1.5.1 Black Box Optical System Based on Cardinal Points and Pupils (Book 2.1.1)

Complex systems reduce to simple box.

Let's determine the paraxial image plane (PIP) and the shape of the cone of light entering and exiting the system.

1.5.2 Wavefront Picture of Imaging (Book 2.1.2)

Each point on the object acts like a point source and radiates a spherical wavefront. A portion of this wavefront is captured by the entrance pupil and is relayed to the exit pupil. For a perfect imaging system (not diffraction-limited), every point in the object plane maps to a perfect point in the paraxial image plane.

In the real world, we never have perfect imaging. Several things disrupt the process:

1. Objects are often not planes, which means \( Z_o(x,y) \) the object distance depends upon the transverse coordinates, so \( Z_F(x,y) \) as well.
2. Diffraction occurs at the aperture stop. The light at the edge of the captured wavefront interacts with the mechanical aperture stop leading to an imperfect wavefront leaving the exit pupil.
3. Aberrations occur due to an imperfect mapping of diverging spherical to converging spherical waves.
1.5.3 Diffraction-limited Systems and Connection to Fresnel Diffraction

We have some control over the object distance \( D \) and the aberrations \( (\theta) \) at least locally based on our lens design and imaging arrangement. We are stuck with diffraction since the light will always interact with the aperture stop or some other surface (regardless) as it passes through the optical system. A "diffraction-limited" system is a system where the object distance \( D \) and only diffraction affects the emerging wavefront.

Recall Fresnel Diffraction

In Fresnel diffraction, we have some incident wavefront with amplitude \( E(x_p, y_p) \) on a screen with an opening in it. The Fresnel diffraction integral relates the amplitude \( E(x, y) \) at another plane located a distance \( z \) downstream.

\[
E(x, y) = e^{\frac{ikz}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x_p, y_p) E(x_p, y_p) e^{\frac{-i\pi}{\lambda z} \left( (x-x_p)^2 + (y-y_p)^2 \right)} \, dx_p \, dy_p
\]

Expanding the exponential gives

\[
E(x, y) = e^{\frac{ikz}{2}} e^{\frac{i\pi}{\lambda z} \left( x^2 + y^2 \right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x_p, y_p) E(x_p, y_p) e^{\frac{-i\pi}{\lambda z} \left( (x-x_p)^2 + (y-y_p)^2 \right)} \, dx_p \, dy_p
\]

Finally, recall the definition of the Fourier transform \( \mathcal{F} \)

\[
\mathcal{F} \left\{ f(x, y) \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{\frac{-i\pi}{\lambda z} \left( x^2 + y^2 \right)} \, dx \, dy = \mathcal{F}(\xi, \eta)
\]

\[
E(x, y) = e^{\frac{ikz}{2}} e^{\frac{i\pi}{\lambda z} \left( x^2 + y^2 \right)} \mathcal{F} \left\{ P(x_p, y_p) E(x_p, y_p) e^{\frac{-i\pi}{\lambda z} \left( x_p^2 + y_p^2 \right)} \right\}
\]

\[
f(x, y) = \frac{2z}{\pi} \frac{\sin(\frac{\pi}{z} (x^2 + y^2)^{1/2})}{(x^2 + y^2)^{1/2}}
\]

\[
\mathcal{F} \left\{ \frac{2z}{\pi} \frac{\sin(\frac{\pi}{z} (x^2 + y^2)^{1/2})}{(x^2 + y^2)^{1/2}} \right\} = \frac{1}{\sqrt{2\pi}} e^{\frac{-i\pi}{\lambda z} \left( \xi^2 + \eta^2 \right)}
\]

\[
\mathcal{F}^{-1} \left\{ \frac{1}{\sqrt{2\pi}} e^{\frac{-i\pi}{\lambda z} \left( \xi^2 + \eta^2 \right)} \right\} = \frac{1}{\sqrt{2\pi}} e^{\frac{i\pi}{\lambda z} \left( \xi^2 + \eta^2 \right)} \frac{\sin(1/2 \pi \sqrt{\xi^2 + \eta^2})}{\sqrt{\xi^2 + \eta^2}}
\]

\[
\mathcal{F} \left\{ f(x, y) \right\} = \frac{1}{\sqrt{2\pi}} e^{\frac{i\pi}{\lambda z} \left( \xi^2 + \eta^2 \right)} \frac{\sin(1/2 \pi \sqrt{\xi^2 + \eta^2})}{\sqrt{\xi^2 + \eta^2}}
\]
Compare the Fresnel diffraction arrangement to our black box system

we have a converging spherical wave incident on a (usually) circular mesh. This is the same arrangement as Fresnel diffraction where \( P(x_p, y_p) \) is the transmission of the exit pupil and \( E(x_p, y_p) \) is the incident converging spherical wave.

The Fresnel diffraction integral can therefore let us predict the amplitude \( E(x,y,z) \) and the irradiance \( \mathcal{I}(x,y,z) = |E(x,y,z)|^2 \) at any plane downstream from the exit pupil (including the paraxial image plane).

