OPTI 435/535 Midterm 2008 Solutions

Problem 1

Write out the Zernike functions explicitly

```
W[ρ_, θ_] = a22 * Sqrt[6] * ρ^2 * Cos[2 * θ] + a31 * Sqrt[8] * (3 * ρ^3 - 2 * ρ) * Cos[θ]
```

$2\sqrt{2}\ a31\ (-2\rho + 3\rho^3)\ Cos[\theta] + \sqrt{6}\ a22\ \rho^2\ Cos[2\theta]$

Next calculate the mean wavefront error by evaluating the integral

$$Wmean = (1/\pi) * \int_0^{2*\pi}\left(\int_0^1 W[\rho, \theta] * \rho\, d\rho\right) d\theta$$

0

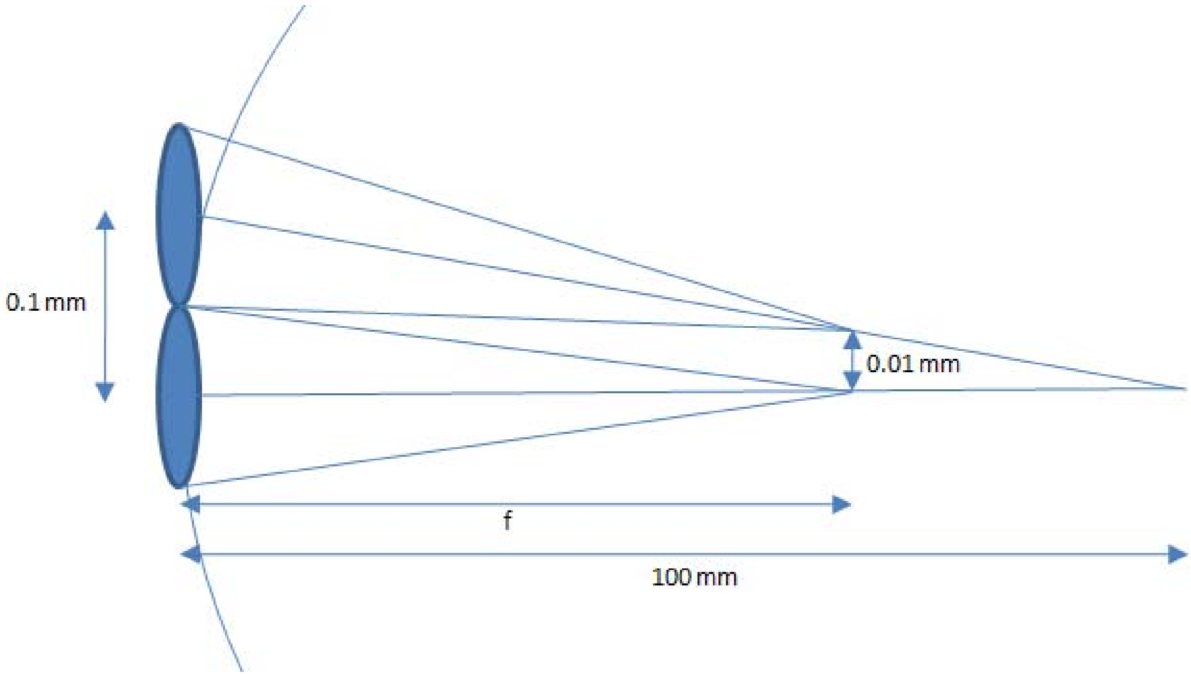
In general, the mean of all of the Zernike polynomials, except Z (0, 0) is zero. Next, calculate the wavefront variance by evaluating

$$Wvariance = (1/\pi) * \int_0^{2*\pi}\left(\int_0^1 (W[\rho, \theta] - Wmean)^2 * \rho\, d\rho\right) d\theta$$

$a22^2 + a31^2$

which is the answer we were looking for.

Problem 2



A wavefront with 10 D of myopia will converge 1/10 D = 100 mm from the lenslet array. Each lenslet is spaced 0.1 mm and the minimum resolution of the sensor is 0.01 mm. The required focal length can be obtained from using similar triangles and the figure above.

```
Solve[0.1 / 100 == 0.01 / (100 - f), f]
```

```
{{f → 90.}}
```

