The Talbot Effect

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The Talbot effect is a self-imaging phenomenon where the diffraction pattern from a periodic structure reforms a copy of the structure at downstream planes. The effect can be derived from the Fresnel diffraction integral.

If (x,y) are the transverse coordinates in the plane of the periodic structure and (x',y') are the transverse coordinates at a plane a distance z from the periodic structure, then the electric field in the (x',y') plane is given by

$$E(x',y') = \frac{\exp[ikz]}{i\lambda z}\exp\left[\frac{i\pi}{\lambda z}\left(x'^2 + y'^2\right)\right]\Im\left\{E(x,y)\exp\left[\frac{i\pi}{\lambda z}\left(x^2 + y^2\right)\right]\right\} \quad (1)$$

where $\Im\{\ \}$ is the Fourier transform with respect to the variables $\xi = x' / \lambda z$ and $\eta = y' / \lambda z$. For convenience, the following constant can be defined

$$A = \frac{\exp[ikz]}{i\lambda z}\exp\left[\frac{i\pi}{\lambda z}\left(x'^2 + y'^2\right)\right] \qquad (2)$$

If E(x,y) is a periodic function with period L, then it can be represented by a Fourier series such that

$$E(x,y) = \sum_{n=-\infty}^{\infty} a_n \exp\left[\frac{i2\pi nx}{L}\right] \quad (3)$$

$$a_n = \frac{1}{L}\int_{-L/2}^{L/2} E(x,y)\exp\left[\frac{-i2\pi nx}{L}\right]dx \quad (4)$$

Substituting equation (3) into equation (1) gives

$$E(x',y') = A\sum a_n \Im\left\{\exp\left[\frac{i2\pi nx}{L}\right]\exp\left[\frac{i\pi}{\lambda z}\left(x^2 + y^2\right)\right]\right\} \qquad (5)$$

Let $b^2 = i\lambda z$ and use the properties of Fourier transform of products being equal to a convolution of the individual Fourier transforms to get

$$E(x',y') = A\sum a_n\left[\Im\left\{\exp\left[\frac{i2\pi nx}{L}\right]\right\} * \Im\left\{\exp\left[\frac{-\pi}{b^2}\left(x^2 + y^2\right)\right]\right\}\right] \qquad (6)$$

Transforming gives

$$E(x',y') = b^2 A \sum a_n \left[\left(\delta\left(\xi - \frac{n}{L} \right) \delta(\eta) \right) * \exp\left[-\pi b^2 \left(\xi^2 + \eta^2 \right) \right] \right] \qquad (7)$$

Carrying out the convolution gives

$$E(x',y') = b^2 A \sum a_n \exp\left[-i\pi\lambda z \left(\left(\xi - \frac{n}{L} \right)^2 + \eta^2 \right) \right] \qquad (8)$$

Simplifying gives

$$E(x',y') = e[ikz] \sum a_n \exp\left[\frac{i2\pi nx'}{L} \right] \exp\left[\frac{-i\pi\lambda zn^2}{L^2} \right] \qquad (9)$$

Note that except for a phase term that is constant over the (x',y') plane, equation (9) is identical to equation (3) when

$$\exp\left[\frac{-i\pi\lambda zn^2}{L^2} \right] = 1 \qquad (10)$$

The condition in equation (10) is only satisfied when

$$\frac{\lambda z}{L^2} = 2m, \text{ m integer} \qquad (11)$$

Consequently, the Talbot planes or the planes where the object is reproduced are located at

$$z_m = \frac{2mL^2}{\lambda} \qquad (12)$$