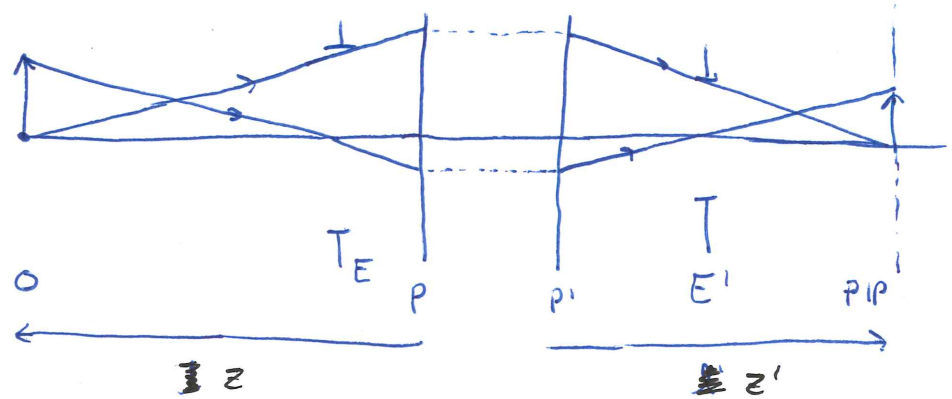
1.5 DIFFRACTION AND ABERRATIONS (Ch. 2 in Book)

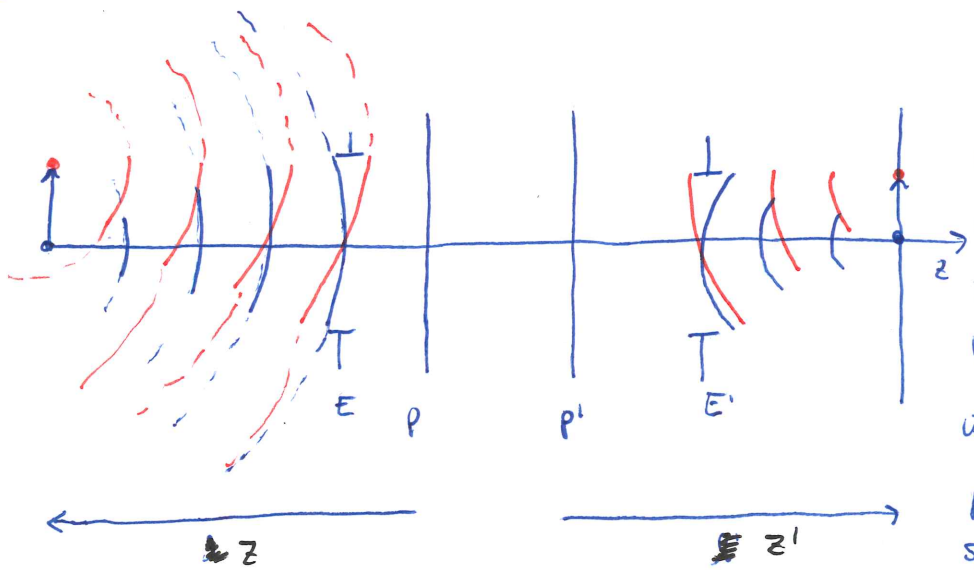
1.5.1 BLACK BOX OPTICAL SYSTEM BASED ON CARDINAL POINTS AND PUPILS (Book 2.1.1)

Complex systems reduce to simple box



Let's us determine the paraxial image plane (PIP) and the shape of the cones of light entering and exiting the system

1.5.2 WAVEFRONT PICTURE OF IMAGING (Book 2.1.2)



Each point on the object acts like a point source and radiates a spherical wavefront. A portion of this wavefront is captured by the entrance pupil and is relayed to the exit pupil. For and imaging system the diverging spherical waves are converted to converging spherical waves which collapse to a single point. For a "perfect" imaging system (Not diffraction limited) every point in the object plane maps to a perfect point in the paraxial image plane.