**Aside: Spherical waves**

\[(x_0, y_0, z_0) \rightarrow (x, y, z)\]

\[R_{01} = \sqrt{x_0^2 + y_0^2 + z_0^2}\]

\[r_{01} = \sqrt{(x-x_0)^2 + (y-y_0)^2}\]

\[R(x,y,z) = 2\sqrt{1 + \frac{r_{01}^2}{z^2}}\]

Binomial expansion

\[(1 + x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \ldots\]
\[ R(z, y, 2) = \frac{1}{z} \left( 1 + \frac{z}{d^2} + \frac{z^2}{d^2} + \frac{z^3}{d^2} + \frac{z^4}{d^2} + \cdots \right) \]

\[ R_o(z, y, 2) = \frac{1}{z} \left( 1 + \frac{z}{d^2} + \frac{z^2}{d^2} + \frac{z^3}{d^2} + \frac{z^4}{d^2} + \cdots \right) \]

For denominator, if \( \frac{z}{d^2} \ll 1 \), then \( R_o(z, y, 2) = \frac{1}{z} \)

\[ |\frac{z}{d^2}| \ll |z| \]

Example: \( z = 100 \text{ mm} \), \( z_o = 3 \text{ mm} \) \( \Rightarrow \frac{z}{d^2} \ll 1 \)

For exponential, more care is needed.

\[ e^{i \frac{2 \pi}{d} R(z, y, 2)} = \cos \left( \frac{2 \pi}{d} R(z, y, 2) \right) + i \sin \left( \frac{2 \pi}{d} R(z, y, 2) \right) \]

Sines and cosines oscillate, so expansion terms need to be included if they represent a significant portion of a period.

\[ \frac{2 \pi}{d} R(z, y, 2) = \frac{2 \pi}{d} z + \frac{2 \pi}{d} \frac{z_o}{d^2} = \frac{2 \pi}{d} \frac{z_o}{d^2} + \cdots \]

Example: \( z = 100 \text{ mm} \), \( z_o = 3 \text{ mm} \), \( d = 0.5 \text{ mm} = 0.5 \times 10^{-3} \text{ mm} \)

2\text{nd} term \( \frac{9 \pi}{0.05} = 180\pi \ll 90 \) more oscillations needed to include

3\text{rd} term \( \frac{81 \pi}{4(0.5 \times 10^{-3})(100)} = 0.0405 \pi \) small fraction of oscillations do not include

So when \( \frac{\pi R_o}{4d^2} \ll 1 \) can ignore 3\text{rd} and higher terms.

So spherical wave is approximately

\[ E = \frac{E_0 \, z \, i \, \frac{\pi}{d} \, z_o}{d^2} \]

where \( E_0 = \frac{A}{R(z, y, 2)} \) a constant
Why is $E_0$ a constant?

In many cases $\cos \theta \approx 1$ so illumination of pupil is constant.

Intensity

$I(x, y) = \left| E \right|^2 = \frac{|E_0|^2}{(R_0^2)^2}$

For point source $I(x, y)$ falls off as $\cos^3 \theta$ across entrance pupil.

For example, DSLR with $f_e = 50$mm taking a picture of an object at 3m

an $f/\# = 2$

$\phi = 0.24^\circ$ $\cos^3 \theta = 0.99997$

$D_e = \frac{f_e}{f/\#} = 25$mm
Back to exit pupil

\[ P(x_p, y_p) = \begin{cases} \frac{1}{k_0} & \frac{r_p}{D_e} = \sqrt{x_p^2 + y_p^2} \leq \frac{D_e}{2} \\ 0 & \text{otherwise} \end{cases} \]

where \( D_e \) is diameter of exit pupil

\[ E(x_p, y_p) = E_0 e^{-ikR} e^{-\frac{ir_p}{r}} \left( x_p, y_p \right) \]

Plug into Fresnel diffraction integral

\[ E(x, y) = \frac{E_0}{ikR} e \frac{-iD_e^2 (x^2 + y^2)}{4} \int \frac{y dl}{r} y_l \left( \frac{r_p}{D_e} \right) \]

Recall \( \int \frac{y dl}{r} y_l \left( \frac{r_p}{D_e} \right) = \left( \frac{2\pi}{q} \right) \text{sn} \left( \frac{q}{r} \right) \)

where \( \text{sn}(r) = \frac{2J_1(\pi r)}{\pi r} \) and \( J_1(r) \) is first order Bessel function

\[ E(x, y) = \frac{E_0}{ikR} e \frac{-iD_e^2 (x^2 + y^2)}{4} D_e \frac{2\pi}{q} \frac{2J_1\left( \frac{D_e}{r_p} \right)}{\pi D_e} \]

Substitute \( r = \sqrt{x^2 + y^2} = \sqrt{x_0^2 + y_0^2} \left( \frac{\pi D_e}{\lambda} \right) = \frac{\pi D_e}{\lambda} \)

The modulation \( I(x, y) \)

\[ I(x, y) = \left| E(x, y) \right|^2 = \frac{E_0^2}{\lambda^2} \left( \frac{\pi D_e}{\lambda} \right)^2 \left[ \frac{2J_1\left( \frac{D_e}{r_p} \right)}{\pi D_e} \right] \left( \frac{\pi D_e}{\lambda} \right) \]

\[ \frac{D_e}{R} = \frac{D_e}{\rho f} = \frac{1}{f/\#} \]
\[
I(x, y) = \left( \frac{E_0^2 \pi D_e^2}{16 N^2 (f/#)^2} \right) \left( \frac{j_1\left(\frac{\pi r}{N f/#}\right)}{\left(\frac{\pi r}{N f/#}\right)} \right)^2
\]

Airy Pattern

\[
E_0^2 \pi D_e^2
\]

is the incoherence (W/m²) falling on the entrance pupil

\[
\frac{(E_0^2 \pi D_e^2)}{4}
\]

is the power (W) falling on the entrance pupil

Some mathematical notes:

\[
I(x, y) = \frac{(E_0^2 \pi D_e^2)}{16 N^2 (f/#)^2} \text{ Some }^2 \left(\frac{r}{N f/#}\right)
\]

zeros of Some²(r) occur when

\[
r = 1.22, 2.23, 3.24, 
\]

