Description of Zernike Polynomials
Jim Schwiegerling

The Zernike polynomials are a set of functions that are orthogonal over the unit circle. They are useful for describing the shape of an aberrated wavefront in the pupil of an optical system or true height corneal topography. There exist several different normalization and numbering schemes for these polynomials. The purpose of this outline is to describe a standard for presenting Zernike data as it relates to aberration theory of the eye. Useful derivations and tables are also included.

1. Definition and Indexing Scheme

1.1 Double Indexing Scheme

The Zernike polynomials are usually defined in polar coordinates (ρ,θ), where ρ is the radial coordinate ranging from 0 to 1 and θ is the azimuthal component ranging from 0 to 2π. Each of the Zernike polynomials consists of three components: a normalization factor, a radial dependent component and an azimuthal dependent component. The radial component is a polynomial, whereas the azimuthal component is sinusoidal. A double indexing scheme is useful for unambiguously describing the functions, with the index n describing the highest power or order of the radial polynomial and the index m describing the azimuthal frequency of the azimuthal component. In general, the Zernike polynomials are defined as

$$Z_n^m(\rho,\theta) = \begin{cases} N_n^m R_n^{|m|}(\rho)\cos m\theta & ;\text{for } m \geq 0 \\ -N_n^m R_n^{|m|}(\rho)\sin m\theta & ;\text{for } m < 0 \end{cases} \quad , \quad (1)$$

Where N_n^m is the normalization factor described in more detail below and $R_n^m(\rho)$ is given by

$$R_n^{|m|}(\rho) = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!\left[0.5(n+|m|)-s\right]!\left[0.5(n-|m|)-s\right]!}\rho^{n-2s} \quad . \quad (2)$$

This definition uniquely describes the Zernike polynomials except for the normalization constant.

The normalization is given by

$$N_n^m = \sqrt{\frac{2(n+1)}{1+\delta_{m0}}} \quad , \quad (3)$$

where δ_{m0} is the Kronecker delta function (i.e . $\delta_{m0} = 1$ for $m = 0$, and $\delta_{m0} = 0$ for $m \neq 0$).

NOTE: The value of n is a positive integer or zero. For a given n, m can only take on values -n, -n + 2, -n + 4, ...n.

When describing individual Zernike terms, the two-index scheme should always be used.

1.2 Single Indexing Scheme

Occasionally, a single indexing scheme is useful for describing Zernike expansion coefficients. Since the polynomials actually depend on two parameters, n and m, ordering of a single indexing scheme is arbitrary. To avoid confusion, a standard single indexing scheme should be used. To obtain the single index, j, it is convenient to lay out the polynomials in a pyramid.

n\m	-5	-4	-3	-2	-1	0	+1	+2	+3	+4	+5
0						j=0					
1					1		2				
2				3		4		5			
3			6		7		8		9		
4		10		11		12		13		14	
5	15		16		17		18		19		20

Thus, the single index, j, starts at the top of the pyramid and steps down from left to right. To convert between j and the values of n and m, the following relationships can be used:

$$j = \frac{n(n+1)}{2} + \frac{n+m}{2} \quad (4)$$

and

$$n = \text{roundup}\left[\frac{-3+\sqrt{9+8j}}{2}\right] \quad (5)$$

$$m = 2j - n(n+2). \quad (6)$$

2. Coordinate System

Typically, a right-handed coordinate system is used in scientific applications. The +z Cartesian axis is taken as pointing out of the eye. The +y axis is oriented vertically and the x axis is therefore horizontal, as shown in Figure 1 below. Conventional definitions of the polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ are shown as well. This definition gives $x = r\cos\theta$ and $y = r\sin\theta$. Malacara[2] uses a polar coordinate system in which $x = r\sin\theta$ and $y = r\cos\theta$. In other words, θ is measured clockwise from the +y axis (Figure 1b), instead of counterclockwise from the +x axis (Figure 1a). This definition stems from early (pre-computer) aberration theory. I am not sure of the purpose other than it leaves only cos mθ terms for a rotationally symmetric optical system.

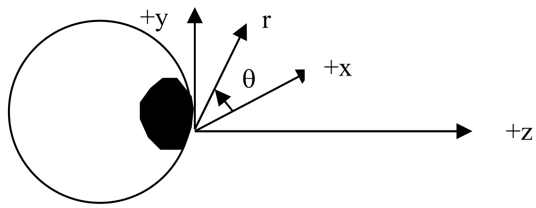

Figure 1. Conventional right-handed coordinate system: Cartesian and Polar.

The Zernike polynomials are only orthogonal over the unit circle, which means that normalized coordinates are typically used when representing functions or surfaces. Normalized polar coordinates (ρ,θ) are related to regular polar coordinates (r,θ) by

$$\rho = r / r_{max} \quad (7)$$

where r_{max} is the maximum radial extent of the function or surface. Note that the θ coordinate is not affected. It is also sometimes useful to consider the Zernike polynomials in Cartesian coordinates. Again, normalized coordinates are useful for taking advantage of the orthogonality properties of Zernikes. The normalized Cartesian coordinate (X,Y) are related to the regular Cartesian coordinates (x,y) by

$$X = x / r_{max} \quad \text{and} \quad Y = y / r_{max} \quad (8)$$

where again r_{max} is the maximum radial extent of the function or surface.

2.1 Conversion of Zernike Polynomials from Polar to Cartesian Coordinates

Since the Zernikes depend on cos mθ and sin mθ, examine the complex relationship $e^{i|m|\theta}$.

$$e^{i|m|\theta} = \cos|m|\theta + i\sin|m|\theta = \left[e^{i\theta}\right]^{|m|} = \left[\cos\theta + i\sin\theta\right]^{|m|} = \left[\frac{X}{\rho} + i\frac{Y}{\rho}\right]^{|m|}$$

From the Binomial Expansion

$$\left[\frac{X}{\rho} + i\frac{Y}{\rho}\right]^{|m|} = \sum_{k=0}^{|m|} \binom{|m|}{k} \left[\frac{iY}{\rho}\right]^{k} \left[\frac{X}{\rho}\right]^{|m|-k} = \sum_{k=0}^{|m|} \rho^{-k} \rho^{-|m|+k} \binom{|m|}{k} (i)^{k} X^{|m|-k} Y^{k}$$

Since cos |m|θ equals the real part of the preceding equation and sin |m|θ equals the imaginary part, the following relationships hold

$$\cos|m|\theta = \sum_{k=0(\text{even})}^{|m|} (-1)^{k/2} \binom{|m|}{k} \rho^{-|m|} X^{|m|-k} Y^{k} \qquad (9)$$

and

$$\sin|m|\theta = \sum_{k=1(\mathrm{odd})}^{|m|} (-1)^{(k-1)/2} \binom{|m|}{k} \rho^{-|m|} X^{|m|-k} Y^{k}, \quad (10)$$

which in turn collapse to

$$\cos|m|\theta = \sum_{k=0}^{|m|/2} (-1)^{k} \binom{|m|}{2k} \rho^{-|m|} X^{|m|-2k} Y^{2k} \qquad (11)$$

and

$$\sin|m|\theta = \sum_{k=0}^{(|m|-1)/2} (-1)^{k} \binom{|m|}{2k+1} \rho^{-|m|} X^{|m|-(2k+1)} Y^{(2k+1)}. \quad (12)$$

Taking advantage of equations (1)-(3), (10-12) and that $\rho^2 = X^2 + Y^2$,

For m>0 (13a)

$$Z_n^m(X,Y) = \sqrt{2(n+1)} \sum_{s=0}^{\frac{n-m}{2}} \sum_{j=0}^{\frac{n-m}{2}-s} \sum_{k=0}^{\frac{m}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+m}{2}-s\right]!\left[\frac{n-m}{2}-s\right]!} \binom{\frac{n-m}{2}-s}{j} \binom{m}{2k} X^{n-2(s+j+k)} Y^{2(j+k)}$$

For m<0 (13b)

$$Z_n^m(X,Y) = \sqrt{2(n+1)} \sum_{s=0}^{\frac{n-|m|}{2}} \sum_{j=0}^{\frac{n-|m|}{2}-s} \sum_{k=0}^{\frac{|m|-1}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+|m|}{2}-s\right]!\left[\frac{n-|m|}{2}-s\right]!} \binom{\frac{n-|m|}{2}-s}{j} \binom{|m|}{2k+1} X^{n-2(s+j+k)-1} Y^{2(j+k)+1}$$

For m=0 (13c)

$$Z_n^0(X,Y) = \sqrt{(n+1)} \sum_{s=0}^{\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}-s} \frac{(-1)^{s}(n-s)!}{s!\left[\frac{n}{2}-s\right]!\left[\frac{n}{2}-s\right]!} \binom{\frac{n}{2}-s}{j} X^{n-2(s+j)} Y^{2j}$$

