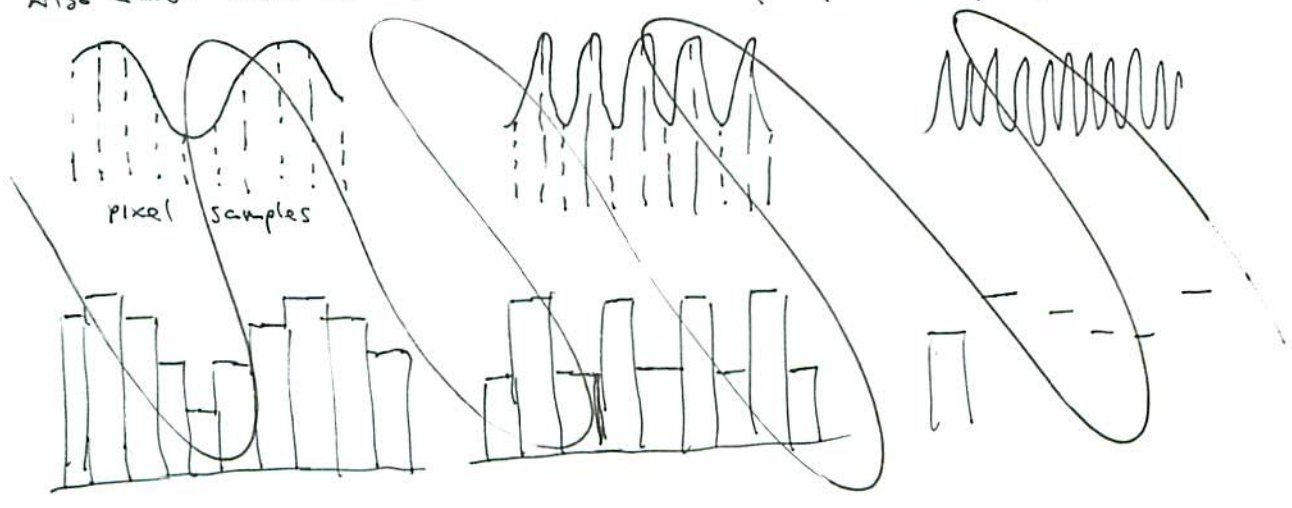
Other diffraction orders could be use too. For example $\pm 2^{nd}$ order would have transitions ~~at~~ every π radians.

Creating the CGH uses lithography techniques





Testing aspheres are especially prone to discussing due to the inability to easily match the aspheric wavefront and the reference wavefront. Also Lens 1 introduces aberrations because of asymmetric pass.



4.5 PHASE SHIFTING INTEROMETRY

Let's examine our basic interferometry expression:

$$I(x,y) = I_1(x,y) + I_2(x,y) + 2\sqrt{I_1(x,y)I_2(x,y)} \cos\left[\frac{2\pi}{\lambda}(\omega_1(x,y) - \omega_2(x,y)) + \phi\right]$$

Let's rewrite this as

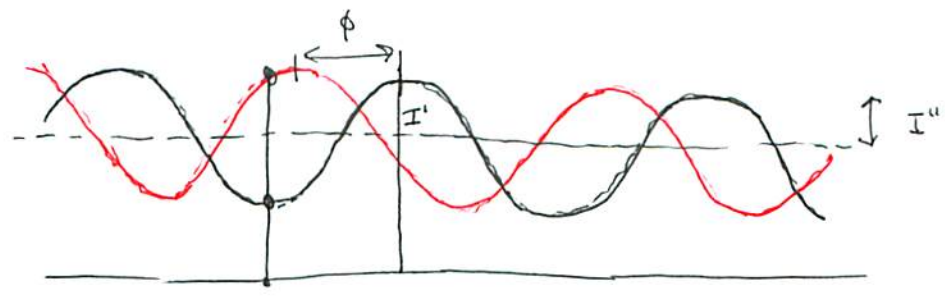
$$I(x,y) = I'(x,y) + I''(x,y) \cos[\Psi(x,y) + \phi]$$