83.9% of energy contained within

\[
r \leq 1.22
\]

Airy disk diameter = diameter of region contained within first dark ring

\[
\text{Airy disk diameter} \leq \frac{r}{N f/#}
\]

Only dark diameter \(d_r = 2.441 d f/#\)

Rule of Thumb: A visible \(d_r \approx 0.5 \text{ mm}\), so the Airy disk diameter

\(d_r \approx 1.22 d f/#\)

Any disk diameter \(d_r = 2.441 d f/#\)

\[
\text{Airy disk diameter} \leq 1.22 d f/#
\]

We have intraocularly assumed a point source at infinity. If point source

\[
\text{at finite distance, use working } f/#w
\]
1.5.4  Point Spread Function (PSF)  (Book 2.1.4)

One metric of the quality of an optical system is the PSF. As the name implies, the PSF is the image of a point source. In the diffraction limited case, the PSF is just the Airy pattern. When aberrations are present, PSF spreads further. For a given F/number, the Airy pattern is the best we can do.

1.5.5  Sign and Coordinate System Conventions  (Book 2.1.5)

Right-handed coordinate systems:
- In we usually look in the negative z direction.
- Positive values on y.

Exit pupil plane: Diameter of exit pupil = 2r_{max}

Normalized coordinates:
\[ p = \frac{p_p}{r_{max}} = \frac{x_p + y_p}{r_{max}} \]
\[ \Theta = \tan^{-1} \left( \frac{y_p}{x_p} \right) \]

Cartesian coordinates:
\[ p_k = r_{max} \cos \Phi = r_{max} \cos \psi \]
\[ p_y = r_{max} \sin \Phi = r_{max} \sin \psi \]

Point \((h_k, h_y)\) Cartesian:
- \(h_k = \frac{h_{img}}{h_{max}} = \frac{k_{img} y_{img}}{h_{max}}\)
- \(h_y = h_{max} \cos \Phi \)
- \(\Phi = \tan^{-1} \left( \frac{y_{img}}{h_{img}} \right)\)
Care must be taken when dealing with pupil coordinates. Most texts on aberration theory use $\alpha$ as an angle (for historical reasons), while most lens design software uses $\Theta$. I will mainly stick with $\Theta$.

1.5.6 Optical Path Length (OPL), Optical Path Difference (OPD), Wavefront Error

**[Book 2.1.6]**

The numerator is the optical path length (OPL). It is related to the amount of time it takes a ray to propagate from one point to another. In the ray picture

$$\text{OPL} = \sum_i n_i e_i$$

We are usually interested in the difference in OPLs between two different paths through a system. For path A and path B, the optical path difference (OPD) is $\text{OPL}_A - \text{OPL}_B$.

**Interested in OPD since it tells us how to combine the rays.**
What is OPD between path A and path B? OPD = 0

Each path has an OPL = radius of wavefront.
In fact, we can turn this around and say a wavefront is a surface of constant OPL and any two points on the wavefront have OPD = 0.

What happens when we have aberrations?

Some paths through the optical system are faster than others. Wavefront is no longer spherical and will not converge to a point.

\[ r_{01} = \sqrt{(x'_{0} - x'_{1})^2 + (y'_{0} - y'_{1})^2} \]

Let's say our aberrated wavefront is given by \( \frac{r_{0}^2}{2R} + W(x'_{0}, y'_{0}) \)

W(x'_{0}, y'_{0}) is called the wavefront error.

The electric field corresponding to the aberrated wavefront is

\[ E(x'_{0}, y'_{0}) = E_{0} e^{-i\phi} \]

\( \phi \) is the phase shift.

W(x'_{0}, y'_{0}) is the wavefront error correction.

The electric field corresponding to the aberrated wavefront is

\[ E(x', y') = E_{0} e^{-i\phi} \left( \frac{r_{0}^2}{2R} + W(x'_{0}, y'_{0}) \right) \]

If we plug this into the Fresnel diffraction integral to calculate

\[ E(x', y') \] at \( z = R \)
The connection between the two planes depends on the point A is present in the plane. Draw a line through the point A and the plane parallel to the plane containing the point A and the line. Then, the points on the line are all on the plane. The distance from the reference plane to the plane is 0.97.

All the ways around to the reference plane depend on the point B. Draw a line through the point B and the plane parallel to the plane containing the point B and the line. Then, the points on the line are all on the plane. The distance from the reference plane to the plane is 0.62.

\[ E(x, y) = \frac{e^{-1} e^{2} (x+1)^{-2}}{e^{2} (x+1)^{2}} \]

\[ E(x, y) = \frac{e^{-1} e^{2} (x+1)^{-2}}{e^{2} (x+1)^{2}} \]

\[ \text{for } x \text{ and } y \]

\[ E(x, y) = \frac{e^{-1} e^{2} (x+1)^{-2}}{e^{2} (x+1)^{2}} \]

\[ \text{for } x \text{ and } y \]

\[ E(x, y) = \frac{e^{-1} e^{2} (x+1)^{-2}}{e^{2} (x+1)^{2}} \]

\[ \text{for } x \text{ and } y \]
Transverse Ray Error and Spot Diagrams

Direction Cosines

\[ \vec{v} = v_1 \hat{x} + v_2 \hat{y} + v_3 \hat{z} = (v_1, v_2, v_3) \]