3. Derivatives of Zernike Polynomials

3.1 Derivative of the Zernike Polynomials in the ρ direction

From equations (1) and (2), it is easy to see that

$$\frac{\partial(Z_n^m(\rho,\theta))}{\partial\rho} = \begin{cases} N_n^m \dfrac{\partial(R_n^{|m|}(\rho))}{\partial\rho}\cos m\theta & ;\text{for } m \ge 0 \\ -N_n^m \dfrac{\partial(R_n^{|m|}(\rho))}{\partial\rho}\sin m\theta & ;\text{for } m < 0 \end{cases} \tag{14}$$

and

$$\frac{\partial(R_n^{|m|}(\rho))}{\partial\rho} = \sum_{s=0}^{(n-|m|)/2} \frac{(-1)^s (n-s)!}{s!\left[0.5(n+|m|)-s\right]!\left[0.5(n-|m|)-s\right]!}(n-2s)\rho^{n-2s-1} \tag{15}$$

3.2 Relationship between ρ and r derivatives of Zernike Polynomials

The chain rule can be used to calculate the r derivative such that

$$\frac{\partial(Z_n^m(r,\theta))}{\partial r} = \frac{\partial(Z_n^m(\rho,\theta))}{\partial\rho}\frac{\partial\rho}{\partial r} \quad , \tag{16}$$

and since $\rho = r / r_{max}$ and subsequently $\frac{\partial\rho}{\partial r} = \frac{1}{r_{max}}$, then

$$\frac{\partial(Z_n^m(r,\theta))}{\partial r} = \frac{1}{r_{max}}\frac{\partial(Z_n^m(\rho,\theta))}{\partial\rho}. \tag{17}$$

3.3 Derivative of the Zernike Polynomials in the θ direction

From equation (1), it is clear that

$$\frac{\partial(Z_n^m(\rho,\theta))}{\partial\theta} = \begin{cases} -mN_n^m R_n^{|m|}(\rho)\sin m\theta & ;\text{for } m \ge 0 \\ -mN_n^m R_n^{|m|}(\rho)\cos m\theta & ;\text{for } m < 0 \end{cases} \tag{18}$$

3.4 Derivative of the Zernike Polynomials in the X direction

From equations 13a-c, it is clear that for m>0

(19a)

$$\frac{dZ_n^m(X,Y)}{dX} = \sqrt{2(n+1)}\times$$
$$\sum_{s=0}^{\frac{n-|m|}{2}} \sum_{j=0}^{\frac{n-|m|}{2}-s} \sum_{k=0}^{\frac{|m|}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+|m|}{2}-s\right]!\left[\frac{n-|m|}{2}-s\right]!} \binom{\frac{n-m}{2}-s}{j}\binom{m}{2k}\left(n-2(s+j+k)\right)X^{n-2(s+j+k)-1}Y^{2(j+k)}$$

and for m<0 (19b)

$$\frac{dZ_n^m(X,Y)}{dX} = \sqrt{2(n+1)} \times$$

$$\sum_{s=0}^{\frac{n-|m|}{2}} \sum_{j=0}^{\frac{n-|m|}{2}-s} \sum_{k=0}^{\frac{|m|-1}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+|m|}{2}-s\right]!\left[\frac{n-|m|}{2}-s\right]!} \binom{\frac{n-m}{2}-s}{j} \binom{m}{2k+1} (n-2(s+j+k)-1) X^{n-2(s+j+k)-2} Y^{2(j+k)+1}$$

and for m=0 (19c)

$$\frac{dZ_n^0(X,Y)}{dX} = \sqrt{(n+1)} \sum_{s=0}^{\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}-s} \frac{(-1)^s (n-s)!}{s!\left[\frac{n}{2}-s\right]!\left[\frac{n}{2}-s\right]!} \binom{\frac{n}{2}-s}{j} (n-2(s+j)) X^{n-2(s+j)-1} Y^{2j} \quad .$$

3.5 Derivative of the Zernike Polynomials in the Y direction

From equations 13a-c, it is clear that for m>0

(20a)

$$\frac{dZ_n^m(X,Y)}{dY} = \sqrt{2(n+1)} \times$$

$$\sum_{s=0}^{\frac{n-|m|}{2}} \sum_{j=0}^{\frac{n-|m|}{2}-s} \sum_{k=0}^{\frac{|m|}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+|m|}{2}-s\right]!\left[\frac{n-|m|}{2}-s\right]!} \binom{\frac{n-m}{2}-s}{j} \binom{m}{2k} (2(j+k)) X^{n-2(s+j+k)} Y^{2(j+k)-1}$$

and for m<0 (20b)

$$\frac{dZ_n^m(X,Y)}{dY} = \sqrt{2(n+1)} \times$$

$$\sum_{s=0}^{\frac{n-|m|}{2}} \sum_{j=0}^{\frac{n-|m|}{2}-s} \sum_{k=0}^{\frac{|m|-1}{2}} \frac{(-1)^{(s+k)}(n-s)!}{s!\left[\frac{n+|m|}{2}-s\right]!\left[\frac{n-|m|}{2}-s\right]!} \binom{\frac{n-m}{2}-s}{j} \binom{m}{2k+1} (2(j+k)+1) X^{n-2(s+j+k)-1} Y^{2(j+k)}$$

and for m=0 (20c)

$$\frac{dZ_n^0(X,Y)}{dY} = \sqrt{(n+1)} \sum_{s=0}^{\frac{n}{2}} \sum_{j=0}^{\frac{n}{2}-s} \frac{(-1)^s (n-s)!}{s!\left[\frac{n}{2}-s\right]!\left[\frac{n}{2}-s\right]!} \binom{\frac{n}{2}-s}{j} (2j) X^{n-2(s+j)} Y^{2j-1} \quad .$$

3.6 Relationship between X and x derivatives of Zernike Polynomials

The chain rule can be used to calculate the x derivative such that

$$\frac{\partial(Z_n^m(x,y))}{\partial x} = \frac{\partial(Z_n^m(X,Y))}{\partial X}\frac{\partial X}{\partial x} \quad , \qquad (21)$$

and since $X = x / r_{max}$ and subsequently $\frac{\partial X}{\partial x} = \frac{1}{r_{max}}$, then

$$\frac{\partial(Z_n^m(x,y))}{\partial x} = \frac{1}{r_{max}}\frac{\partial(Z_n^m(X,Y))}{\partial X} . \qquad (22)$$

3.7 Relationship between Y and y derivatives of Zernike Polynomials

The chain rule can be used to calculate the y derivative such that

$$\frac{\partial(Z_n^m(x,y))}{\partial y} = \frac{\partial(Z_n^m(X,Y))}{\partial Y}\frac{\partial Y}{\partial y} \quad , \qquad (21)$$

and since $Y = y / r_{max}$ and subsequently $\frac{\partial Y}{\partial y} = \frac{1}{r_{max}}$, then

$$\frac{\partial(Z_n^m(x,y))}{\partial y} = \frac{1}{r_{max}}\frac{\partial(Z_n^m(X,Y))}{\partial Y} . \qquad (22)$$

3.8 X and Y derivatives of Zernikes expressed as Zernikes(from Charlie Campbell)

The following equations give the relationships between the Zernike functions and their first derivatives. The expressions here are a modification of the expressions Charlie Campbell derived for the un-normalized case.

$$\frac{\partial}{\partial X} Z_n^m = \sqrt{(1+\delta_{m0})(n+1)}\left[\sum_{\substack{n'=|m|+1 \\ \text{Step2}}}^{n-1} \sqrt{(n'+1)} Z_{n'}^{\frac{m}{|m|}(|m|+1)} + (1-\delta_{m0})(1-\delta_{m,-1})\sqrt{1+\delta_{m1}} \sum_{\substack{n'=|m|-1 \\ \text{Step2}}}^{n-1} \sqrt{(n'+1)} Z_{n'}^{\frac{m}{|m|}(|m|-1)} \right]$$

(23)

$$\frac{\partial}{\partial Y} Z_n^m = \sqrt{(1+\delta_{m0})(n+1)}\frac{m}{|m|}\left[\sum_{\substack{n'=|m|+1 \\ \text{Step2}}}^{n-1} \sqrt{(n'+1)} Z_{n'}^{-\frac{m}{|m|}(|m|+1)} - (1-\delta_{m0})(1-\delta_{m1})\sqrt{1+\delta_{m,-1}} \sum_{\substack{n'=|m|-1 \\ \text{Step2}}}^{n-1} \sqrt{(n'+1)} Z_{n'}^{-\frac{m}{|m|}(|m|-1)} \right]$$

(24)

4. Mirroring and Rotating Zernikes

4.1 Mirror image of surface fit

In order to flip the Zernike patterns about the Y axis, the value of X in equations 13a-c with –X. This flip is useful for comparing wavefronts from right and left eyes.

For cases where $m \ge 0$, if X is replaced with –X, then a factor of $(-1)^{n-2(s+j+k)}$ appears. When the exponent is even, then $Z_n^m(X,Y) = Z_n^m(-X,Y)$. When the exponent is odd, then $Z_n^m(X,Y) = -Z_n^m(-X,Y)$. Since 2(s+j+k) is always even, then the exponent is even if n is even and odd, if n is odd.