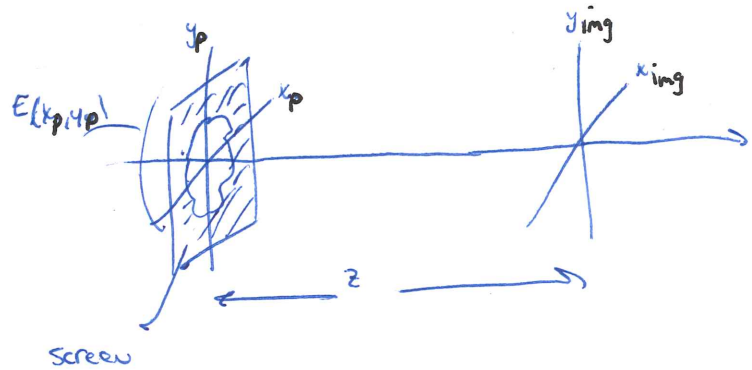
In the real world, we never have perfect imaging. Several things disrupt the process

1. Objects are often not planes, which means $z(x,y)$ the object distance depends upon the transverse coordinates, so $z'(x,y)$ as well.
2. Diffraction occurs at the aperture stop. The light at the edge of the captured wavefront interacts with the mechanical aperture stop leading to a imperfect wavefront leaving the exit pupil.
3. Aberrations occur due to an imperfect mapping of diverging spherical to converging spherical

1.5.3 Diffraction-limited Systems and Connection to Fresnel Diffraction. [Bode 2.1.3]

We have some control over the object distance ① and the aberrations ③ at least locally based on our lens design and imaging arrangement. We are stuck with diffraction since the light will always interact with the aperture stop or some other surface (vignetting) as it passes through the optical system. A "diffraction-limited" system is a system where the ~~object distance and aber~~ only diffraction affects the emerging wavefront.

Recall Fresnel Diffraction



For Fresnel diffraction, we had some incident wavefront with amplitude $E(x_p, y_p)$ on a screen with an opening in it $P(x_p, y_p)$. The Fresnel diffraction integral relates the amplitude $E(x_{img}, y_{img})$ at another plane located a distance z downstream.

$$E(x_{img}, y_{img}) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{\infty} P(x_p, y_p) E(x_p, y_p)\, e^{\frac{i\pi}{\lambda z}\left[(x_p - x_{img})^2 + (y_p - y_{img})^2\right]} dx_p\, dy_p$$

$k = \frac{2\pi}{\lambda}$

Expanding the exponential gives

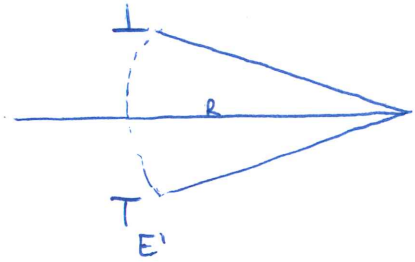
$$E(x_{img}, y_{img}) = \frac{e^{ikz}}{i\lambda z} e^{\frac{i\pi}{\lambda z}(x_{img}^2 + y_{img}^2)} \iint_{-\infty}^{\infty} P(x_p, y_p) E(x_p, y_p)\, e^{\frac{i\pi}{\lambda z}(x_p^2 + y_p^2)}\, e^{\frac{-i2\pi}{\lambda z}(x_p x_{img} + y_p y_{img})} dx_p\, dy_p$$

Finally, recall the definition of the Fourier transform $\mathcal{F}\{\ \}$

$$\mathcal{F}\{f(x,y)\} = \iint_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(x\xi + y\eta)} dx\, dy = F(\xi, \eta)$$

$$\boxed{E(x_{img}, y_{img}) = \frac{e^{ikz}}{i\lambda z} e^{\frac{i\pi}{\lambda z}(x_{img}^2 + y_{img}^2)}\, \mathcal{F}\left\{ P(x_p, y_p) E(x_p, y_p)\, e^{\frac{i\pi}{\lambda z}(x_p^2 + y_p^2)} \right\}_{\xi = \frac{x_{img}}{\lambda z},\ \eta = \frac{y_{img}}{\lambda z}}}$$

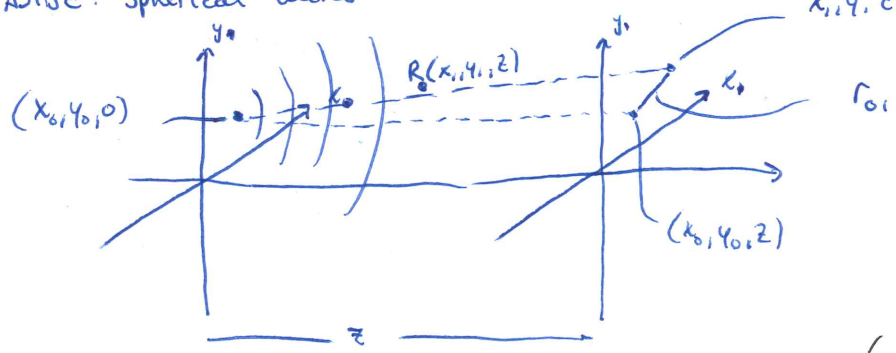
Compare the Fresnel diffraction arrangement to our black box imaging system