Look at the \( \hat{x} \) axis

\[ \cos a = \frac{\vec{v} \cdot \hat{x}}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = \alpha = \frac{v_1}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \]

Similarly,

\[ \cos b = \frac{\vec{v} \cdot \hat{y}}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = \beta = \frac{v_2}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \]

\[ \cos c = \frac{\vec{v} \cdot \hat{z}}{\sqrt{v_1^2 + v_2^2 + v_3^2}} = \gamma = \frac{v_3}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \]

Note that \( a^2 + b^2 + c^2 = 1 \)

So vector \((a, b, c)\) is just a unit vector pointing in the same direction as \( \vec{v} \)

Rays - Rays are extensions of the concept of a vector. They are described by two vectors: \((x_0, y_0, z_0)\) denoting the ray origin and \((a, b, c)\) denoting the ray's trajectory.

\[ (x_0 + t a, y_0 + t b, z_0 + t c) \]

Since \((a, b, c)\) is a unit vector, \(t\) is a scalar that extends the ray in the direction of the trajectory.

Denote the true aberrated wavefront as \(W_0(x', y')\)

\[ W_0(x', y') = \frac{x'^2 + y'^2}{2R} + W(x_0, y_0) \]

Reference Sphere Wavefront Error

For on-axis case
A ray on this wavefront has its origin at \((x_0, y_0, \omega_0(x, y))\) and its trajectory is defined by the normal at point \((x_1, y_1)\):

\[
\mathbf{n} = \left( \frac{\partial \omega_0}{\partial x} \mathbf{i} + \frac{\partial \omega_0}{\partial y} \mathbf{j} + 1 \mathbf{k} \right) \quad \text{wavefront normal}
\]

\[
\hat{n} = \left( \frac{x_p}{R} \frac{1}{\sqrt{\frac{x_p}{R} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{R} \frac{\partial \omega_0}{\partial y} + 1}} \mathbf{i} + \frac{y_p}{R} \frac{1}{\sqrt{\frac{x_p}{R} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{R} \frac{\partial \omega_0}{\partial y} + 1}} \mathbf{j} \right) \quad \text{unit normal}
\]

where \( \sqrt{1} = \sqrt{\left(\frac{x_p}{R} \frac{\partial \omega_0}{\partial x}\right)^2 + \left(\frac{y_p}{R} \frac{\partial \omega_0}{\partial y}\right)^2 + 1} \).

Ray is given by

\[
(x_p, y_p, \omega_0(x, y)) + t \hat{n}
\]

Look at z component in range plane \((z \equiv R)\):

\[
\omega_0(x, y) + \frac{5}{R \sqrt{\left(\frac{x_p}{R} \frac{\partial \omega_0}{\partial x}\right)^2 + \left(\frac{y_p}{R} \frac{\partial \omega_0}{\partial y}\right)^2 + 1}} = 0
\]

\[
t = \left( R - \omega_0(x, y) \right) R
\]

Approximation: \( R - \omega_0(x, y) \approx R \)

So \( t \approx R \sqrt{1} \)

and ray \((x_0, y_0, \omega_0(x, y)) + \left( -\frac{R \frac{\partial \omega_0}{\partial x}}{\sqrt{\frac{x_p}{R} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{R} \frac{\partial \omega_0}{\partial y} + 1}}, -\frac{R \frac{\partial \omega_0}{\partial y}}{\sqrt{\frac{x_p}{R} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{R} \frac{\partial \omega_0}{\partial y} + 1}}, R \right) \)

\[
\approx \left( -\frac{R \mathbf{d} \omega}{\sqrt{x_p} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{\sqrt{y_p}} \frac{\partial \omega_0}{\partial y}}, \frac{R \mathbf{d} \omega}{\sqrt{x_p} \frac{\partial \omega_0}{\partial x} + \frac{y_p}{\sqrt{y_p}} \frac{\partial \omega_0}{\partial y}}, R \right) \quad \text{when } \omega_0(x, y) = 0\text{, ray strikes at }(0, 0, R)
\]

\[
\mathbf{e}_x, \mathbf{e}_y
\]

For off-axis case, reference plane is \( (x_p, y_p) (x, y) \)

\[
\frac{(x - x_p)^2 + (y - y_p)^2}{2R} \quad \text{Similar derivation with same results}.
\]
If we use normalized pupil coordinates instead:

\[ P_x = \frac{x}{\theta_{\text{max}}} \quad P_y = \frac{y}{\theta_{\text{max}}} \]

\[ \frac{dP_x}{dx_P} = \frac{1}{\theta_{\text{max}}} \quad \frac{dP_y}{dy_P} = \frac{1}{\theta_{\text{max}}} \]

From chain rule:

\[ E_x = -R \frac{dW(P_x, P_y)}{dP_x} \frac{dP_x}{dx_P} \quad E_y = -R \frac{dW(P_x, P_y)}{dP_y} \frac{dP_y}{dy_P} \]

\[ E_x = \frac{-R}{N_{\text{max}}} \frac{dW(P_x, P_y)}{dP_x} \quad E_y = \frac{-R}{N_{\text{max}}} \frac{dW(P_x, P_y)}{dP_y} \]

**Spot Diagram**: A scatterplot of \( E_x \) vs. \( E_y \). Values of \( P_x \) and \( P_y \) are chosen to determine \( (E_x, E_y) \). This point is marked and the process repeated.

Contrast, color and random values of \( P_x, P_y \) typical.

---

**1.5.8 Aberrations of Rotationally Symmetric Optical Systems**

**Point of Interest**: In image plane is \((h_x', h_y')\).

Rotate coordinate system by angle \( \omega \) about 2-axis.