For cases where $m < 0$, if X is replaced with –X, then a factor of $(-1)^{n-2(s+j+k)-1}$ appears. When the exponent is even, then $Z_n^m(X,Y) = Z_n^m(-X,Y)$. When the exponent is odd, then $Z_n^m(X,Y) = -Z_n^m(-X,Y)$. Since 2(s+j+k) is always even, then the exponent is even if n is odd and the exponent is odd, if n is even.

To flip the Zernikes, if n is even, negate the coefficients where $m < 0$. If n is odd negate the coefficients where $m > 0$.

4.2 Rotation of Zernike functions

To rotate the Zernike polynomials through an angle θ_o, replace θ with $\theta - \theta_o$ in equation 1.

$$Z_n^m(\rho, \theta - \theta_o) = \begin{cases} N_n^m R_n^{|m|}(\rho) \cos m(\theta - \theta_o) & ;\text{for } m \ge 0 \\ -N_n^m R_n^{|m|}(\rho) \sin m(\theta - \theta_o) & ;\text{for } m < 0 \end{cases} \qquad (25)$$

In a linear expansion, if pairs of polynomials with the same n and equal and azimuthal frequencies of equal magnitude and opposite signs are considered

$$\begin{aligned} & a_{n,|m|} Z_n^{|m|}(\rho, \theta - \theta_o) + a_{n,-|m|} Z_n^{-|m|}(\rho, \theta - \theta_o) \\ &= a_{n,|m|} N_n^m R_n^{|m|}(\rho)\left[\cos|m|\theta \cos|m|\theta_o + \sin|m|\theta \sin|m|\theta_o\right] \\ &\qquad - a_{n,-|m|} N_n^m R_n^{|m|}(\rho)\left[-\sin|m|\theta \cos|m|\theta_o + \cos|m|\theta \sin|m|\theta_o\right] \qquad (26) \\ &= \left[a_{n,|m|} \cos|m|\theta_o - a_{n,-|m|} \sin|m|\theta_o\right] N_n^m R_n^{|m|}(\rho) \cos|m|\theta \\ &\qquad + \left[a_{n,|m|} \sin|m|\theta_o + a_{n,-|m|} \cos|m|\theta_o\right] N_n^m R_n^{|m|}(\rho) \sin|m|\theta \end{aligned}$$

Thus, the polynomials can be rotated by modifying the coefficients so that

$$\begin{aligned} b_{n,|m|} &= \left[a_{n,|m|}\cos|m|\theta_o - a_{n,-|m|}\sin|m|\theta_o\right] \\ b_{n,-|m|} &= \left[a_{n,|m|}\sin|m|\theta_o + a_{n,-|m|}\cos|m|\theta_o\right] \end{aligned} , \qquad (27)$$

where the b's are the expansion coefficients for the rotated polynomials.

5. Variable Power Elements

Decentering coma and trefoil terms, Z(3,1) and Z(3,3), introduces defocus and astigmatism terms, and that the magnitude of the defocus and astigmatism is linearly proportional to the amount of decentration $\Delta\rho$. Consider two complementary thin elements whose phase is given by

$$\phi_1(x,y) = a_{31}Z_3^1(x,y) + a_{33}Z_3^3(x,y) \qquad (28)$$

And

$$\phi_2(x,y) = -a_{31}Z_3^1(x,y) - a_{33}Z_3^3(x,y) \qquad (29)$$

If one element is shifted along the x-axis by an amount Δx and the other element is shifted an equal distance in the opposite direction, then variable levels of defocus and astigmatism can be produced. Specifically,

$$\begin{aligned} &\phi_1(x+\Delta x,y) \\ &\quad= a_{31}Z_3^1(x+\Delta x,y) + a_{33}Z_3^3(x+\Delta x,y) \\ &\quad= a_{31}\left[-2\sqrt{2}\left(\Delta x + 3\Delta x^3\right)Z_0^0(x,y) + 9\sqrt{2}\Delta x^2 Z_1^1(x,y) - 2\sqrt{6}\Delta x Z_2^0(x,y) - 2\sqrt{3}\Delta x Z_2^2(x,y) + Z_3^1(x,y)\right] \\ &\qquad + a_{33}\left[-2\sqrt{2}\Delta x^3 Z_0^0(x,y) + 3\sqrt{2}\Delta x^2 Z_1^1(x,y) - 2\sqrt{3}\Delta x Z_2^2(x,y) + Z_3^3(x,y)\right] \end{aligned} \qquad (30)$$

and

$$\begin{aligned} &\phi_2(x-\Delta x,y) \\ &\quad= -a_{31}Z_3^1(x+\Delta x,y) - a_{33}Z_3^3(x+\Delta x,y) \\ &\quad= -a_{31}\left[2\sqrt{2}\left(\Delta x + 3\Delta x^3\right)Z_0^0(x,y) + 9\sqrt{2}\Delta x^2 Z_1^1(x,y) + 2\sqrt{6}\Delta x Z_2^0(x,y) + 2\sqrt{3}\Delta x Z_2^2(x,y) + Z_3^1(x,y)\right] \\ &\qquad - a_{33}\left[2\sqrt{2}\Delta x^3 Z_0^0(x,y) + 3\sqrt{2}\Delta x^2 Z_1^1(x,y) + 2\sqrt{3}\Delta x Z_2^2(x,y) + Z_3^3(x,y)\right] \end{aligned} \qquad (31)$$

where the decentered version of the coma and trefoil terms are simply obtained from calculating the displaced versions of coma and trefoil terms in Table 4 and projecting the resultant polynomial onto the lower order Zernike terms. Note that in combining these two phase-elements, terms with even powers of Δx cancel, and that the piston terms $Z_0^0(x,y)$ can be ignored since they represent only a constant phase shift of the net wavefront passing through the elements. The phase of a plane wave passing through the elements is then given by

$$\phi_1(x+\Delta x, y) + \phi_2(x-\Delta x, y) = -4\sqrt{6}\Delta x a_{31} Z_2^0(x,y) - 4\sqrt{3}\Delta x (a_{31} + a_{33}) Z_2^2(x,y) \quad (32)$$

There are two special cases of this result. First, when $a_{33} = -a_{31}$. In this case, only pure defocus with a magnitude that is linearly dependent upon Δx appears. The phase of the first element is given by

$$\phi_1(x,y) = a_{31}\left(Z_3^1(x,y) - Z_3^3(x,y)\right) = \frac{a_{31}\sqrt{2}}{3}\left(\frac{x^3}{3} + xy^2 - \frac{x}{3}\right) \quad . (33)$$

The phase of the second element is just the negative of the first. Equation (33) describes one form of a Variable Focus Lens (VFL) first described by Alvarez. The generalized form for the VFL such that the phase of one of the elements is given by

$$\phi(x,y) = A\left(\frac{x^3}{3} + xy^2\right) + Bx^2 + Cxy + Dx + E + F(y) \quad , (34)$$

where B, C, D, and E are arbitrary constants and F(y) is an arbitrary function of y. Interestingly, the specific example of the VFL given in equation (33) is unique in that it minimizes the rms phase difference between the phase plate and a plane wave. This minimization is due to the rms being related to the sum of the squares of the Zernike expansion coefficients. The function $\phi_1(x,y)$ requires only two Zernike terms ($Z_3^1(x,y)$ and $Z_3^3(x,y)$) to represent its shape.

Additional Zernike terms would be necessary to represent the generalized VFL shape in equation (34). These additional terms can only increase the rms phase difference, since the squares of the coefficients are necessarily positive.

The second special case of the variable power elements is when $a_{31} = 0$. In this case, pure 90/180 astigmatism results in equation (32). The phase of the first element is given by

$$\phi_1(x,y) = a_{33} Z_3^3(x,y) = a_{33} 3\sqrt{8}\left(\frac{x^3}{3} - xy^2\right) \quad (35)$$

Again, the phase of the second element is just the negative of the first. This element is exactly the Variable Astigmatic Lens (VAL) described by Humphrey, Alvarez's student and colleague. The element can be generalized as well such that the phase is given by

$$\phi(x,y) = A\left(\frac{x^3}{3} - xy^2\right) + Bx^2 + Cxy + Dy^2 + Ex + Fy + G \quad (36)$$

where B, C, D, E, F and G are arbitrary constants. As with the VFL, the phase surface described in equation (25) represents the minimum rms phase surface for the VAF.

6. Relationship of Zernikes to Ophthalmic Terms

Positive Z(2,0) defocus means myopia
Negative Z(2,0) defocus means hyperopia

Positive Z(4,0) spherical aberration means rays at the edge of the pupil are myopic relative to rays in the center of the pupil.
Negative Z(4,0) spherical aberration means rays at the edge of the pupil are hyperopic relative to rays in the center of the pupil.

