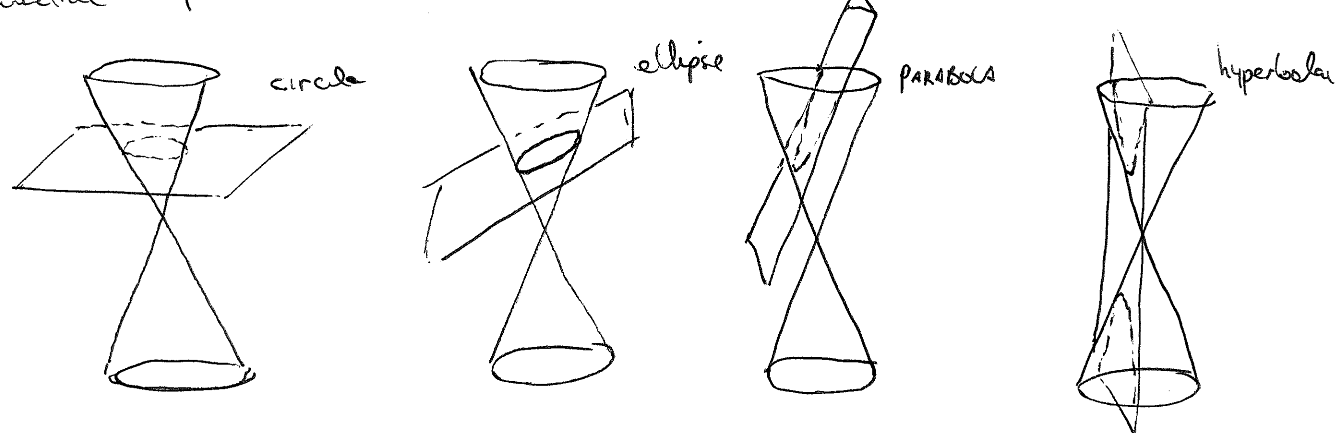
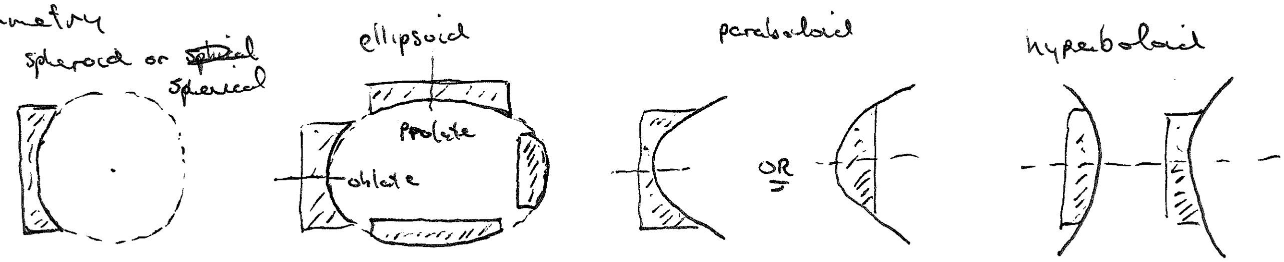
1.7 Aspheric Surfaces

1.7.1 Conics [BOOK 4.2.3] - In general, spherical surfaces are the easiest to fabricate and test. We'll talk more about fabrication and testing in the future, but essentially when grinding and polishing a surface it tends towards a spherical shape. In testing, we can match the spherical shape to a spherical wavefront.

An asphere is a more general surface that may or may not be rotationally symmetric. The most common form of rotationally symmetric asphere is the conic section.