So this point is aligned with \( y' \)-axis. New point is \((0, h')\).
\[ P_x = P_x' \cos \omega + P_y' \sin \omega \]
\[ P_y = P_x' \cos \omega - P_y' \sin \omega \]
\[ h_x = h_x' \cos \omega + h_y' \sin \omega \]
\[ h_y = h_x' \cos \omega - h_y' \sin \omega \]

Choose \( \omega \) so that
\[ 0 = h_x' \cos \omega + h_y' \sin \omega \implies h_x' = -h_y', \tan \omega \]
\[ h = h_x' \cos \omega - h_y' \sin \omega \implies h = h_x' \cos \omega + h_y' \sin \omega \frac{\sin \omega}{\cos \omega} \]
\[ h \cos \omega = h_x' \]
\[ \tan \omega = -\frac{h_x'}{h_y'} \]
\[ \omega = \tan^{-1} \left( -\frac{h_x'}{h_y'} \right) = -\tan^{-1} \left( \frac{h_x'}{h_y'} \right) \]

Note that pupil rotates with image plane.

Point \((P_x, P_y)\) in Cartesian is point \((r, \theta)\) in polar

where \( r = \sqrt{P_x^2 + P_y^2} \)
\( \theta = \tan^{-1} \left( \frac{P_y}{P_x} \right) \)

\( \psi = \frac{\pi}{2} - \theta \) from aberration theory

\( P_x = ps \cos \psi \quad P_x^* = P_x^2 + P_y^2 \)
\( P_y = ps \sin \psi \quad \psi = \tan^{-1} \left( \frac{P_x}{P_y} \right) \)

Easier just to look at:

Ideal image point located at \((0, h)\)
In general wavefront error is a function of \( P_x, P_y, h_x, h_y \). For rotationally symmetric system

\[
\omega(P_x, P_y, h_x, h_y) = \omega(P'_x, P'_y, h'_x, h'_y)
\]

In other words, shape of wavefront independent of how we rotate coordinate system. Look for combinations of \( P_x, P_y, h_x, h_y \) that satisfy this requirement.

\[
P_x^2 + P_y^2 = P'_x \cos^2 \phi - 2P'_x P'_y \sin \phi \cos \phi + P'_y^2 \sin^2 \phi
\]

\[
P_x^2 \sin^2 \phi + 2P'_x P'_y \sin \phi \cos \phi + P'_y^2 \cos^2 \phi
\]

\[
= P_x^2 + P_y^2 \quad \text{i.e independent of } \phi
\]

Same for \( h_x^2 + h_y^2 \)

\[
P_x h_x + P_y h_y = P'_x h'_x \cos^2 \phi - P'_x h'_y \sin \phi \cos \phi - P'_y h'_y \sin \phi \cos \phi + P'_y h'_x \sin^2 \phi
\]

\[
P'_x h'_x \sin \phi \cos \phi + P'_x h'_y \sin \phi \cos \phi - P'_y h'_x \sin \phi \cos \phi + P'_y h'_y \cos \phi \cos \phi
\]

\[
= P'_x h'_x + P'_y h'_y
\]

For a given point \( (h'_x, h'_y) \) in the image plane, we can always rotate our coordinate system such that the point is \((0, h'_y)\). We represent our wavefront as

\[
\omega(P_x, P_y, h_x, h_y) = \omega(p, h_x, h_y \cos \psi)
\]

where \( p^2 = P_x^2 + P_y^2 \), \( h_x^2 + h_y^2 = h'_y \), \( h y \cos \psi = h'_y \).

Note \( \psi \) measured clockwise from \( P_y \) axis.

**Power Series**

1-D \( f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots \) converges if \( f(x) \) "nice", smooth

2-D \( f(x, y) = \sum_{i,j=0}^{\infty} a_{i,j} x^i y^j = a_{0,0} + a_{1,0} x + a_{0,1} y + a_{2,0} x^2 + a_{0,2} y^2 + a_{1,1} x y + \ldots \)

Want to represent \( \omega(p, h_x, h_y \cos \psi) \) as power series.
\[ \omega(p^2 h^2, \rho \cos \Psi) = \sum_{i,j,k=0}^{\infty} a_{ijk} (h^2)^i (\rho^2)^j (\rho \cos \Psi)^k \]

Let's modify the numbering scheme on \( a_{ijk} \):

Define \( w_{i+j+k, i+j+k, k} = a_{ijk} \)

Regrouping like terms:

\[ \omega(p^2 h^2, \rho \cos \Psi) = \sum_{i,j,k=0}^{\infty} w_{i+j+k, i+j+k, k} h^i \rho^j \cos^k \Psi \]

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>k</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( w_{000} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( w_{001} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( w_{100} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( w_{010} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>( w_{002} )</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>( w_{111} )</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>( w_{101} )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>( w_{220} )</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>( w_{020} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>3</td>
<td>( w_{003} )</td>
</tr>
</tbody>
</table>
| ... | ... | ... | ...