Positive Z(2,2) horizontal astigmatism is against-the-rule astigmatism
Negative Z(2,2) horizontal astigmatism is with-the-rule astigmatism

Positive Z(3,-1) vertical coma has superior steepening
Negative Z(3,-1) vertical coma has inferior steepening

Positive Z(3,1) horizontal coma has steepening along 0 degree semi-meridian
Negative Z(3,1) horizontal coma has steening along 180 degree semi-meridian

Table 1. Zernike Polynomials in Polar Coordinates up to 7th order (36 terms)

j	n	m	$Z_n^m(\rho,\theta)$
0	0	0	1
1	1	-1	$2\ \rho \sin\theta$
2	1	1	$2\ \rho \cos\theta$
3	2	-2	$\sqrt{6}\ \rho^2 \sin 2\theta$
4	2	0	$\sqrt{3}\ (2\rho^2-1)$
5	2	2	$\sqrt{6}\ \rho^2 \cos 2\theta$
6	3	-3	$\sqrt{8}\ \rho^3 \sin 3\theta$
7	3	-1	$\sqrt{8}\ (3\rho^3-2\rho) \sin\theta$
8	3	1	$\sqrt{8}\ (3\rho^3-2\rho) \cos\theta$
9	3	3	$\sqrt{8}\ \rho^3 \cos 3\theta$
10	4	-4	$\sqrt{10}\ \rho^4 \sin 4\theta$
11	4	-2	$\sqrt{10}\ (4\rho^4-3\rho^2) \sin 2\theta$
12	4	0	$\sqrt{5}\ (6\rho^4-6\rho^2+1)$
13	4	2	$\sqrt{10}\ (4\rho^4-3\rho^2) \cos 2\theta$
14	4	4	$\sqrt{10}\ \rho^4 \cos 4\theta$
15	5	-5	$\sqrt{12}\ \rho^5 \sin 5\theta$
16	5	-3	$\sqrt{12}\ (5\rho^5-4\rho^3) \sin 3\theta$
17	5	-1	$\sqrt{12}\ (10\rho^5-12\rho^3+3\rho) \sin\theta$
18	5	1	$\sqrt{12}\ (10\rho^5-12\rho^3+3\rho) \cos\theta$
19	5	3	$\sqrt{12}\ (5\rho^5-4\rho^3) \cos 3\theta$
20	5	5	$\sqrt{12}\ \rho^5 \cos 5\theta$
21	6	-6	$\sqrt{14}\ \rho^6 \sin 6\theta$
22	6	-4	$\sqrt{14}\ (6\rho^6-5\rho^4) \sin 4\theta$
23	6	-2	$\sqrt{14}\ (15\rho^6-20\rho^4+6\rho^2) \sin 2\theta$
24	6	0	$\sqrt{7}\ (20\rho^6-30\rho^4+12\rho^2-1)$
25	6	2	$\sqrt{14}\ (15\rho^6-20\rho^4+6\rho^2) \cos 2\theta$
26	6	4	$\sqrt{14}\ (6\rho^6-5\rho^4) \cos 4\theta$
27	6	6	$\sqrt{14}\ \rho^6 \cos 6\theta$
28	7	-7	$4\ \rho^7 \sin 7\theta$
29	7	-5	$4\ (7\rho^7-6\rho^5) \sin 5\theta$
30	7	-3	$4\ (21\rho^7-30\rho^5+10\rho^3) \sin 3\theta$
31	7	-1	$4\ (35\rho^7-60\rho^5+30\rho^3-4\rho) \sin\theta$
32	7	1	$4\ (35\rho^7-60\rho^5+30\rho^3-4\rho) \cos\theta$
33	7	3	$4\ (21\rho^7-30\rho^5+10\rho^3) \cos 3\theta$
34	7	5	$4\ (7\rho^7-6\rho^5) \cos 5\theta$
35	7	7	$4\ \rho^7 \cos 7\theta$

Table 2. ρ Derivative of Zernike Polynomials in Polar Coordinates up to 7th order

j	n	m	$d(Z_n^m(\rho,\theta))/d\rho$
0	0	0	0
1	1	-1	$2 \sin\theta$
2	1	1	$2 \cos\theta$
3	2	-2	$\sqrt{6}\, 2\rho \sin 2\theta$
4	2	0	$\sqrt{3}\, 4\rho$
5	2	2	$\sqrt{6}\, 2\rho \cos 2\theta$
6	3	-3	$\sqrt{8}\, 3\rho^2 \sin 3\theta$
7	3	-1	$\sqrt{8}\, (9\rho^2-2) \sin\theta$
8	3	1	$\sqrt{8}\, (9\rho^2-2) \cos\theta$
9	3	3	$\sqrt{8}\, 3\rho^2 \cos 3\theta$
10	4	-4	$\sqrt{10}\, 4\rho^3 \sin 4\theta$
11	4	-2	$\sqrt{10}\, (16\rho^3-6\rho) \sin 2\theta$
12	4	0	$\sqrt{5}\, (24\rho^3-12\rho)$
13	4	2	$\sqrt{10}\, (16\rho^3-6\rho) \cos 2\theta$
14	4	4	$\sqrt{10}\, 4\rho^3 \cos 4\theta$
15	5	-5	$\sqrt{12}\, 5\rho^4 \sin 5\theta$
16	5	-3	$\sqrt{12}\, (25\rho^4-12\rho^2) \sin 3\theta$
17	5	-1	$\sqrt{12}\, (50\rho^4-36\rho^2+3) \sin\theta$
18	5	1	$\sqrt{12}\, (50\rho^4-36\rho^2+3) \cos\theta$
19	5	3	$\sqrt{12}\, (25\rho^4-12\rho^2) \cos 3\theta$
20	5	5	$\sqrt{12}\, 5\rho^4 \cos 5\theta$
21	6	-6	$\sqrt{14}\, 6\rho^5 \sin 6\theta$
22	6	-4	$\sqrt{14}\, (36\rho^5-20\rho^3) \sin 4\theta$
23	6	-2	$\sqrt{14}\, (90\rho^5-80\rho^3+12\rho) \sin 2\theta$
24	6	0	$\sqrt{7}\, (120\rho^5-120\rho^3+24\rho)$
25	6	2	$\sqrt{14}\, (90\rho^5-80\rho^3+12\rho) \cos 2\theta$
26	6	4	$\sqrt{14}\, (36\rho^5-20\rho^3) \cos 4\theta$
27	6	6	$\sqrt{14}\, 6\rho^5 \cos 6\theta$
28	7	-7	$4\ (7\rho^6 \sin 7\theta)$
29	7	-5	$4\ (49\rho^6-30\rho^4) \sin 5\theta$
30	7	-3	$4\ (147\rho^6-150\rho^4+30\rho^2) \sin 3\theta$
31	7	-1	$4\ (245\rho^6-300\rho^4+90\rho^2-4) \sin\theta$
32	7	1	$4\ (245\rho^6-300\rho^4+90\rho^2-4) \cos\theta$
33	7	3	$4\ (147\rho^6-150\rho^4+30\rho^2) \cos 3\theta$
34	7	5	$4\ (49\rho^6-30\rho^4) \cos 5\theta$
35	7	7	$4\ (7\rho^6 \cos 7\theta)$

Table 3. θ derivative of Zernike Polynomials in Polar Coordinates up to 7th order

j	n	m	$d(Z_n^m(\rho,\theta))\,/\,d\theta$
0	0	0	0
1	1	-1	$2\ \rho\cos\theta$
2	1	1	$-2\ \rho\sin\theta$
3	2	-2	$2\sqrt{6}\,\rho^2\cos 2\theta$
4	2	0	0
5	2	2	$-2\sqrt{6}\,\rho^2\sin 2\theta$
6	3	-3	$3\sqrt{8}\,\rho^3\cos 3\theta$
7	3	-1	$\sqrt{8}\,(3\rho^3-2\rho)\cos\theta$
8	3	1	$-\sqrt{8}\,(3\rho^3-2\rho)\sin\theta$
9	3	3	$-3\sqrt{8}\,\rho^3\sin 3\theta$
10	4	-4	$4\sqrt{10}\,\rho^4\cos 4\theta$
11	4	-2	$2\sqrt{10}\,(4\rho^4-3\rho^2)\cos 2\theta$
12	4	0	0
13	4	2	$-2\sqrt{10}\,(4\rho^4-3\rho^2)\sin 2\theta$
14	4	4	$-4\sqrt{10}\,\rho^4\sin 4\theta$
15	5	-5	$5\sqrt{12}\,\rho^5\cos 5\theta$
16	5	-3	$3\sqrt{12}\,(5\rho^5-4\rho^3)\cos 3\theta$
17	5	-1	$\sqrt{12}\,(10\rho^5-12\rho^3+3\rho)\cos\theta$
18	5	1	$-\sqrt{12}\,(10\rho^5-12\rho^3+3\rho)\sin\theta$
19	5	3	$-3\sqrt{12}\,(5\rho^5-4\rho^3)\sin 3\theta$
20	5	5	$-5\sqrt{12}\,\rho^5\sin 5\theta$
21	6	-6	$6\sqrt{14}\,\rho^6\cos 6\theta$
22	6	-4	$4\sqrt{14}\,(6\rho^6-5\rho^4)\cos 4\theta$
23	6	-2	$2\sqrt{14}\,(15\rho^6-20\rho^4+6\rho^2)\cos 2\theta$
24	6	0	0
25	6	2	$-2\sqrt{14}\,(15\rho^6-20\rho^4+6\rho^2)\sin 2\theta$
26	6	4	$-4\sqrt{14}\,(6\rho^6-5\rho^4)\sin 4\theta$
27	6	6	$-6\sqrt{14}\,\rho^6\sin 6\theta$
28	7	-7	$28\ \rho^7\cos 7\theta$
29	7	-5	$20\ (7\rho^7-6\rho^5)\cos 5\theta$
30	7	-3	$12\ (21\rho^7-30\rho^5+10\rho^3)\cos 3\theta$
31	7	-1	$4\ (35\rho^7-60\rho^5+30\rho^3-4\rho)\cos\theta$
32	7	1	$-4\ (35\rho^7-60\rho^5+30\rho^3-4\rho)\sin\theta$
33	7	3	$-12\ (21\rho^7-30\rho^5+10\rho^3)\sin 3\theta$
34	7	5	$-20\ (7\rho^7-6\rho^5)\sin 5\theta$
35	7	7	$-28\ \rho^7\sin 7\theta$