where $I'(x,y) = I_1(x,y) + I_2(x,y)$

$$I''(x,y) = 2\sqrt{I_1(x,y)I_2(x,y)}$$

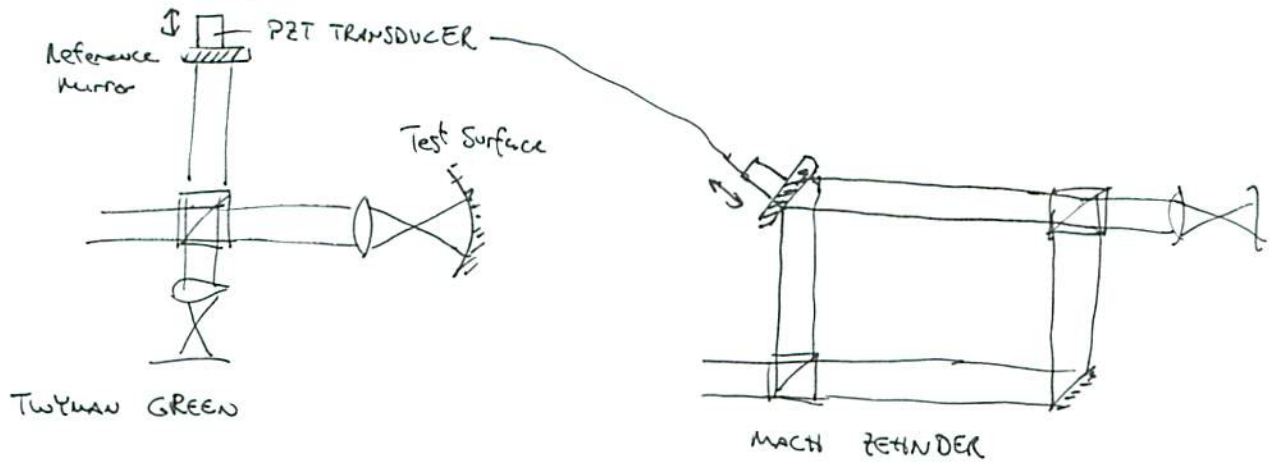
$$\Psi(x,y) = \frac{2\pi}{\lambda}(\omega_1(x,y) - \omega_2(x,y))$$

For any given point (x_0, y_0) , there are 3 unknowns $I'(x_0, y_0)$, $I''(x_0, y_0)$ and $\psi(x_0, y_0)$. If we make at least 3 measurements with different values of ϕ , then we can solve exactly for I' , I'' and ψ . Furthermore, this holds for every pixel in the interferogram

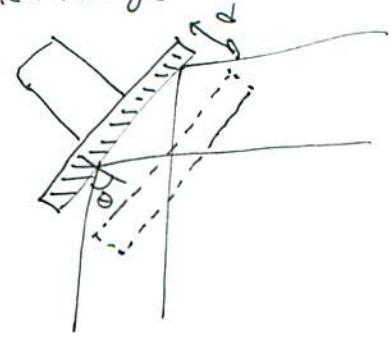


ϕ shifts the cosine pattern, a given pixel sees different intensity values as a function of ϕ . The pattern repeats for ϕ equal to multiples of 2π .

How to change ϕ between measurements



PZT = Piezo electric Transducer - ceramic that expands and contracts with applied voltage.



wavefront displacement = $2d \cos \theta$
 phase shift = $\frac{4\pi}{\lambda} d \cos \theta$

For simplicity let's start with four measurements with

$$\phi = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$I_1(x, y) = I'(x, y) + I''(x, y) \cos[\psi(x, y)]$$

$$I_2(x, y) = I'(x, y) + I''(x, y) \cos\left[\psi(x, y) + \frac{\pi}{2}\right]$$

$$I_3(x, y) = I'(x, y) + I''(x, y) \cos[\psi(x, y) + \pi]$$

$$I_4(x, y) = I'(x, y) + I''(x, y) \cos\left[\psi(x, y) + \frac{3\pi}{2}\right]$$