We have a converging spherical wave incident on a (usually) circular mask. This is the same arrangement as Fresnel diffraction where $P(x_p, y_p)$ is the transmission of the ~~exit~~ exit pupil and $E(x_p, y_p)$ is the incident converging spherical wave.

The Fresnel diffraction integral can therefore let us predict the amplitude $E(x_{img}, y_{img})$ and the irradiance $I(x_{img}, y_{img}) = |E(x_{img}, y_{img})|^2$ at any plane downstream from the exit pupil (including the paraxial image plane).

ASIDE: Spherical waves



Amplitude

A spherical wave is given by $E = \dfrac{A}{R(x_1, y_1, z)} e^{\pm i k R_0(x_1, y_1, z)}$ $\qquad k = \dfrac{2\pi}{\lambda}$

falls off as $\dfrac{1}{\text{distance}}$

minus for converging, plus for diverging

$$R_0(x_1, y_1, z) = \sqrt{r_{01}^2 + z^2} \quad \text{where}$$

$$r_{01}^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2$$

$$R_0(x_1, y_1, z) = z\sqrt{1 + \frac{r_{01}^2}{z^2}}$$

Binomial expansion

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$$

$$\cancel{R(x,y,z) = z\left(1 + \frac{(x_1-x_0)^2 + (y_1-y_0)^2}{2z^2} - \frac{[(x-x_0)^2 + (y_1-y_0)^2]^2}{8z^4} + \dots\right)}$$

$$R_0(x,y,z) = z\left(1 + \frac{r_{01}^2}{2z^2} - \frac{r_{01}^4}{8z^4} + \dots\right)$$

~~For Denominator if $\frac{r_{01}^2}{2z^2} \ll 1$ then $R_0(x,y,z) \simeq z$~~

~~$|r_{01}| \ll \sqrt{2}|z|$~~

~~Example $z = 100$ mm $r_{01} = 3$ mm $\Rightarrow 3 \ll 141$~~

For exponent, more care is needed

$$e^{\pm i\frac{2\pi}{\lambda}R_0(x,y,z)} = \cos\left(\frac{2\pi}{\lambda}R_0(x,y,z)\right) \pm i\sin\left(\frac{2\pi}{\lambda}R_0(x,y,z)\right)$$

Sines and cosines oscillate, so expansion terms need to be included if they represent a significant portion of a period.

$$\frac{2\pi}{\lambda}R_0(x,y,z) = \frac{2\pi}{\lambda}z + \frac{\cancel{2}\pi r_{01}^2}{\cancel{2}\lambda z} - \frac{2\pi r_{01}^4}{8\lambda z^3} + \dots$$

Example $z = 100$ mm $r_{01} = 3$ mm $\lambda = 0.5\ \mu m = 0.5 \times 10^{-3}$ mm

2nd term $\frac{9\pi}{0.05} = 180\pi \Leftarrow$ 90 more oscillations need to include

3rd term $\frac{81\pi}{4(0.5\times10^{-3})(100)^3} = 0.0405\pi$ small fraction of oscillation do not include

so when $\frac{\pi r_{01}^4}{4\lambda z^3} \ll 1$ can ignore 3rd and higher terms

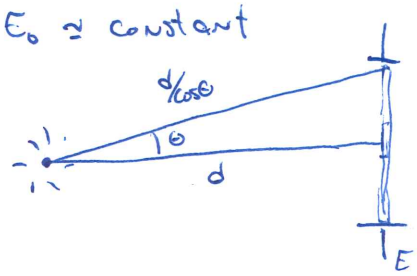
So spherical wave is approximately

$$E = \frac{E_0}{z} e^{\pm ikz} e^{\pm i\frac{\pi}{\lambda z}r_{01}^2}$$

minus converging
plus diverging

where $E_0 = \frac{A}{R(x,y,z)} \simeq$ Constant

Why is $E_0 \simeq$ constant



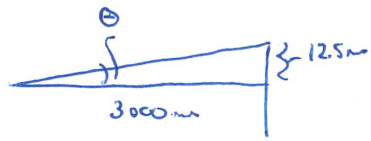
In many cases $\cos\theta \simeq 1$ so illumination at exit pupil $\simeq$ constant