Table 4. Zernike Polynomials in Cartesian Coordinates up to 7th order (36 terms)

j	n	m	$Z_n^m(X,Y)$ where $X^2+Y^2=\rho^2\le 1$
0	0	0	1
1	1	-1	$2Y$
2	1	1	$2X$
3	2	-2	$\sqrt{6}\ 2XY$
4	2	0	$\sqrt{3}(2X^2+2Y^2-1)$
5	2	2	$\sqrt{6}(X^2-Y^2)$
6	3	-3	$\sqrt{8}\ (3X^2Y-Y^3)$
7	3	-1	$\sqrt{8}\ (3X^2Y+3Y^3-2Y)$
8	3	1	$\sqrt{8}\ (3X^3+3XY^2-2X)$
9	3	3	$\sqrt{8}\ (X^3-3XY^2)$
10	4	-4	$\sqrt{10}\ (4X^3Y-4XY^3)$
11	4	-2	$\sqrt{10}\ (8X^3Y+8XY^3-6XY)$
12	4	0	$\sqrt{5}\ (6X^4+12X^2Y^2+6Y^4-6X^2-6Y^2+1)$
13	4	2	$\sqrt{10}\ (4X^4-4Y^4-3X^2+3Y^2)$
14	4	4	$\sqrt{10}\ (X^4-6X^2Y^2+Y^4)$
15	5	-5	$\sqrt{12}\ (5X^4Y-10X^2Y^3+Y^5)$
16	5	-3	$\sqrt{12}\ (15X^4Y+10X^2Y^3-5Y^5-12X^2Y+4Y^3)$
17	5	-1	$\sqrt{12}\ (10X^4Y+20X^2Y^3+10Y^5-12X^2Y-12Y^3+3Y)$
18	5	1	$\sqrt{12}\ (10X^5+20X^3Y^2+10XY^4-12X^3-12XY^2+3X)$
19	5	3	$\sqrt{12}\ (5X^5-10X^3Y^2-15XY^4-4X^3+12XY^2)$
20	5	5	$\sqrt{12}\ (X^5-10X^3Y^2+5XY^4)$
21	6	-6	$\sqrt{14}\ (6X^5Y-20X^3Y^3+6XY^5)$
22	6	-4	$\sqrt{14}\ (24X^5Y-24XY^5-20X^3Y+20XY^3)$
23	6	-2	$\sqrt{14}\ (30X^5Y+60X^3Y^3+30XY^5-40X^3Y-40XY^3+12XY)$
24	6	0	$\sqrt{7}\ (20X^6+60X^4Y^2+60X^2Y^4+20Y^6-30X^4-60X^2Y^2-30Y^4+12X^2+12Y^2-1)$
25	6	2	$\sqrt{14}\ (15X^6+15X^4Y^2-15X^2Y^4-15Y^6-20X^4+20Y^4+6X^2-6Y^2)$
26	6	4	$\sqrt{14}\ (6X^6-30X^4Y^2-30X^2Y^4+6Y^6-5X^4+30X^2Y^2-5Y^4)$
27	6	6	$\sqrt{14}\ (X^6-15X^4Y^2+15X^2Y^4-Y^6)$
28	7	-7	$4\ (7X^6Y-35X^4Y^3+21X^2Y^5-Y^7)$
29	7	-5	$4\ (35X^6Y-35X^4Y^3-63X^2Y^5+7Y^7-30X^4Y+60X^2Y^3-6Y^5)$
30	7	-3	$4\ (63X^6Y+105X^4Y^3+21X^2Y^5-21Y^7-90X^4Y-60X^2Y^3+30Y^5+30X^2Y-10Y^3)$
31	7	-1	$4\ (35X^6Y+105X^4Y^3+105X^2Y^5+35Y^7-60X^4Y-120X^2Y^3-60Y^5+30X^2Y+30Y^3-4Y)$
32	7	1	$4\ (35X^7+105X^5Y^2+105X^3Y^4+35XY^6-60X^5-120X^3Y^2-60XY^4+30X^3+30XY^2-4X)$
33	7	3	$4\ (21X^7-21X^5Y^2-105X^3Y^4-63XY^6-30X^5+60X^3Y^2+90XY^4+10X^3-30XY^2)$
34	7	5	$4\ (7X^7-63X^5Y^2-35X^3Y^4+35XY^6-6X^5+60X^3Y^2-30XY^4)$
35	7	7	$4\ (X^7-21X^5Y^2+35X^3Y^4-7XY^6)$

Table 5. X Derivative of Zernike Polynomials in Cartesian Coordinates up to 7th order

j	n	m	$d(Z_n^m(X,Y))/dX$ where $X^2+Y^2=\rho^2\le 1$
0	0	0	0
1	1	-1	0
2	1	1	2
3	2	-2	$\sqrt{6}\ 2Y$
4	2	0	$\sqrt{3}\ 4X$
5	2	2	$\sqrt{6}\ 2X$
6	3	-3	$\sqrt{8}\ 6XY$
7	3	-1	$\sqrt{8}\ 6XY$
8	3	1	$\sqrt{8}\ (9X^2+3Y^2-2)$
9	3	3	$\sqrt{8}\ (3X^2-3Y^2)$
10	4	-4	$\sqrt{10}\ (12X^2Y-4Y^3)$
11	4	-2	$\sqrt{10}\ (24X^2Y+8Y^3-6Y)$
12	4	0	$\sqrt{5}\ (24X^3+24XY^2-12X)$
13	4	2	$\sqrt{10}\ (16X^3-6X)$
14	4	4	$\sqrt{10}\ (4X^3-12XY^2)$
15	5	-5	$\sqrt{12}\ (20X^3Y-20XY^3)$
16	5	-3	$\sqrt{12}\ (60X^3Y+20XY^3-24XY)$
17	5	-1	$\sqrt{12}\ (40X^3Y+40XY^3-24XY)$
18	5	1	$\sqrt{12}\ (50X^4+60X^2Y^2+10Y^4-36X^2-12Y^2+3)$
19	5	3	$\sqrt{12}\ (25X^4-30X^2Y^2-15Y^4-12X^2+12Y^2)$
20	5	5	$\sqrt{12}\ (5X^4-30X^2Y^2+5Y^4)$
21	6	-6	$\sqrt{14}\ (30X^4Y-60X^2Y^3+6Y^5)$
22	6	-4	$\sqrt{14}\ (120X^4Y-24Y^5-60X^2Y+20Y^3)$
23	6	-2	$\sqrt{14}\ (150X^4Y+180X^2Y^3+30Y^5-120X^2Y-40Y^3+12Y)$
24	6	0	$\sqrt{7}\ (120X^5+240X^3Y^2+120XY^4-120X^3-120XY^2+24X)$
25	6	2	$\sqrt{14}\ (90X^5+60X^3Y^2-30XY^4-80X^3+12X)$
26	6	4	$\sqrt{14}\ (36X^5-120X^3Y^2-60XY^4-20X^3+60XY^2)$
27	6	6	$\sqrt{14}\ (6X^5-60X^3Y^2+30XY^4)$
28	7	-7	$4\ (42X^5Y-140X^3Y^3+42XY^5)$
29	7	-5	$4\ (210X^5Y-140X^3Y^3-126XY^5-120X^3Y+120XY^3)$
30	7	-3	$4\ (378X^5Y+420X^3Y^3+42XY^5-360X^3Y-120XY^3+60XY)$
31	7	-1	$4\ (210X^5Y+420X^3Y^3+210XY^5-240X^3Y-240XY^3+60XY)$
32	7	1	$4\ (245X^6+525X^4Y^2+315X^2Y^4+35Y^6-300X^4-360X^2Y^2-60Y^4+90X^2+30Y^2-4)$
33	7	3	$4\ (147X^6-105X^4Y^2-315X^2Y^4-63Y^6-150X^4+180X^2Y^2+90Y^4+30X^2-30Y^2)$
34	7	5	$4\ (49X^6-315X^4Y^2-105X^2Y^4+35Y^6-30X^4+180X^2Y^2-30Y^4)$
35	7	7	$4\ (7X^6-105X^4Y^2+105X^2Y^4-7Y^6)$