A conoid is the 3-D equivalent of each of these surfaces (spheroid, ellipsoid, paraboloid, hyperboloid) spun about an axis of symmetry

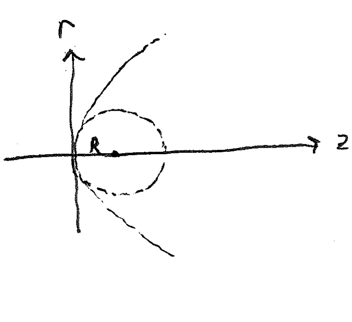


∞ axes of symmetry — 2 axes of symmetry — 1 axes of symmetry — 1 axes of symmetry

The sag of a conoid is given by

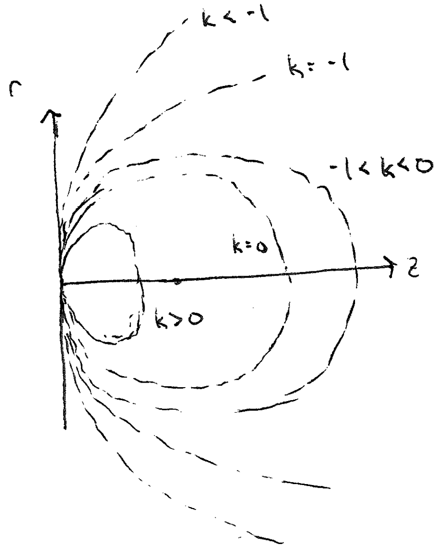
$\boxed{\text{FORM 1}}$

$$z = \frac{r^2/R}{1 + \sqrt{1 - (1+k)\, r^2/R^2}}$$



R = apical radius

K = conic constant



CONIC CONSTANT

$K > 0$	OBLATE ELLIPSOID
$K = 0$	SPHERE
$-1 < K < 0$	PROLATE ELLIPSOID
$K = -1$	PARABOLOID
$K < -1$	HYPERBOLOID

Different forms of sag formula

$$z = \frac{r^2/R}{1 + \sqrt{1-(k+1)r^2/R^2}} \cdot \frac{1 - \sqrt{1-(k+1)r^2/R^2}}{1 - \sqrt{1-(k+1)r^2/R^2}}$$

$$z = \frac{\frac{r^2}{R} - \frac{r^2}{R}\sqrt{1-(k+1)r^2/R^2}}{1 - \left(1 - (k+1)\frac{r^2}{R^2}\right)}$$

FORM 2

$$z = \frac{1}{(k+1)}\left[R - \sqrt{R^2 - (k+1)r^2}\right]$$ ok except when $k = -1$

$$z = \frac{r^2}{2R}$$ when $k = -1$ paraboloid

Another one

$$\sqrt{R^2 - (k+1)r^2} = R - (k+1)z$$

$$\cancel{R^2} - (k+1)r^2 = \cancel{R^2} - 2(k+1)Rz + (k+1)^2 z^2$$

FORM 3 $\boxed{r^2 + (k+1)z^2 - 2Rz = 0}$

Q commonly used instead of K

$p = K+1 \Rightarrow$ p-value

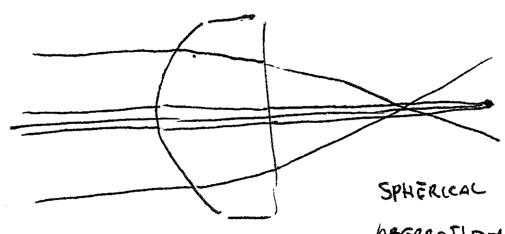
$K = -e^2$ where e is eccentricity

eccentricity of ellipse

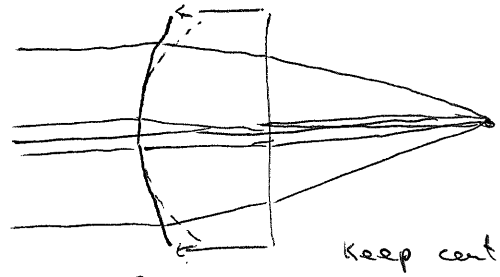
$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

Why use a conic if they're harder to fabricate and test?