[1.5.8.1] Piston, Tilt and Defocus

Piston is a constant offset between the reference sphere and the wavefront:

\[ \omega(0, h^4, 0) = 0 \]

Typically, we define \( \omega(0, h^4, 0) = 0 \) i.e. no wavefront error in center of pupil.
This definition means
\[ \omega_{\omega} = \omega_{2\omega} = \omega_{4\omega} = 0 \]

Tilt is the portion of the wavefront error given by the \( \omega_{11\text{p}} = \omega_{12\text{p}} \) term.
\[ \omega_{11\text{p}} \text{ or } \omega_{12\text{p}} \text{ (we'll drop the primes on } \omega) \]

- Linear \( \omega_{11\text{p}} \) i.e. a plane rotated about \( \omega_{11\text{p}} \) axis

CASE: \( \omega_{11\text{p}} < 0 \)

Transverse Ray Error
\[ E_x = \frac{-R}{\max} \frac{\partial \omega}{\partial \omega_{11\text{p}}} = 0 \]
\[ E_y = \frac{-R}{\max} \frac{\partial \omega}{\partial \omega_{12\text{p}}} = \frac{-R\omega_{11\text{p}} h}{\max} \text{ (constant path length)} \]

Spot Diagram
\[ E_y \] always forms point regardless of where ray is
w exit pupil.

Defocus is the portion of the wavefront error given by the \( \omega_{20\text{p}} \rho^2 \) term
\[ \omega_{20\text{p}} \rho^2 = \omega_{20\text{p}} (\omega_{11\text{p}} + \omega_{12\text{p}}^2) \]

- (in air)
\[ E_x = \frac{-R}{\max} \frac{\partial \omega_{20\text{p}}}{\partial \omega_{11\text{p}}} \rho \]
\[ E_y = \frac{-R}{\max} \frac{\partial \omega_{20\text{p}}}{\partial \omega_{12\text{p}}} \rho \]

Linear - both directions

Longitudinal aberration
- Spherical wavefront
- T wavefront

For \( \rho = 1 \)
\[ E_y = \frac{-2R\omega_{20\text{p}}}{N\max} \]

Apo index after in image space
\[
\frac{R_{\text{max}}}{R + 5x} = \frac{2R_{\text{wolo}}}{N_{\text{max}}} \quad \Rightarrow \quad \frac{5x}{N_{\text{max}}} = \frac{R_{\text{max}}}{2R_{\text{wolo}}} - 1
\]

Note: \( R_{\text{wolo}} \) is positive.

\[
5x = \frac{2R_{\text{wolo}}}{N_{\text{max}}} (R + 5x) = -\frac{2R_{\text{wolo}}}{N_{\text{max}}} = -\frac{2R_{\text{wolo}}}{N_{\text{max}}} \left( \frac{R}{\text{wolo}} \right)^2
\]

\[
\frac{R}{\text{wolo}} = \frac{\rho_{\text{c}}}{D_{\text{g}}} = \frac{f}{f^{'}}
\]

In \( wolo > 0 \) shift is towards the exit pupil.

\[
\Delta x = -8 \left( \frac{f}{f^{'}} \right)^2 wolo
\]

This relationship will come up repeatedly when we do non-interferometric testing (book 2.2.2).

1.5.3.2. Seidel Aberrations - Seidel defined aberrations much in the way we have previously. They are often labeled \( S_1, S_2, S_3, S_4, S_5 \), and are proportional to \( \omega(1), (2), (3) \), and \( (4) \), coefficients (called wavefront coefficients from here on out).

\( S_1 = 8 \omega_{100} \text{ : SPHERICAL ABERRATION} \)

\( S_2 = 2 \omega_{111} \text{ : COMA} \)

\( S_3 = 2 \omega_{200} \text{ : ASTIGMATISM} \)

\( S_4 = 4 \omega_{220} \text{ : FIELD CURVATURE} \)

\( S_5 = 2 \omega_{311} \text{ : DISTORTION} \)
Spherical Aberration

\( W(x, y) = w_0 w_0 p^4 = w_0 w_0 (p_x^2 + p_y^2) \)

1. Transverse Ray Errors (in air)

\[
\begin{align*}
\hat{e}_x & = -\frac{R}{\gamma_{\infty}} \omega_0 (p_x^2 + p_y^2) (2p_x) = -\frac{4R}{\gamma_{\infty}} \omega_0 (p_x^3 + p_x p_y^2) = -\frac{4R}{\gamma_{\infty}} \omega_0 p_x^3 \sin \gamma \sin \gamma \\
\hat{e}_y & = -\frac{R}{\gamma_{\infty}} \omega_0 (p_x^2 + p_y^2) (2p_y) = -\frac{4R}{\gamma_{\infty}} \omega_0 (p_x p_y + p_y^3) = -\frac{4R}{\gamma_{\infty}} \omega_0 p_y^3 \cos \gamma
\end{align*}
\]

Longitudinal Errors

2. Longitudinal Error

\[
\hat{e}_x = -\frac{4R}{\gamma_{\infty}} \omega_0 p_y^3
\]

\[
\hat{e}_y = \frac{4R}{\gamma_{\infty}} \omega_0 p_x^3
\]

Essentially same deviation as defocus case

\[
\frac{\text{Focus } p_y}{R + \hat{e}_z} = \frac{4R}{\gamma_{\infty}} \frac{\omega_0 p_y^3}{R + \hat{e}_z} = \frac{4R}{\gamma_{\infty}} \omega_0 p_y^3
\]

but \( R + \hat{e}_z \neq R \)

\[ \xi_2 < 0 \text{ for } w_0 > 0 \]

Can think of spherical aberration as defocus which depends on \( p_y^2 \)

Definition of Marginal Focus \( \xi_2 \) when \( p_y = 0 \) \( \Rightarrow \xi_2 = 16 (f/#)^2 \omega_0 \)