Table 6. Y Derivative of Zernike Polynomials in Cartesian Coordinates up to 7th order

j	n	m	$d(Z_n^m(X,Y))/dY$ where $X^2+Y^2=\rho^2\leq 1$
0	0	0	0
1	1	-1	2
2	1	1	0
3	2	-2	$\sqrt{6}\ 2X$
4	2	0	$\sqrt{3}\ 4Y$
5	2	2	$\sqrt{6}\ -2Y$
6	3	-3	$\sqrt{8}\ (3X^2-3Y^2)$
7	3	-1	$\sqrt{8}\ (3X^2+9Y^2-2)$
8	3	1	$\sqrt{8}\ 6XY$
9	3	3	$\sqrt{8}\ -6XY$
10	4	-4	$\sqrt{10}\ (4X^3-12XY^2)$
11	4	-2	$\sqrt{10}\ (8X^3+24XY^2-6X)$
12	4	0	$\sqrt{5}\ (24X^2Y+24Y^3-12Y)$
13	4	2	$\sqrt{10}\ (-16Y^3+6Y)$
14	4	4	$\sqrt{10}\ (-12X^2Y+4Y^3)$
15	5	-5	$\sqrt{12}\ (5X^4-30X^2Y^2+5Y^4)$
16	5	-3	$\sqrt{12}\ (15X^4+30X^2Y^2-25Y^4-12X^2+12Y^2)$
17	5	-1	$\sqrt{12}\ (10X^4+60X^2Y^2+50Y^4-12X^2-36Y^2+3)$
18	5	1	$\sqrt{12}\ (40X^3Y+40XY^3-24XY)$
19	5	3	$\sqrt{12}\ (-20X^3Y-60XY^3+24XY)$
20	5	5	$\sqrt{12}\ (-20X^3Y+20XY^3)$
21	6	-6	$\sqrt{14}\ (6X^5-60X^3Y^2+30XY^4)$
22	6	-4	$\sqrt{14}\ (24X^5-120XY^4-20X^3+60XY^2)$
23	6	-2	$\sqrt{14}\ (30X^5+180X^3Y^2+150XY^4-40X^3-120XY^2+12X)$
24	6	0	$\sqrt{7}\ (120X^4Y+240X^2Y^3+120Y^5-120X^2Y-120Y^3+24Y)$
25	6	2	$\sqrt{14}\ (30X^4Y-60X^2Y^3-90Y^5+80Y^3-12Y)$
26	6	4	$\sqrt{14}\ (-60X^4Y-120X^2Y^3+36Y^5+60X^2Y-20Y^3)$
27	6	6	$\sqrt{14}\ (-30X^4Y+60X^2Y^3-6Y^5)$
28	7	-7	$4\ (7X^6-105X^4Y^2+105X^2Y^4-7Y^6)$
29	7	-5	$4\ (35X^6-105X^4Y^2-315X^2Y^4+49Y^6-30X^4+180X^2Y^2-30Y^4)$
30	7	-3	$4\ (63X^6+315X^4Y^2+105X^2Y^4-147Y^6-90X^4-180X^2Y^2+150Y^4+30X^2-30Y^2)$
31	7	-1	$4\ (35X^6+315X^4Y^2+525X^2Y^4+245Y^6-60X^4-360X^2Y^2-300Y^4+30X^2+90Y^2-4)$
32	7	1	$4\ (210X^5Y+420X^3Y^3+210XY^5-240X^3Y-240XY^3+60XY)$
33	7	3	$4\ (-42X^5Y-420X^3Y^3-378XY^5+120X^3Y+360XY^3-60XY)$
34	7	5	$4\ (-126X^5Y-140X^3Y^3+210XY^5+120X^3Y-120XY^3)$
35	7	7	$4\ (-42X^5Y+140X^3Y^3-42XY^5)$

Table 7. X Derivative of Zernike Polynomials in terms of Zernike Polynomials to 7th order

j	n	m	$d(Z_n^m(X,Y))/dX$ where $X^2+Y^2=\rho^2 \le 1$
0	0	0	0
1	1	-1	0
2	1	1	2 Z00
3	2	-2	$\sqrt{6}$ Z1-1
4	2	0	$2\sqrt{3}$ Z11
5	2	2	$\sqrt{6}$ Z11
6	3	-3	$2\sqrt{3}$ Z2-2
7	3	-1	$2\sqrt{3}$ Z2-2
8	3	1	$2\sqrt{2}$ Z00 + $2\sqrt{6}$ Z20 + $2\sqrt{3}$ Z22
9	3	3	$2\sqrt{3}$ Z22
10	4	-4	$2\sqrt{5}$ Z3-3
11	4	-2	$\sqrt{10}$ Z1-1 + $2\sqrt{5}$ Z3-3 + $2\sqrt{5}$ Z3-1
12	4	0	$2\sqrt{5}$ Z11 +$2\sqrt{10}$ Z31
13	4	2	$\sqrt{10}$ Z11 + $2\sqrt{5}$ Z31 + $2\sqrt{5}$ Z33
14	4	4	$2\sqrt{5}$ Z33
15	5	-5	$\sqrt{30}$ Z4-4
16	5	-3	$3\sqrt{2}$ Z2-2 + $\sqrt{30}$ Z4-4 + $\sqrt{30}$ Z4-2
17	5	-1	$3\sqrt{2}$ Z2-2 + $\sqrt{30}$ Z4-2
18	5	1	$2\sqrt{3}$ Z00 + 6 Z20 + $3\sqrt{2}$ Z22 + $2\sqrt{15}$ Z40 + $\sqrt{30}$ Z42
19	5	3	$3\sqrt{2}$ Z22 + $\sqrt{30}$ Z42 + $\sqrt{30}$ Z44
20	5	5	$\sqrt{30}$ Z44
21	6	-6	$\sqrt{42}$ Z5-5
22	6	-4	$2\sqrt{7}$ Z3-3 + $\sqrt{42}$ Z5-5 + $\sqrt{42}$ z5-3
23	6	-2	$\sqrt{14}$ Z1-1 + $2\sqrt{7}$ Z3-3 + $2\sqrt{7}$ Z3-1 + $\sqrt{42}$ Z5-3 + $\sqrt{42}$ Z5-1
24	6	0	$2\sqrt{7}$ Z11 + $2\sqrt{14}$ Z31 + $2\sqrt{21}$ Z51
25	6	2	$\sqrt{14}$ Z11 + $2\sqrt{7}$ Z31 + $2\sqrt{7}$ Z33 + $\sqrt{42}$ Z51 + $\sqrt{42}$ Z53
26	6	4	$2\sqrt{7}$ Z33 + $\sqrt{42}$ Z53 + $\sqrt{42}$ Z55
27	6	6	$\sqrt{42}$ Z55
28	7	-7	$2\sqrt{14}$ Z6-6
29	7	-5	$2\sqrt{10}$ Z4-4 + $2\sqrt{14}$ Z6-6 + $2\sqrt{14}$ Z6-4
30	7	-3	$2\sqrt{6}$ Z2-2 + $2\sqrt{10}$ Z4-4 + $2\sqrt{10}$ Z4-2 + $2\sqrt{14}$ Z6-4 + $2\sqrt{14}$ Z6-2
31	7	-1	$2\sqrt{6}$ Z2-2 + $2\sqrt{10}$ Z4-2 + $2\sqrt{14}$ Z6-2
32	7	1	4 Z00 + $4\sqrt{3}$ Z20 + $2\sqrt{6}$ Z22 + $4\sqrt{5}$ Z40 + $2\sqrt{10}$ Z42 + $4\sqrt{7}$ Z60 + $2\sqrt{14}$ Z62
33	7	3	$2\sqrt{6}$ Z22 + $2\sqrt{10}$ Z42 + $2\sqrt{10}$ Z44 + $2\sqrt{14}$ Z62 + $2\sqrt{14}$ Z64
34	7	5	$2\sqrt{10}$ Z44 + $2\sqrt{14}$ Z64 + $2\sqrt{14}$ Z66
35	7	7	$2\sqrt{14}$ Z66