Problem 3

```
f = (R - Sqrt[R^2 - (K + 1) * (x^2 + y^2)]) / (K + 1)
```

$$\frac{R - \sqrt{R^2 - (1 + K)\,(x^2 + y^2)}}{1 + K}$$

Calculate the first and second derivatives in x and y

```
dfdx = D[f, x] /. {x^2 + y^2 → r^2}
dfdy = D[f, y] /. {x^2 + y^2 → r^2}
d2fdx2 = Simplify[D[f, {x, 2}]] /. {x^2 + y^2 → r^2}
d2fdxdy = D[D[f, y], x] /. {x^2 + y^2 → r^2}
d2fdy2 = Simplify[D[f, {y, 2}]] /. {x^2 + y^2 → r^2}
```

$$\frac{x}{\sqrt{-(1+K)\, r^2 + R^2}}$$

$$\frac{y}{\sqrt{-(1+K)\, r^2 + R^2}}$$

$$\frac{R^2 - (1+K)\, y^2}{(-(1+K)\, r^2 + R^2)^{3/2}}$$

$$\frac{(1+K)\, x\, y}{(-(1+K)\, r^2 + R^2)^{3/2}}$$

$$\frac{R^2 - (1+K)\, x^2}{(-(1+K)\, r^2 + R^2)^{3/2}}$$

Calculate the first fundamental forms

```
E0 = Simplify[Together[1 + dfdx^2]]
F0 = dfdx * dfdy
G0 = Simplify[Together[1 + dfdy^2]]
```

$$-\frac{-(1+K)\, r^2 + R^2 + x^2}{(1+K)\, r^2 - R^2}$$

$$\frac{x\, y}{-(1+K)\, r^2 + R^2}$$

$$-\frac{-(1+K)\, r^2 + R^2 + y^2}{(1+K)\, r^2 - R^2}$$

Calculate the quantitiy EG - F^2

```
rad = Simplify[E0 * G0 - F0^2] /. {x^2 + y^2 → r^2}
```

$$-\frac{r^2 - (1+K)\, r^2 + R^2}{(1+K)\, r^2 - R^2}$$

$$\texttt{rad} = -\frac{\texttt{Collect}[r^2 - (1+K)\, r^2 + R^2, r]}{(1+K)\, r^2 - R^2}$$

$$-\frac{-K\, r^2 + R^2}{(1+K)\, r^2 - R^2}$$

Calculate the Second fundamental Form

```
L0 = d2fdx2 / Sqrt[rad]
M0 = d2fdxdy / Sqrt[rad]
N0 = d2fdy2 / Sqrt[rad]
```

$$\frac{R^2 - (1+K)\, y^2}{\sqrt{-\frac{-K r^2+R^2}{(1+K) r^2-R^2}}\; \left(-(1+K)\, r^2 + R^2\right)^{3/2}}$$

$$\frac{(1+K)\, x\, y}{\sqrt{-\frac{-K r^2+R^2}{(1+K) r^2-R^2}}\; \left(-(1+K)\, r^2 + R^2\right)^{3/2}}$$

$$\frac{R^2 - (1+K)\, x^2}{\sqrt{-\frac{-K r^2+R^2}{(1+K) r^2-R^2}}\; \left(-(1+K)\, r^2 + R^2\right)^{3/2}}$$

Simplifying gives

$$\mathbf{L0} = \frac{R^2 - (1+K)\, y^2}{\sqrt{-K * r^2 + R^2}\; \left(-(1+K)\, r^2 + R^2\right)}$$

$$\mathbf{M0} = \frac{(1+K) * x * y}{\sqrt{-K * r^2 + R^2}\; \left(-(1+K)\, r^2 + R^2\right)}$$

$$\mathbf{N0} = \frac{R\verb|^|2 - (1+K) * x\verb|^|2}{\sqrt{-K * r^2 + R^2}\; \left(-(1+K)\, r^2 + R^2\right)}$$

$$\frac{R^2 - (1+K)\, y^2}{\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}}$$

$$\frac{(1+K)\, x\, y}{\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}}$$

$$\frac{R^2 - (1+K)\, x^2}{\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}}$$

Let's now play with the mean curvature numerator

```
E0 * N0 + G0 * L0 + 2 * F0 * M0
```

$$-\frac{\left(-(1+K)\, r^2 + R^2 + x^2\right)\left(R^2 - (1+K)\, x^2\right)}{\left((1+K)\, r^2 - R^2\right)\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}} + \frac{2\,(1+K)\, x^2\, y^2}{\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}\;\left(-(1+K)\, r^2 + R^2\right)} - \frac{\left(-(1+K)\, r^2 + R^2 + y^2\right)\left(R^2 - (1+K)\, y^2\right)}{\left((1+K)\, r^2 - R^2\right)\left((-1-K)\, r^2 + R^2\right)\sqrt{-K r^2 + R^2}}$$