Irradiance

$$I(x,y) = |E|^2 = \frac{|E_0|^2}{(R_0)^2}$$

For point source $I(x,y)$ falls off as $\cos^3\theta$ across entrance pupil

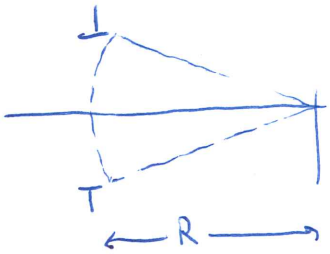
For example DSLR with $f_E = 50$mm taking a picture of an object at 3m and $f/\# = 2$



$$D_E = \frac{f_E}{f/\#} = 25\text{mm}$$

$\theta = 0.24°$ $\quad \cos^3\theta = 0.99997$

Back to exit pupil



$$P(x_p, y_p) = \begin{cases} 1 & r_p = \sqrt{x_p^2 + y_p^2} \le \frac{D_{E'}}{2} \\ \frac{1}{2} & \\ 0 & \text{otherwise} \end{cases} = \mathrm{cyl}\left(\frac{\sqrt{x_p^2+y_p^2}}{D_{E'}}\right)$$

where $D_{E'}$ is diameter of exit pupil

$$E(x_p, y_p) = E_0 \, e^{-ikR} \, e^{-\frac{i\pi}{\lambda R}(x_p^2 + y_p^2)}$$

Plug into Fresnel diffraction integral

$$E(x_{img}, y_{img}) = \frac{E_0}{i\lambda R} e^{\frac{i\pi}{\lambda R}(x_{img}^2 + y_{img}^2)} \mathcal{F}\left\{ \mathrm{cyl}\left(\frac{r_p}{D_{E'}}\right) \right\}$$

Recall $\mathcal{F}\left\{ \mathrm{cyl}\left(\frac{r}{d}\right) \right\} = |d|^2 \frac{\pi}{4} \, \mathrm{somb}(d\rho)$ $\qquad r = \sqrt{x^2+y^2}$, $\rho = \sqrt{\xi^2 + \eta^2}$

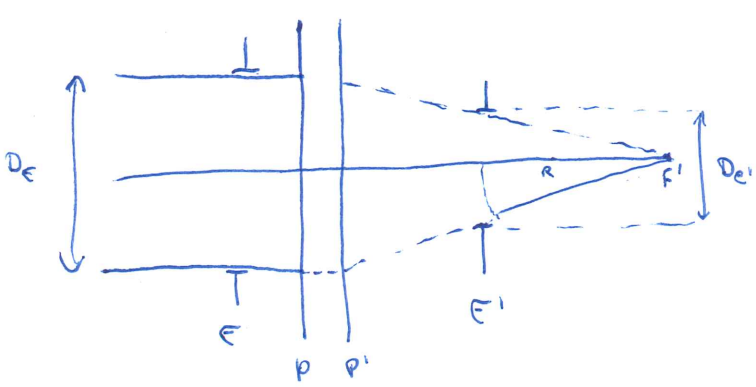
where $\mathrm{somb}(r) = \frac{2 J_1(\pi r)}{\pi r}$ and $J_1()$ is first order Bessel function

$$E(x_{img}, y_{img}) = \frac{E_0}{i\lambda R} e^{\frac{i\pi}{\lambda R}(x_{img}^2 + y_{img}^2)} D_{E'}^2 \frac{\pi}{4} \cdot \frac{2 J_1(\pi D_{E'} \rho)}{\pi D_{E'} \rho}$$

Substitute $\rho = \sqrt{\xi^2 + \eta^2} = \sqrt{\frac{x_{img}^2}{\lambda^2 R^2} + \frac{y_{img}^2}{\lambda^2 R^2}} = \frac{r_{img}}{\lambda R}$

The irradiance $I(x_{img}, y_{img})$

$$I(x_{img}, y_{img}) = |E(x_{img}, y_{img})|^2 = \frac{|E_0|^2}{\lambda^2 R^2} \left[\frac{\pi^2 D_{E'}^4}{16}\right] \left[\frac{2 J_1\left(\frac{\pi D_{E'} r_{img}}{\lambda R}\right)}{\left(\frac{\pi D_{E'} r_{img}}{\lambda R}\right)} \right]^2$$