Paraxial Focus \( \xi_2 \) when \( p_y = 0 \) i.e. gaussian image plane

For \( w_0 > 0 \) marginal focus is before paraxial focus.
\[ \omega(x, y) = \omega_{x3} h p^3 \cos \Psi = \omega_{x3} h (p_x^2 p_y + p_y^3) \]

\[ \varepsilon_x = -\frac{R}{h} \omega_{x3} h (2 p_x p_y) = -\frac{R}{h} \omega_{x3} h p^3 \sin 2\Psi \]

\[ \varepsilon_y = -\frac{R}{h} \omega_{x3} h (p_x^2 + 3 p_y^2) = -\frac{R}{h} \omega_{x3} h p^4 (2 + \cos^2 \Psi) \]

zero with \( P_x \)

\[ C_S = \varepsilon_x \text{ for } h = 1 \text{ and } P_x = \pm \frac{1}{\sqrt{2}} \]

\[ P_y = \pm \frac{1}{\sqrt{2}} \]

\[ C_S = -\frac{R \omega_{x3}}{h_{max}} \]

\[ C_T = \varepsilon_y \text{ for } h = 1 \text{ and } P_x = 0 \]

\[ P_y = 1 \]

\[ C_T = -\frac{3 \omega_{x3}}{h_{max}} = 3 C_S \]

\[ \sin \theta = \frac{C_T}{2C_S} = \frac{1}{2} \]

\[ \theta = 30^\circ \]

\[ 2\theta = 60^\circ \]
\[ w(p_x, p_y) = w_{022} h^2 p^2 \cos^2 \psi = w_{222} h^2 p_y \]

Transverse
\[ \varepsilon_x = 0 \]
Ray Error
\[ \varepsilon_y = \frac{-R}{r_{\text{max}}} (w_{222} h^2 p_y) \]

\[ \psi_0 = p_x \]

---

Since \( \varepsilon_x = 0 \), pure astigmatism puts the sagittal focus on the Gaussian image plane. We need to add defocus to get to other foci.

\[ w(p_x, p_y) = w_{020} p^2 + w_{222} h^2 p_y \]

\[ \varepsilon_x = \frac{-2R}{r_{\text{max}}} w_{020} p_x \]

\[ \varepsilon_y = \frac{-2R}{r_{\text{max}}} w_{020} p_y - \frac{2R}{r_{\text{max}}} w_{222} h^2 p_y = \frac{-2R}{r_{\text{max}}} p_y \left( w_{020} + w_{222} h^2 \right) \]

When does \( \varepsilon_x \) (for \( p_x = 1 \)) equal \( -\varepsilon_y \) (for \( p_y = 1 \))?

\[ \frac{-2R}{r_{\text{max}}} w_{020} p_x = \frac{2R}{r_{\text{max}}} \left( w_{020} + w_{222} h^2 \right) \]
\( \omega_0 = -\frac{1}{2} \omega_{122} h^2 \)
\[
\varepsilon_2 = 4 \left( \frac{f}{\#} \right)^2 h^2 \omega_{122}
\]

When does \( \varepsilon_2 = 0 \)?

\( \varepsilon_2 = 0 \Rightarrow \omega_{122} = -\frac{1}{8} \omega_{222} h^2 \)

**Tangential focus**

The sagittal focus is on Gaussian image plane. The medial and tangential focus on spherical (parabolic approx.) surfaces.

\[
S_t = -4 \left( \frac{f}{\#} \right)^2 \omega_{222} h^2 = \frac{h^2}{d R_m}
\]

**Medical surface**

\[
R_m = -\frac{1}{8 \left( \frac{f}{\#} \right)^2 \omega_{222}}
\]

**Tangential surface**

\[
R_t = -\frac{1}{16 \left( \frac{f}{\#} \right)^2 \omega_{222}}
\]

The radius of curvature.

**Dimensional length of line focus**

\( L = 2 |\varepsilon_2| \) at tangential focus for \( f_x = 1 \)

\[
L = 2 |\varepsilon_2| = \left| \frac{d R}{R_{max}} \omega_{222} h^2 \right| \quad \text{but} \quad \omega_{222} = -\frac{1}{8} \omega_{122} h^2 \quad \text{at tangential focus}
\]

\[
L = 2 |\varepsilon_2| = \left| \frac{d R}{R_{max}} \omega_{222} h^2 \right| = \left| \frac{2 h^2}{R_{max}} |\omega_{222}| \right|
\]

**Diameter of circle of least confusion**

\( D = 2 |\varepsilon_2| \) at medial focus for \( f_x = 1 \)

\[
D = 2 |\varepsilon_2| = \left| \frac{d R}{R_{max}} \omega_{222} h^2 \right| \quad \text{but} \quad \omega_{222} = -\frac{1}{2} \frac{h^2}{2} \quad \text{at medial focus}
\]

\[
D = \frac{2 h^2}{R_{max}} |\omega_{222}| = \frac{L}{2}
\]
Field Curvature

$$\omega(p_x, p_y) = \omega_{220} h^2 p^2 = \omega_{220} h^2 (p_x^2 + p_y^2)$$

$$\varepsilon_x = -\frac{R}{\hbar} (\omega_{220} h^2 p_x) \quad \text{(in air)}$$

$$\varepsilon_y = -\frac{R}{\hbar} (\omega_{220} h^2 p_y)$$

Looks like defocus, but with \( h^2 \) dependence, can calculate \( S_2 \) in same manner

$$S_2 = 8 (F/#)^2 \omega_{220} h^2 = \frac{h^2}{2R_p}$$

Again, looks like spherical surface

$$R_p = \frac{1}{16 (F/#)^2 \omega_{220}}$$

Geodesic image surface
Distortion

\[ \omega(f_x, f_y) = \omega_{\text{III}} h^3 \cos \psi = \omega_{\text{III}} h^3 f_y \]

\[ \varepsilon_x = 0 \]

\[ \varepsilon_y = \frac{-R}{r_{\text{max}}} \omega_{\text{III}} h^3 \]

**Barrel Distortion**

\[ \omega_{\text{III}} > 0 \quad \varepsilon_y < 0 \quad f_y > 0 \]

**Pincushion Distortion**

\[ \omega_{\text{III}} < 0 \quad \varepsilon_y > 0 \quad f_y > 0 \]

Looks like it's with \( h^3 \) dependence instead of \( h \) dependence.