Refractive surfaces



SPHERE

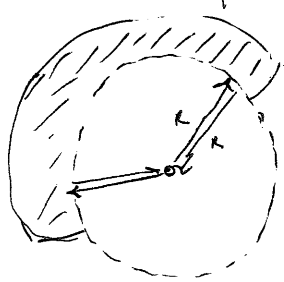


CONIC

Keep center the same, but change periphery (flatten)

Reflective Surfaces

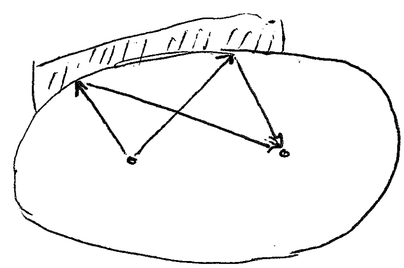
- Conics have two foci which have the property that all rays leaving one foci have equal path length upon reflection to other foci. In other words, no aberrations



SPHERE

TWO COINCIDENT FOCI

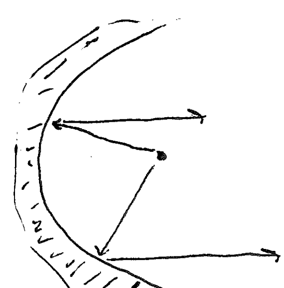
we use this to measure the radius of spherical surfaces



ELLIPSOID

TWO FOCI ALONG MAJOR AXIS

Image a real point to a real point



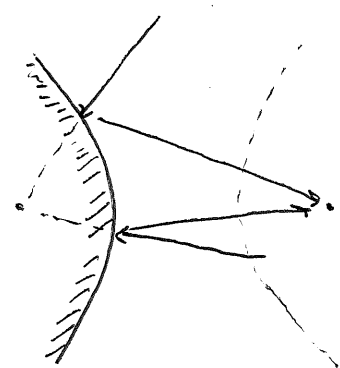
PARABOLOID

ONE FOCI AT ∞

ONE FOCI AT finite distance

SEARCH LIGHT

SATELLITE DISH

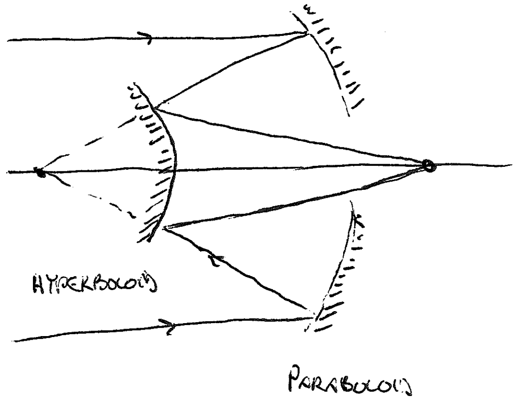


HYPERBOLOID

TWO FOCI ALONG AXIS OF SYMMETRY

Image virtual point to real point.

Cassegrain Telescope



match the focus of the paraboloid to the focus of the hyperboloid. ~~Perfect~~ ⇒ Unaberrated imaging to other focus of hyperboloid.

This unaberrated imaging only occurs between foci. Aberrations appear when object is finite and/or we look off axis.

$\boxed{1.7.2}$ QUADRICS

A quadric is a general 2^{nd} order surface that encompasses conics plus astigmatic surfaces such as biconics (defined further below) plus rotations and decentrations. A quadric surface has the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

If we compare this to the 3^{rd} form of the conic

$$r^2 + (k+1)z^2 - 2Rz = 0$$

we see that this is just a quadric with

$$A = B = 1 \; ; \; C = k+1 \; ; \; I = -2R \; ; \quad D = E = F = G = H = J = 0$$

Suppose we decenter the conic by y_0 in the y direction

$$x^2 + (y-y_0)^2 + (k+1)z^2 - 2Rz = 0$$

$$x^2 + y^2 + (k+1)z^2 - 2y_0 y - 2Rz + y_0^2 = 0$$

Quadric with

$$A = B = 1 \; ; \; C = k+1 \; ; \; H = -2y_0 \; ; \; I = -2R \; ; \; J = y_0^2 \quad D = E = F = G = 0$$

1.7.3 Higher order aspheres [Book 4.2.4]

Raytracing programs provide higher order aspheres

Even Asphere

even powers of ρ

$$z = \underbrace{\frac{r^2/R}{1+\sqrt{1-(k+1)r^2/R^2}}}_{\text{conic}} + \underbrace{\sum_{m=1}^{M} \alpha_{2m}\rho^{2m}}_{} \quad \text{where } \rho = \frac{r}{r_{max}}$$

additional terms ρ^2, ρ^4, ρ^6 ---

ρ^2 term not always available

Odd Asphere

all powers of ρ

$$z = \frac{r^2/R}{1+\sqrt{1-(k+1)r^2/R^2}} + \underbrace{\sum_{m=1}^{M} \alpha_m \rho^m}_{}$$