$$-\frac{(-(1+K)\,r^2+R^2+x^2)\,(R^2-(1+K)\,x^2)}{\text{Together}[((1+K)\,r^2-R^2)\,((-1-K)\,r^2+R^2)]\,\sqrt{-K\,r^2+R^2}}-\frac{(-(1+K)\,r^2+R^2+y^2)\,(R^2-(1+K)\,y^2)}{\text{Together}[((1+K)\,r^2-R^2)\,((-1-K)\,r^2+R^2)]\,\sqrt{-K\,r^2+R^2}}+\frac{2\,(1+K)\,x^2\,y^2}{\text{Together}[((-1-K)\,r^2+R^2)\,(-(1+K)\,r^2+R^2)]\,\sqrt{-K\,r^2+R^2}}$$

$$\frac{((-1-K)\,r^2+R^2+x^2)\,(R^2-(1+K)\,x^2)}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}+\frac{2\,(1+K)\,x^2\,y^2}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}+\frac{((-1-K)\,r^2+R^2+y^2)\,(R^2-(1+K)\,y^2)}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}$$

$$\frac{1}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}\Big(\text{Distribute}\big[\big(-(1+K)\;*\,(x\text{^}2+y\text{^}2)+R^2\big)*\big(R^2-(1+K)\,x^2\big)\big]+\text{Distribute}\big[x^2*\big(R^2-(1+K)\,x^2\big)\big]+2\,(1+K)\,x^2\,y^2+\text{Distribute}\big[\big(-(1+K)\;*\,(x\text{^}2+y\text{^}2)+R^2\big)*\big(R^2-(1+K)\,y^2\big)\big]+\text{Distribute}\big[y^2*\big(R^2-(1+K)\,y^2\big)\big]\Big)$$

$$\frac{1}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}\,\text{Simplify}\Big[\big(2\,R^4+R^2\,x^2-(1+K)\,R^2\,x^2-(1+K)\,x^4+R^2\,y^2-(1+K)\,R^2\,y^2+2\,(1+K)\,x^2\,y^2-(1+K)\,y^4+2\,(-1-K)\,R^2\,(x^2+y^2)-(-1-K)\,(1+K)\,x^2\,(x^2+y^2)-(-1-K)\,(1+K)\,y^2\,(x^2+y^2)\big)\Big]$$

$$\frac{2\,R^4+4*(K+1)*x^2\,y^2-(2+3\,K)\,R^2*r\text{^}2+K^2\;*r\text{^}4+K*\text{Factor}[\,(x^4+2\,x^2\,y^2+y^4)\,]}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}$$

$$\text{num1}=\frac{K^2\,r^4-(2+3\,K)\,r^2\,R^2+2\,R^4+4\,(1+K)\,x^2\,y^2+K\;*r\text{^}4}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}$$

$$\frac{K\,r^4+K^2\,r^4-(2+3\,K)\,r^2\,R^2+2\,R^4+4\,(1+K)\,x^2\,y^2}{(r^2+K\,r^2-R^2)^2\,\sqrt{-K\,r^2+R^2}}$$

num1 / rad

$$-\frac{((1+K)\,r^2-R^2)\,(K\,r^4+K^2\,r^4-(2+3\,K)\,r^2\,R^2+2\,R^4+4\,(1+K)\,x^2\,y^2)}{(r^2+K\,r^2-R^2)^2\,(-K\,r^2+R^2)^{3/2}}$$

The Mean Curvatures is given by

H0 = Apart[Simplify[num1 / rad]] /. {x^2 + y^2 → r^2}

$$\text{Simplify}\left[\frac{R^2\sqrt{-K r^2+R^2}}{(K r^2-R^2)^2}\right]-\text{Simplify}\left[\frac{\sqrt{-K r^2+R^2}}{K r^2-R^2}\right]-$$
$$\text{Simplify}\left[\frac{4\sqrt{-K r^2+R^2}\ x^2 y^2}{r^2\ (K r^2-R^2)^2}\right]-\text{Simplify}\left[\frac{4 R^2\sqrt{-K r^2+R^2}\ x^2 y^2}{r^4\ (K r^2-R^2)^2}\right]+$$
$$\text{Simplify}\left[\frac{4 R^2\sqrt{-K r^2+R^2}\ x^2 y^2}{r^6\ (K r^2-R^2)}\right]-\text{Simplify}\left[\frac{4 R^2\sqrt{-K r^2+R^2}\ x^2 y^2}{r^6\ (r^2+K r^2-R^2)}\right]$$