Table 8. 2^{nd} x Derivative of Zernike Polynomials in terms of Zernikes to 7^{th} order

j	n	m	$d^2(Z_n^m(X,Y))/dX^2$ where $X^2+Y^2=\rho^2 \le 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	$4\sqrt{3}$ Z00
5	2	2	$2\sqrt{6}$ Z00
6	3	-3	$6\sqrt{2}$ Z1-1
7	3	-1	$6\sqrt{2}$ Z1-1
8	3	1	$18\sqrt{2}$ Z11
9	3	3	$6\sqrt{2}$ Z11
10	4	-4	$4\sqrt{15}$ Z2-2
11	4	-2	$8\sqrt{15}$ Z2-2
12	4	0	$12\sqrt{5}$ Z00 +$8\sqrt{15}$ Z20 + $4\sqrt{30}$ Z22
13	4	2	$6\sqrt{10}$ Z00 + $4\sqrt{30}$ Z20 + $8\sqrt{15}$ Z22
14	4	4	$4\sqrt{15}$ Z22
15	5	-5	$10\sqrt{6}$ Z3-3
16	5	-3	$16\sqrt{3}$ Z1-1 + $20\sqrt{6}$ Z3-3 + $10\sqrt{6}$ Z3-1
17	5	-1	$16\sqrt{3}$ Z1-1 + $10\sqrt{6}$ Z3-3 + $10\sqrt{6}$ Z3-1
18	5	1	$48\sqrt{3}$ Z11 + $30\sqrt{6}$ Z31 + $10\sqrt{6}$ Z33
19	5	3	$16\sqrt{3}$ Z11 + $10\sqrt{6}$ Z31 + $20\sqrt{6}$ Z33
20	5	5	$10\sqrt{6}$ Z33
21	6	-6	$6\sqrt{35}$ Z4-4
22	6	-4	$10\sqrt{21}$ Z2-2 + $12\sqrt{35}$ Z4-4 + $6\sqrt{35}$ Z4-2
23	6	-2	$20\sqrt{21}$ Z2-2 + $6\sqrt{35}$ Z4-4 + $12\sqrt{35}$ Z4-2
24	6	0	$24\sqrt{7}$ Z00 + $20\sqrt{21}$ Z20 + $10\sqrt{42}$ Z22 + $12\sqrt{35}$ Z40 + $6\sqrt{70}$ Z42
25	6	2	$12\sqrt{14}$ Z00 + $10\sqrt{42}$ Z20 + $20\sqrt{21}$ Z22 + $6\sqrt{70}$ Z40 + $12\sqrt{35}$ Z42 + $6\sqrt{35}$ Z44
26	6	4	$10\sqrt{21}$ Z22 + $6\sqrt{35}$ Z42 + $12\sqrt{35}$ Z44
27	6	6	$6\sqrt{35}$ Z44
28	7	-7	$28\sqrt{3}$ Z5-5
29	7	-5	$48\sqrt{2}$ Z3-3 + $56\sqrt{3}$ Z5-5 + $28\sqrt{3}$ Z5-3
30	7	-3	60Z1-1 + $96\sqrt{2}$ Z3-3 + $48\sqrt{2}$ Z3-1 + $28\sqrt{3}$ Z5-5 + $56\sqrt{3}$ Z5-3 + $28\sqrt{3}$ Z5-1
31	7	-1	60Z1-1 + $48\sqrt{2}$ Z3-3 + $48\sqrt{2}$ Z3-1 +$28\sqrt{3}$ Z5-3 + $28\sqrt{3}$ Z5-1
32	7	1	180Z11 + $144\sqrt{2}$ Z31 + $48\sqrt{2}$ Z33 + $84\sqrt{3}$ Z51 + $28\sqrt{3}$ Z53
33	7	3	60Z11 + $48\sqrt{2}$ Z31 + $96\sqrt{2}$ Z33 + $28\sqrt{3}$ Z51 + $56\sqrt{3}$ Z53 + $28\sqrt{3}$ Z55
34	7	5	$48\sqrt{2}$ Z33 + $28\sqrt{3}$ Z53 + $56\sqrt{3}$ Z55
35	7	7	$28\sqrt{3}$ Z55

Table 9. 3rd X Derivative of Zernike Polynomials in terms of Zernikes to 7th order

j	n	m	$d^3(Z_n^m(X,Y))/dX^3$ where $X^2+Y^2=\rho^2\le 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	0
5	2	2	0
6	3	-3	0
7	3	-1	0
8	3	1	$36\sqrt{2}$ Z00
9	3	3	$12\sqrt{2}$ Z00
10	4	-4	$12\sqrt{10}$ Z1-1
11	4	-2	$24\sqrt{10}$ Z1-1
12	4	0	$72\sqrt{5}$ Z11
13	4	2	$48\sqrt{10}$ Z11
14	4	4	$12\sqrt{10}$ Z11
15	5	-5	$60\sqrt{2}$ Z2-2
16	5	-3	$180\sqrt{2}$ Z2-2
17	5	-1	$120\sqrt{2}$ Z2-2
18	5	1	$216\sqrt{3}$ Z00 + 360Z20 + $240\sqrt{2}$ Z22
19	5	3	$72\sqrt{3}$ Z00+ 120Z20 + $180\sqrt{2}$ Z22
20	5	5	$60\sqrt{2}$ Z22
21	6	-6	$60\sqrt{7}$ Z3-3
22	6	-4	$60\sqrt{14}$ Z1-1 + $180\sqrt{7}$ Z3-3 + $60\sqrt{7}$ Z3-1
23	6	-2	$120\sqrt{14}$ Z1-1 + $180\sqrt{7}$ Z3-3 + $120\sqrt{7}$ Z3-1
24	6	0	$360\sqrt{7}$ Z11 + $180\sqrt{14}$ Z31 + $60\sqrt{14}$ Z33
25	6	2	$240\sqrt{14}$ Z11 + $240\sqrt{7}$ Z31 + $180\sqrt{7}$ Z33
26	6	4	$60\sqrt{14}$ Z11 + $60\sqrt{7}$ Z31 + $180\sqrt{7}$ Z33
27	6	6	$60\sqrt{7}$ Z33
28	7	-7	$84\sqrt{10}$ Z4-4
29	7	-5	$180\sqrt{6}$ Z2-2 + $252\sqrt{10}$ Z4-4 + $84\sqrt{10}$ Z4-2
30	7	-3	$540\sqrt{6}$ Z2-2 + $252\sqrt{10}$ Z4-4 + $252\sqrt{10}$ Z4-2
31	7	-1	$360\sqrt{6}$ Z2-2 + $84\sqrt{10}$ Z4-4 + $168\sqrt{10}$ Z4-2
32	7	1	1440 Z00 + $1080\sqrt{3}$ Z20 + $720\sqrt{6}$ Z22 + $504\sqrt{5}$ Z40 + $336\sqrt{10}$ Z42 + $84\sqrt{10}$ Z44
33	7	3	480 Z00 + $360\sqrt{3}$ Z20 + $540\sqrt{6}$ Z22 + $168\sqrt{5}$ Z40 + $252\sqrt{10}$ Z42 + $252\sqrt{10}$ Z44
34	7	5	$180\sqrt{6}$ Z22 + $84\sqrt{10}$ Z42 + $252\sqrt{10}$ Z44
35	7	7	$84\sqrt{10}$ Z44

Table 10. 4^{th} X Derivative of Zernike Polynomials in terms of Zernikes to 7^{th} order

j	n	m	$d^4(Z_n^m(X,Y))/dX^4$ where $X^2+Y^2=\rho^2\leq 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	0
5	2	2	0
6	3	-3	0
7	3	-1	0
8	3	1	0
9	3	3	0
10	4	-4	0
11	4	-2	0
12	4	0	$144\sqrt{5}$ Z00
13	4	2	$96\sqrt{10}$ Z00
14	4	4	$24\sqrt{10}$ Z00
15	5	-5	$120\sqrt{3}$ Z1-1
16	5	-3	$360\sqrt{3}$ Z1-1
17	5	-1	$240\sqrt{3}$ Z1-1
18	5	1	$1200\sqrt{3}$ Z11
19	5	3	$600\sqrt{3}$ Z11
20	5	5	$120\sqrt{3}$ Z11
21	6	-6	$120\sqrt{21}$ Z2-2
22	6	-4	$480\sqrt{21}$ Z2-2
23	6	-2	$600\sqrt{21}$ Z2-2
24	6	0	$1440\sqrt{7}$ Z00 + $720\sqrt{21}$ Z20 + $480\sqrt{42}$ Z22
25	6	2	$960\sqrt{14}$ Z00 + $480\sqrt{42}$ Z20 + $840\sqrt{21}$ Z22
26	6	4	$240\sqrt{14}$ Z00 + $120\sqrt{42}$ Z20 + $480\sqrt{21}$ Z22
27	6	6	$120\sqrt{21}$ Z22
28	7	-7	$840\sqrt{2}$ Z3-3
29	7	-5	1920 Z1-1 + $3360\sqrt{2}$ Z3-3 + $840\sqrt{2}$ Z3-1
30	7	-3	5760 Z1-1 + $5040\sqrt{2}$ Z3-3 + $2520\sqrt{2}$ Z3-1
31	7	-1	3840 Z1-1 + $2520\sqrt{2}$ Z3-3 + $1680\sqrt{2}$ Z3-1
32	7	1	19200 Z11 + $8400\sqrt{2}$ Z31 + $4200\sqrt{2}$ Z33
33	7	3	9600 Z11 + $4200\sqrt{2}$ Z31 + $5040\sqrt{2}$ Z33
34	7	5	1920 Z11 + $840\sqrt{2}$ Z31 + $3360\sqrt{2}$ Z33
35	7	7	$840\sqrt{2}$ Z33