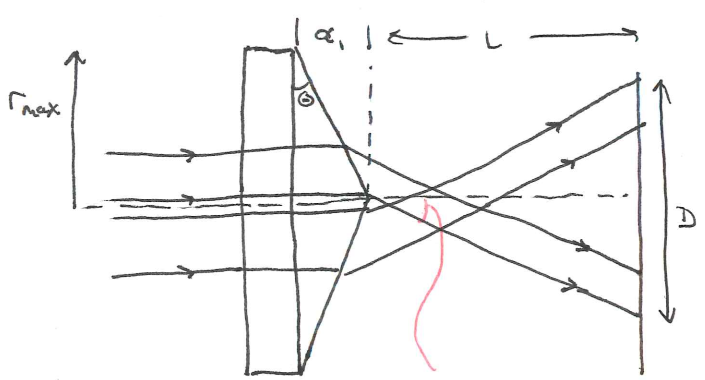
additional terms ρ, ρ^2, ρ^3 ----

odd powers produce a point at the origin

e.g. Axicon $R = \infty$ α_1 = constant $N = 1$

Converts collimated beam into ring

$z = \alpha_1 \rho$

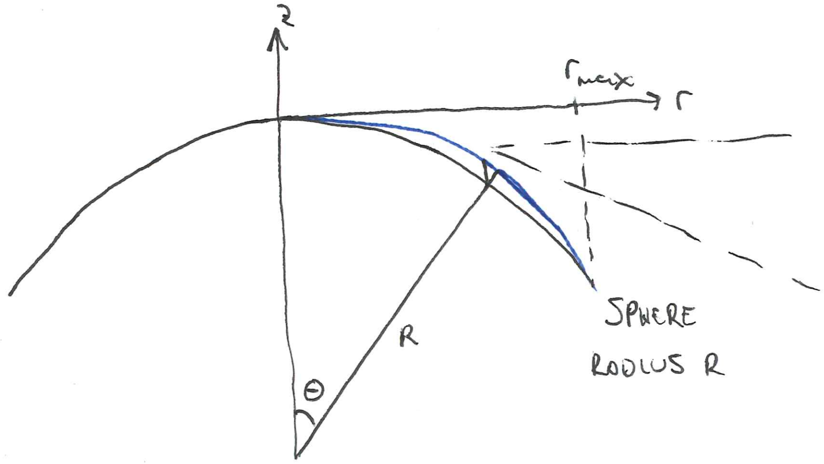


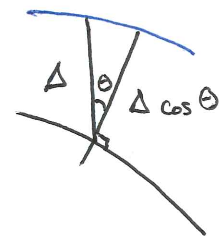
$$\tan\Theta = \frac{\alpha_1}{r_{max}}$$

Cone angle = $180° - 2\Theta$

$$D \simeq 2L\tan[(N-1)\Theta] \simeq 2L(N-1)\Theta$$

FOR THIN PRISM APPROXIMATION THIS ANGLE IS $(N-1)\Theta$ where N is index of axicon





$$z(r) = \frac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r^2}{R^2}}}\left(\frac{r^2}{r_{max}^2}\right)\left(1-\frac{r^2}{r_{max}^2}\right)\sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

$\rho = \dfrac{r}{r_{max}}$ ↖ sag of a sphere ↑ additional sag of asphere

$Q_m^{bfs}(\rho^2)$ are set of ~~other~~ polynomials which will be defined below

a_m are expansion coefficients

$\left(\frac{r^2}{r_{max}^2}\right)\left(1-\frac{r^2}{r_{max}^2}\right)$ ensures sum = 0 when $r = 0$ and $r = \pm r_{max}$

bfs = "Best Fit Sphere" := sphere that matches origin and edge

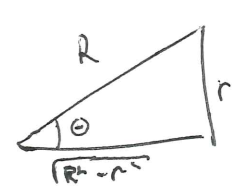
Δ = sag difference between asphere and sphere

$$\Delta = \frac{1}{\sqrt{1-\frac{r^2}{R^2}}}(\rho^2)(1-\rho^2)\sum a_m Q_m^{bfs}(\rho^2)$$