Using our trig identities, this reduces to

$$I_1 = I' + I'' \cos[\psi]$$

drop (x, y) for now

$$I_2 = I' - I'' \sin[\psi]$$

$$I_3 = I' - I'' \cos[\psi]$$

$$I_4 = I' + I'' \sin[\psi]$$

Note

$$I_4 - I_2 = 2I'' \sin[\psi]$$

$$I_1 - I_3 = 2I'' \cos[\psi]$$

Taking the ratio

$$\frac{I_4 - I_2}{I_1 - I_3} = \tan \psi \Rightarrow$$

$$\psi(x, y) = \tan^{-1} \left[\frac{I_4 - I_2}{I_1 - I_3} \right]$$

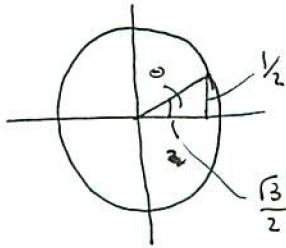
This can be related back to OPD as well

$$\text{OPD}(x, y) = \frac{\lambda}{2\pi} \psi(x, y)$$

Recall that Visibility = $\frac{\sqrt{I_1 I_2}}{I_1 + I_2} = \frac{I''(x,y)}{I'(x,y)}$

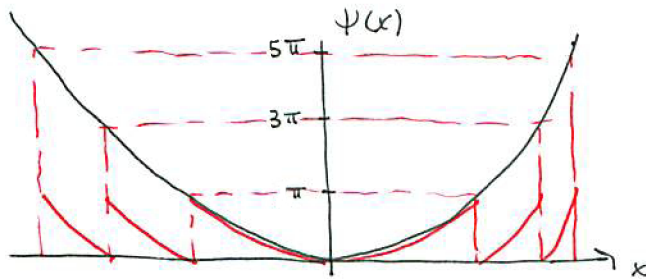
Visibility = $\frac{2\sqrt{(I_4 - I_2)^2 + (I_1 - I_3)^2}}{I_1 + I_2 + I_3 + I_4}$ Holds for all pixels (x,y)

WRAPPED PHASE



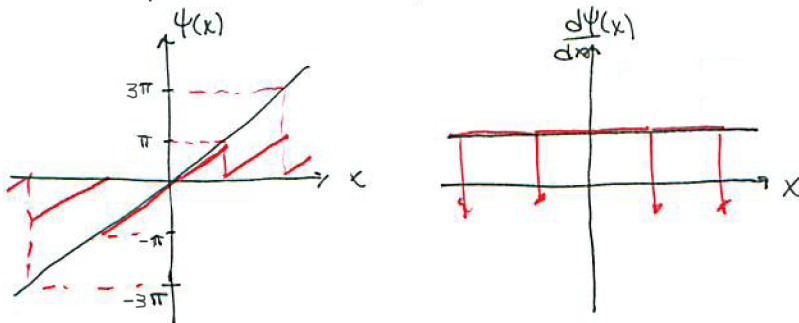
What's $\tan^{-1}\left[\frac{1/2}{\sqrt{3}/2}\right] = \dots -330^\circ, \underline{30^\circ}, 390^\circ \dots$

The problem with $\psi(x,y) = \tan^{-1}\left[\frac{I_4 - I_2}{I_1 - I_3}\right]$ is that we don't know how many 360's (2π's) to add or subtract.



Phase Unwrapping Algorithms - These are a general class of algorithms which take the wrapped phase and tries to unwrap it to recover the true, smooth $\psi(x,y)$. Conceptually, find every place with a discontinuity and add 2π's until the surface is smooth. In practice, for discrete data with noise, this problem gets more complex. Many algorithms exist.

Example: Zernike slope fit



(1) Approximate slope as $\frac{\psi(x) - \psi(x+\Delta x)}{\Delta x}$

(2) Fit slope to Zernike derivatives

$\frac{d\psi(x)}{dx} = \sum a_{nm} \frac{dZ_n^m}{dx}$

(3) $\psi^{(1)}(x) = \sum a_{nm} Z_n^m$

(4) $z(x) = \psi(x) - \psi^{(1)}(x)$

(5) Repeat with $z(x)$

$\psi(x) \approx \psi^{(1)}(x) + \psi^{(2)}(x) + \dots + \psi^{(n)}(x)$

Three Step Algorithm

(55)

$$\phi = -\alpha, 0, \alpha$$

$$\Psi(x, y) = \tan^{-1} \left[\left(\frac{1 - \cos \alpha}{\sin \alpha} \right) \frac{I_1 - I_3}{2I_2 - I_1 - I_3} \right]$$