$$\text{Together}\left[\frac{R^2}{(-K r^2+R^2)^{3/2}}+\frac{1}{\sqrt{-K r^2+R^2}}\right]-\frac{4 R^2 x^2 y^2}{r^4\ (-K r^2+R^2)^{3/2}}-$$
$$\frac{4 R^2 x^2 y^2}{r^6\sqrt{-K r^2+R^2}}-\frac{4\sqrt{-K r^2+R^2}\ x^2 y^2}{(K r^3-r R^2)^2}-\frac{4 R^2\sqrt{-K r^2+R^2}\ x^2 y^2}{(1+K)\ r^8-r^6 R^2}$$

$$\frac{-K r^2+2 R^2}{(-K r^2+R^2)^{3/2}}-$$
$$\text{Together}\left[\frac{4 R^2 x^2 y^2}{r^4\ (-K r^2+R^2)^{3/2}}+\frac{4 R^2 x^2 y^2}{r^6\sqrt{-K r^2+R^2}}+\frac{4\sqrt{-K r^2+R^2}\ x^2 y^2}{(K r^3-r R^2)^2}+\frac{4 R^2\sqrt{-K r^2+R^2}\ x^2 y^2}{(1+K)\ r^8-r^6 R^2}\right]$$

$$\frac{-K r^2+2 R^2}{(-K r^2+R^2)^{3/2}}+\frac{\text{Simplify}[4\ (x^2 y^2+K x^2 y^2)]}{(-K r^2+R^2)^{3/2}\ (-r^2-K r^2+R^2)}$$

$$\text{H0}=\frac{-K r^2+2 R^2}{(-K r^2+R^2)^{3/2}}+\frac{4\ (1+K)\ x^2 y^2}{(-K r^2+R^2)^{3/2}\ \text{Simplify}[(-r^2-K r^2+R^2)]}$$

$$\text{Together}\left[\frac{-K r^2+2 R^2}{(-K r^2+R^2)^{3/2}}+\frac{4\ (1+K)\ x^2 y^2}{(-K r^2+R^2)^{3/2}\ (-(1+K)\ r^2+R^2)}\right]$$

$$\frac{K r^4+K^2 r^4-2 r^2 R^2-3 K r^2 R^2+2 R^4+4 x^2 y^2+4 K x^2 y^2}{(K r^2-R^2)\ (r^2+K r^2-R^2)\ \sqrt{-K r^2+R^2}}$$

The Gaussian Curvature is given by

```
K0 = Simplify[(L0 * N0 - M0^2) / rad] /. {x^2 + y^2 → r^2}
```

$$\frac{R^2}{(-K r^2+R^2)^2}$$

Let' s examine H0^2 - K0

```
rad2 = FullSimplify[H0^2 - K0]
```

$$\frac{K r^2 R^2-R^4+\left(K r^2-2 R^2+\frac{4\ (1+K)\ x^2 y^2}{(1+K)\ r^2-R^2}\right)^2}{(-K r^2+R^2)^3}$$

Finally, the Principal Curvatures are given by

```
κ1 = Simplify[H0 + Sqrt[rad2]]
κ2 = Simplify[H0 - Sqrt[rad2]]
```

$$\frac{-K r^2+2 R^2}{\left(-K r^2+R^2\right)^{3/2}}+\frac{4 (1+K) x^2 y^2}{\left(-K r^2+R^2\right)^{3/2} \left(-(1+K) r^2+R^2\right)}+\sqrt{\frac{K r^2 R^2-R^4+\left(K r^2-2 R^2+\frac{4 (1+K) x^2 y^2}{(1+K) r^2-R^2}\right)^2}{\left(-K r^2+R^2\right)^3}}$$

$$\frac{-K r^2+2 R^2}{\left(-K r^2+R^2\right)^{3/2}}+\frac{4 (1+K) x^2 y^2}{\left(-K r^2+R^2\right)^{3/2} \left(-(1+K) r^2+R^2\right)}-\sqrt{\frac{K r^2 R^2-R^4+\left(K r^2-2 R^2+\frac{4 (1+K) x^2 y^2}{(1+K) r^2-R^2}\right)^2}{\left(-K r^2+R^2\right)^3}}$$

Problem 4

	Sphere	Cylinder	Axis	J0	J45	M
Rx1	1	2	40	-0.174	-0.985	2.000
Rx2	1	2	30	-0.500	-0.866	2.000
Net	**2.03**	**3.94**	**125.00**	0.674	1.851	4.000
Net	**5.97**	**-3.94**	**35.00**	0.674	1.851	4.000