$\Delta\cos\Theta$ = difference between asphere and sphere along normal to sphere

slope of sphere $\dfrac{d}{dr}\left(\dfrac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}}\right) = \dfrac{d}{dr}\left(R - \sqrt{R^2-r^2}\right)$

$$= -\frac{1}{\cancel{2}}\frac{-\cancel{2}r}{\sqrt{R^2-r^2}} = \frac{r}{\sqrt{R^2-r^2}} = \tan\Theta$$



$\cos\Theta = \dfrac{\sqrt{R^2-r^2}}{R}$

$\cos\Theta = \sqrt{1-\dfrac{r^2}{R^2}}$

$$\Delta \cos\theta = \rho^2(1-\rho^2) \sum a_m Q_m^{bfs}(\rho^2)$$

so the $\{Q_m^{bfs}\}$ describes the difference between the asphere and sphere along the normal to the sphere

Definition of $Q_m^{bfs}(x)$

Iterative technique used

Constants

$$f_0 = 2 \qquad f_1 = \frac{\sqrt{19}}{2} \qquad g_0 = \frac{-1}{2}$$

for $m \geq 2$

$$h_{m-2} = \frac{-m(m-1)}{2 f_{m-2}}$$

$$g_{m-1} = \frac{-(1 + g_{m-2} h_{m-2})}{f_{m-1}}$$

$$f_m = \sqrt{m(m+1) + 3 - g_{m-1}^2 - h_{m-2}^2}$$

Functions

$$\left.\begin{aligned} P_0(x) &= 2 \\ P_1(x) &= 6 - 8x \\ P_{m+1}(x) &= (2-4x)P_m(x) - P_{m-1}(x) \end{aligned}\right\} \text{Scaled Jacobi Polynomials}$$

$$Q_0^{bfs}(x) = 1$$

$$Q_1^{bfs}(x) = \frac{1}{\sqrt{19}}(13 - 16x)$$

$$Q_{m+1}^{bfs}(x) = \frac{\left[P_{m+1}(x) - g_m Q_m^{bfs}(x) - h_{m-1} Q_{m-1}^{bfs}(x)\right]}{f_{m+1}}$$

EXAMPLE: CALCULATE $Q_2^{bfs}(x)$

For $Q_2^{bfs}(x) \Rightarrow m+1 = 2 \Rightarrow m = 1$

Need to know $P_2(x)$, g_1, $Q_1^{bfs}(x)$, h_0, $Q_0^{bfs}(x)$

$$P_2(x) = (2-4x)\,P_1(x) - P_0(x)$$

$$= (2-4x)(6-8x) - 2$$

$$= 12 - 40x + 32x^2 - 2$$

$$\boxed{P_2(x) = 32x^2 - 40x + 10}$$

$$g_1 = \frac{-(1+g_0 h_0)}{f_1}$$

$$\boxed{h_0 = \frac{-2(2-1)}{2f_0} = \frac{-1}{2}}$$

$$g_1 = \frac{-\left(1+\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\right)}{\frac{\sqrt{19}}{2}}$$

$$\boxed{g_1 = -\frac{5}{4}\,\frac{2}{\sqrt{19}} = \frac{-5}{2\sqrt{19}}}$$

$$f_2 = \sqrt{2(3) + 3 - \left(\frac{-5}{2\sqrt{19}}\right)^2 - \left(\frac{-1}{2}\right)^2}$$

$$f_2 = \sqrt{9 - \frac{25}{4\cdot 19} - \frac{1}{4}}$$

$$f_2 = \sqrt{\frac{9\cdot 4\cdot 19}{4\cdot 19} - \frac{25}{4\cdot 19} - \frac{19}{4\cdot 19}}$$

$$\boxed{f_2 = \sqrt{\frac{640}{76}} = \sqrt{\frac{160}{19}}}$$

$\cancel{Q_2^{bfs}(x)}$

$$Q_2^{bfs}(x) = \sqrt{\frac{19}{160}}\left[32x^2 - 40x + 10 + \frac{5}{2\sqrt{19}}\frac{1}{\sqrt{19}}(13-16x) + \frac{1}{2}\right]$$

$$\boxed{Q_2^{bfs}(x) = 4\sqrt{\frac{38}{5}}\,x^2 - 20\sqrt{\frac{10}{19}}\,x + 29\sqrt{\frac{2}{95}}}$$