Table 11. 5th X Derivative of Zernike Polynomials in terms of Zernikes to 7th order

j	n	m	$d^5(Z_n^m (X,Y)) / dX^5$ where $X^2 + Y^2 = \rho^2 \le 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	0
5	2	2	0
6	3	-3	0
7	3	-1	0
8	3	1	0
9	3	3	0
10	4	-4	0
11	4	-2	0
12	4	0	0
13	4	2	0
14	4	4	0
15	5	-5	0
16	5	-3	0
17	5	-1	0
18	5	1	$2400\sqrt{3}$ Z00
19	5	3	$1200\sqrt{3}$ Z00
20	5	5	$240\sqrt{3}$ Z00
21	6	-6	$360\sqrt{14}$ Z1-1
22	6	-4	$1440\sqrt{14}$ Z1-1
23	6	-2	$1800\sqrt{14}$ Z1-1
24	6	0	$7200\sqrt{7}$ Z11
25	6	2	$5400\sqrt{14}$ Z11
26	6	4	$2160\sqrt{14}$ Z11
27	6	6	$360\sqrt{14}$ Z11
28	7	-7	$1680\sqrt{6}$ Z2-2
29	7	-5	$8400\sqrt{6}$ Z2-2
30	7	-3	$15120\sqrt{6}$ Z2-2
31	7	-1	$8400\sqrt{6}$ Z2-2
32	7	1	72000 Z00 + $33600\sqrt{3}$ Z20 + $25200\sqrt{6}$ Z22
33	7	3	36000 Z00 + $16800\sqrt{3}$ Z20 + $18480\sqrt{6}$ Z22
34	7	5	7200 Z00 + $3360\sqrt{3}$ Z20 + $8400\sqrt{6}$ Z22
35	7	7	$1680\sqrt{6}$ Z22

Table 12. 6th X Derivative of Zernike Polynomials in terms of Zernikes to 7th order

j	n	m	$d^6(Z_n^m(X,Y)) / dX^6$ where $X^2 + Y^2 = \rho^2 \leq 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	0
5	2	2	0
6	3	-3	0
7	3	-1	0
8	3	1	0
9	3	3	0
10	4	-4	0
11	4	-2	0
12	4	0	0
13	4	2	0
14	4	4	0
15	5	-5	0
16	5	-3	0
17	5	-1	0
18	5	1	0
19	5	3	0
20	5	5	0
21	6	-6	0
22	6	-4	0
23	6	-2	0
24	6	0	$14400\sqrt{7}$ Z00
25	6	2	$10800\sqrt{14}$ Z00
26	6	4	$4320\sqrt{14}$ Z00
27	6	6	$720\sqrt{14}$ Z00
28	7	-7	10080 Z1-1
29	7	-5	50400 Z1-1
30	7	-3	90720 Z1-1
31	7	-1	50400 Z1-1
32	7	1	352800 Z11
33	7	3	211680 Z11
34	7	5	70560 Z11
35	7	7	10080 Z11

Table 13. 7th X Derivative of Zernike Polynomials in terms of Zernikes to 7th order

j	n	m	$d^7(Z_n^m(X,Y)) / dX^7$ where $X^2 + Y^2 = \rho^2 \le 1$
0	0	0	0
1	1	-1	0
2	1	1	0
3	2	-2	0
4	2	0	0
5	2	2	0
6	3	-3	0
7	3	-1	0
8	3	1	0
9	3	3	0
10	4	-4	0
11	4	-2	0
12	4	0	0
13	4	2	0
14	4	4	0
15	5	-5	0
16	5	-3	0
17	5	-1	0
18	5	1	0
19	5	3	0
20	5	5	0
21	6	-6	0
22	6	-4	0
23	6	-2	0
24	6	0	0
25	6	2	0
26	6	4	0
27	6	6	0
28	7	-7	0
29	7	-5	0
30	7	-3	0
31	7	-1	0
32	7	1	705600 Z00
33	7	3	423360 Z00
34	7	5	141120 Z00
35	7	7	20160 Z00

Table 14. Y Derivative of Zernike Polynomials in terms of Zernike Polynomials to 7th order

j	n	m	$d(Z_n^m(X,Y))/dY$ where $X^2+Y^2=\rho^2 \le 1$
0	0	0	0
1	1	-1	2Z00
2	1	1	0
3	2	-2	$\sqrt{6}$ Z11
4	2	0	$2\sqrt{3}$ Z1-1
5	2	2	$-\sqrt{6}$ Z1-1
6	3	-3	$2\sqrt{3}$ Z22
7	3	-1	$2\sqrt{2}$ Z00 + $2\sqrt{6}$ Z20 - $2\sqrt{3}$ Z22
8	3	1	$2\sqrt{3}$ Z2-2
9	3	3	$-2\sqrt{3}$ Z2-2
10	4	-4	$2\sqrt{5}$ Z33
11	4	-2	$\sqrt{10}$ Z11 +$2\sqrt{5}$ Z31 - $2\sqrt{5}$ Z33
12	4	0	$2\sqrt{5}$ Z1-1 + $2\sqrt{10}$ Z3-1
13	4	2	$-\sqrt{10}$ Z1-1 + $2\sqrt{5}$ Z3-3 $-2\sqrt{5}$ Z3-1
14	4	4	$-2\sqrt{5}$ Z3-3
15	5	-5	$\sqrt{30}$ Z44
16	5	-3	$3\sqrt{2}$ Z22 + $\sqrt{30}$ Z42 - $\sqrt{30}$ Z44
17	5	-1	$2\sqrt{3}$ Z00+ 6 Z20 - $3\sqrt{2}$ Z22 + $2\sqrt{15}$ Z40 $-\sqrt{30}$ Z42
18	5	1	$3\sqrt{2}$ Z2-2 + $\sqrt{30}$ Z4-2
19	5	3	$-3\sqrt{2}$ Z2-2 + $\sqrt{30}$ Z4-4 $-\sqrt{30}$ Z4-2
20	5	5	$-\sqrt{30}$ Z4-4
21	6	-6	$\sqrt{42}$ Z55
22	6	-4	$2\sqrt{7}$ Z33 + $\sqrt{42}$ Z53 - $\sqrt{42}$ Z55
23	6	-2	$\sqrt{14}$ Z11 + $2\sqrt{7}$ Z31 - $2\sqrt{7}$ Z33 + $\sqrt{42}$ Z51 $-\sqrt{42}$ Z53
24	6	0	$2\sqrt{7}$ Z1-1 + $2\sqrt{14}$ Z3-1 + $2\sqrt{21}$ Z5-1
25	6	2	$-\sqrt{14}$ Z1-1 + $2\sqrt{7}$ Z3-3 $-2\sqrt{7}$ Z3-1 +$\sqrt{42}$ Z5-3 $-\sqrt{42}$ Z5-1
26	6	4	$-2\sqrt{7}$ Z3-3 +$\sqrt{42}$ Z5-5 $-\sqrt{42}$ Z5-3
27	6	6	$-\sqrt{42}$ Z5-5
28	7	-7	$2\sqrt{14}$ Z66
29	7	-5	$2\sqrt{10}$ Z44 +$2\sqrt{14}$ Z64 - $2\sqrt{14}$ Z66
30	7	-3	$2\sqrt{6}$ Z22 + $2\sqrt{10}$ Z42 - $2\sqrt{10}$ Z44 +$2\sqrt{14}$ Z62 $-2\sqrt{14}$ Z64
31	7	-1	4 Z00 + $4\sqrt{3}$ Z20 - $2\sqrt{6}$ Z22 + $4\sqrt{5}$ Z40 $-2\sqrt{10}$ Z42 + $4\sqrt{7}$ Z60 $-2\sqrt{14}$ Z62
32	7	1	$2\sqrt{6}$ Z2-2 +$2\sqrt{10}$ Z4-2 +$2\sqrt{14}$ Z6-2
33	7	3	$-2\sqrt{6}$ Z2-2 + $2\sqrt{10}$ Z4-4 $-2\sqrt{10}$ Z4-2 + $2\sqrt{14}$ Z6-4 $-2\sqrt{14}$ Z6-2
34	7	5	$-2\sqrt{10}$ Z4-4 + $2\sqrt{14}$ Z6-6 $-2\sqrt{14}$ Z6-4
35	7	7	$-2\sqrt{14}$ Z6-6