Show orders of polynomial studies
Through-focus PSF and Star Test

Show slides.

The star test is a qualitative test for examining the quality of an optical system. A travelling microscope or camera is used to look at the image of a point source through focus. Different aberrations have different characteristic patterns.

- Note from slides, for spherical aberration and astigmatism, apertures resemble defects to bring real foci to paraxial foci. In cone, tilt is incorporated to cause cone pattern.

Measurement of Distortion

Conventional Case

If height of real chief ray is \( y \), then the percent distortion is given by

\[
\text{% Distortion} = 100 \times \frac{y - y'}{y'}
\]

Show distortion slides.

Special cases: Anamorphic, f/0 lens, Scheimpflug

For lens, incorporate distortion such that \( y' = \frac{p}{f_1} \tan \theta \)

(i.e. linear in \( \theta \))
The imaging formula is given by

\[ \frac{1}{z'} - \frac{1}{z} = \frac{1}{f_{\mathcal{E}}}. \quad (1) \]

In conventional imaging, the object and image planes are parallel to one another with \( z = L \) (\( L \) is negative in the figure above) and \( z' = L' \). If the object plane is tilted by an angle \( \theta \), then the Scheimpflug condition says the image plane is tilted as well. The tilted object and image planes become functions of \( y \), so the Lensmaker's formula becomes

\[ \frac{1}{z'(y)} - \frac{1}{z(y)} = \frac{1}{f_{\mathcal{E}}}. \quad (2) \]

From the geometry in the image above, the object plane is described by a plane tilted about the \( x \) axis such that

\[ z(y) = L - y \tan \theta, \quad (3) \]

where \( L = z(\alpha) \)

where a counterclockwise rotation of the object plane corresponds to a positive value of \( \theta \). Plugging this expression (3) into equation (2) and solving for \( z'(y) \) leads to
\[ z'(y) = \frac{f (L - y \tan \theta)}{f_e + L - y \tan \theta} \approx \frac{f (L - y \tan \theta)}{f_e + L} \]  

where the assumption that \( L >> y \tan \theta \) has been made. Equation (4) also describes a plane tilted about the x axis.

**Location of Image Plane**

The location of the image plane can be found by evaluating \( z'(0) \).

\[
\frac{1}{\frac{f + L}{f_L}} = \frac{1}{z'(0)}
\]

Equation (5) is just a statement of the Lensmaker’s formula, requiring \( z'(0) = L' \).

**Intersection of the Object and Image Planes**

The object and image planes intersect when \( z(y) = z'(y) \). This intersection occurs when

\[
y = \frac{L}{\tan \theta}
\]

Plugging (6) back into the expressions for the object and image planes leads to

\[
z\left(\frac{L}{\tan \theta}\right) = L - L = 0 \quad \text{and} \quad z'\left(\frac{L}{\tan \theta}\right) = \frac{f_L}{f_e + L + f_e} = 0
\]

In other words, the object and image plane intersect at the plane of the lens.

**Image Plane Tilt**

Equation (4) can be rewritten as

\[
z'(y) = \frac{L}{f_e + L - y \tan \theta} - \frac{f \tan \theta}{f_e + L} \quad y = L' - y \tan \theta' \quad (8)
\]

where

\[
\tan \theta' = \frac{f \tan \theta}{L + f_e} \quad (9)
\]

**Magnification**
The magnification $m_o$ for the axial object and image points is given by

$$m_o = \frac{L'}{L} = \frac{f_{\varepsilon}}{L + f_{\varepsilon}}. \quad (10)$$

To calculate the magnification $m$ as a function of $y$ for the tilted system, equation (4) without the approximation $L \gg y \tan \theta$ needs to be used.

$$z'(y) = \frac{f_{\varepsilon}(L - y \tan \ell)}{f_{\varepsilon} + L - y \tan \theta} = \frac{f_{\varepsilon}z(y)}{f_{\varepsilon} + L - y \tan \theta} \Rightarrow m = \frac{z'(y)}{z(y)} = \frac{f_{\varepsilon}}{f_{\varepsilon} + L} \left[ \frac{1}{1 - \frac{y \tan \theta}{f_{\varepsilon} + L}} \right] \quad (11)$$

$$m = \frac{m_o}{\tan \theta} \left( 1 - \frac{y}{f_{\varepsilon} + L} \right). \quad (12)$$

Using a binomial expansion on equation (12) leads to

$$m = m_o \left[ 1 + \frac{\tan \theta}{f_{\varepsilon} + L} y + \ldots \right]. \quad (13)$$

In other words, the magnification is linear in $y$ or there is keystone distortion in the system (at where the truncated binomial expansion closely approximates equation (12)).

---

**Measurement of Distortion**

Place a target with known levels of distortions and examine image

<table>
<thead>
<tr>
<th>Object</th>
<th>Image</th>
</tr>
</thead>
</table>

**Barrel distortion**