Slope Difference Along Normal

$$\frac{d}{d\rho}\left[\Delta\cos\Theta\right] = \frac{d}{d\rho}\left[\rho^2(1-\rho^2)\sum a_m Q_m^{bfs}(\rho^2)\right] = S_m(\rho)$$

$S_m(\rho)$ is the difference in surface slope between the "best-fit" sphere and the asphere along the normal to the sphere.

The $\{S_m(\rho)\}$ satisfy

$$\int_0^1 S_m(\rho)\,S_n(\rho)\,\frac{1}{\sqrt{1-\rho^2}}\,d\rho = \frac{\pi}{2}\delta_{mn}$$

The $\{S_m(\rho)\}$ are orthogonal! The $\{Q_m^{bfs}(\rho^2)\}$ were specifically chosen so that the $\{S_m(\rho)\}$ would satisfy the above orthogonality requirement.

Recall for Zernike Radial Polynomials we had

$$\int_0^1 R_N^n(\rho)\,R_{N'}^{n'}(\rho)\,\rho\,d\rho = \frac{1}{2N+2}\delta_{NN'}$$

This ρ is called a weighting function

For the Zernike Radial polynomials the weighting function ensures equal areas of the pupil contribute equally to the function shape.

For the Forbes Q polynomials the weighting function is $\frac{1}{\sqrt{1-\rho^2}}$.
This choice of weighting function tends to minimize the maximum slope that appears in $\{S_m(\rho)\}$.

Mean Square Slope

$$\left[\int_0^1 \left[\frac{1}{r_{max}} \sum_{m=0}^{M} a_m S_m(\rho) \right]^2 \frac{1}{\sqrt{1-\rho^2}} \, d\rho \right]$$

$$= \frac{1}{r_{max}^2} \sum_m a_m^2$$ METRIC OF SLOPE DEPARTURE FROM A SPHERE

This will be important when we talk about optical testing

Fitting Q Polynomials

Suppose we have aspheric surface $z(r)$ with $z(0) = 0$

First, find radius R of "Best-fit" sphere which is sphere with same origin and end points as $z(r)$

$$z(r_{max}) = R - \sqrt{R^2 - r_{max}^2} \Rightarrow R = \frac{r_{max}^2 + z^2(r_{max})}{2z(r_{max})}$$

Second, rearrange sag equation

$$z(r) = \frac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r^2}{R^2}}} \left(\frac{r^2}{r_{max}^2}\right)\left(1 - \frac{r^2}{r_{max}^2}\right) \sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

replace $r \Rightarrow r_{max}\rho$ to have everything in normalized coordinates

$$z(\rho) = \frac{r_{max}^2 \rho^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}} \rho^2(1-\rho^2) \sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

$$\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}\left[Z(\rho)-\frac{r_{max}^2\rho^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}}\right]=\sum_{m=0}^{M}a_m\,\rho^2(1-\rho^2)\,Q_m^{bfs}(\rho^2)$$

Third, write as a matrix equation assuming a set of points $\{\rho_i\}$ $i=1,N$

$$\begin{array}{ccc} \overleftrightarrow{A} & \vec{x} & \vec{b} \\ \begin{pmatrix} \rho_1^2(1-\rho_1^2)Q_0^{bfs}(\rho_1^2) & \rho_1^2(1-\rho_1^2)Q_1^{bfs}(\rho_1^2) & \cdots & \rho_1^2(1-\rho_1^2)Q_M^{bfs}(\rho_1^2) \\ \vdots & & & \vdots \\ \rho_N^2(1-\rho_N^2)Q_0^{bfs}(\rho_N^2) & \cdots & \cdots & \rho_N^2(1-\rho_N^2)Q_M^{bfs}(\rho_N^2) \end{pmatrix} & \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{pmatrix} & = \begin{pmatrix} F(\rho_1) \\ F(\rho_2) \\ \vdots \\ F(\rho_N) \end{pmatrix} \end{array}$$

where $F(\rho_i)=\sqrt{1-\frac{r_{max}^2\rho_i^2}{R^2}}\left[Z(\rho_i)-\frac{r_{max}^2\rho_i^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho_i^2}{R^2}}}\right]$

Exclude points where $\rho=0$ and $\rho=1$ as these will make matrices singular!

$\overleftrightarrow{A}\vec{x}=\vec{b}$ Matrix equation

Least Squares fit

$\vec{x}=[\overleftrightarrow{A}^T\overleftrightarrow{A}]^{-1}\overleftrightarrow{A}^T\vec{b}$

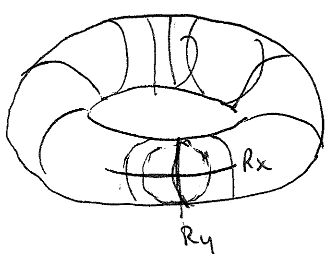
Related Sets

$Q_m^{con}(\rho^2)$ - Similar to $Q_m^{bfs}(\rho^2)$ but base surface is a conic instead of a sphere

Can also define over ~~a~~ an annular region

1.7.4 TORICS AND BICONICS [Book 4.2.6]

Torics and biconics are surfaces with axial astigmatism (i.e. they have two distinct powers along two orthogonal directions. A toric is a section of a donut.



Torus has one radius of curvature R_x (long in this case) and a second Radius of curvature R_y in the orthogonal direction (R_y is short and same sign as R_x in our drawing)

The sag of a toric surface

$$\boxed{z = R_x - \sqrt{\left(R_x - R_y + \sqrt{R_y^2 - y^2}\right)^2 - x^2}}$$

when $x = 0$

$$z = R_x - \left(R_x - R_y + \sqrt{R_y^2 - y^2}\right) = R_y - \sqrt{R_y^2 - y^2} \; : \text{ circle of radius } R_y$$

when $y = 0$

$$z = R_x - \sqrt{(R_x - \cancel{R_y} + \cancel{R_y})^2 - x^2} = R_x - \sqrt{R_x^2 - x^2} = \text{circle of radius } R_x$$

along other directions, the surface shape is more complex.

Biconic is an alternative astigmatic with radii R_x and R_y along orthogonal directions. This surface also has conic constants K_x and K_y along those axes as well. The sag of a biconic is given by

$$z = \frac{\left(\dfrac{x^2}{R_x} + \dfrac{y^2}{R_y}\right)}{1 + \sqrt{1 - (1+K_x)\dfrac{x^2}{R_x^2} - (1+K_y)\dfrac{y^2}{R_y^2}}}$$

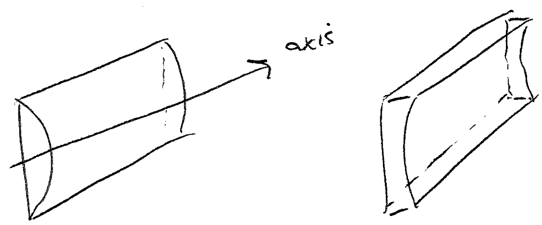
NOTE! TORIC IS <u>NOT</u> A BICONIC WITH $K_x = K_y = 0$!

1.7.5 Cylinders

Cylinder lenses have power along one direction and no power along the other direction. Can be thought of a ~~toric~~ biconic with $R_x \to \infty$ and $K_y = 0$

$$\text{Sag} \qquad z = \frac{\frac{y^2}{R_y}}{1 + \sqrt{1 - \frac{y^2}{R_y^2}}} = \frac{y^2}{R_y + \sqrt{R_y^2 - y^2}}$$

Focuses light to a line.



We define the cylinder axis as the axis along the zero power direction of the lens.