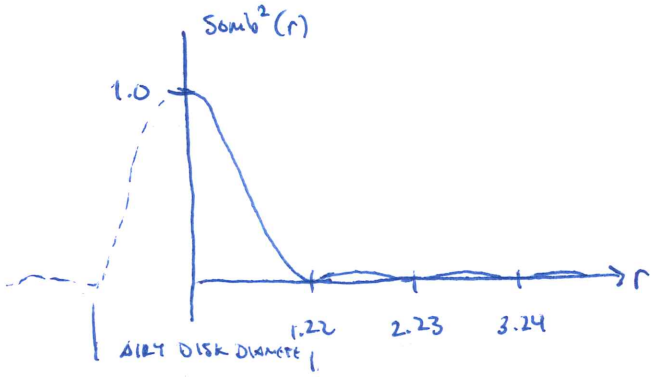
$$\frac{D_{E'}}{R} = \frac{D_E}{\overline{P'F'}} = \frac{1}{f/\#}$$

$$I(x_{img}, y_{img}) = \frac{|E_0|^2 \pi^2 D_E^2}{16 \lambda^2 (f/\#)^2} \left[\frac{2 J_1\left(\frac{\pi r_{img}}{\lambda f/\#}\right)}{\left(\frac{\pi r_{img}}{\lambda f/\#}\right)} \right]^2$$ Airy Pattern

$|E_0|^2$ is the irradiance (W/m^2) falling on the entrance pupil

$|E_0|^2 \frac{\pi D_E^2}{4}$ is the power (W) falling on the entrance pupil

~~Sombrero function~~ ~~$\mathrm{somb}\left(\frac{r}{a}\right)$~~ $$I(x_{img}, y_{img}) = \frac{|E_0|^2 \pi^2 D_E^2}{16 \lambda^2 (f/\#)^2} \mathrm{somb}^2\left(\frac{r_{img}}{\lambda f/\#}\right)$$



zeroes of $\mathrm{somb}^2(r)$ occur when

$r = 1.22, 2.23, 3.24 \ldots$

83.9% of energy contained within $r \leq 1.22$

Airy disk diameter - diameter of region contained within first dark ring

~~Airy disk diameter~~ $$\frac{r_{img}}{\lambda f/\#} = 1.22$$

$$r_{img} = 1.22 \lambda f/\#$$

$$\boxed{\text{Airy disk diameter} \quad d_{r_1} = 2.44 \lambda f/\#}$$

<u>RULE OF THUMB:</u> In visible $\lambda \sim 0.5\ \mu m$, so the Airy disk diameter in microns is approximately the f/#.

f/# = 2 "fast" system $\sim 2\ \mu m$ Airy disk diameter

f/# = 22 "slow" system $\sim 22\ \mu m$ Airy disk diameter

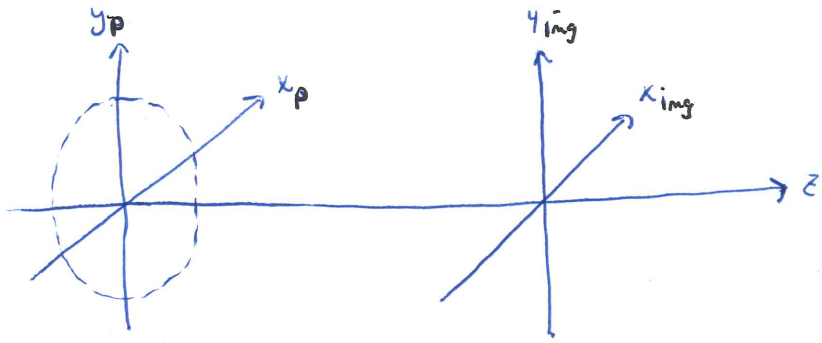
CCD/CMOS pixels $\sim 3 - 15\ \mu m$.

We have intuitively assumed a point source at infinity. If point source at finite distance, use working $f/\#_w$.

1.5.4 Point Spread Function (PSF) (Book 2.1.4)

One metric of ~~how good~~ the quality of an optical system is the PSF. As the name implies, the PSF is the image of a point source. In the diffraction limited case, the PSF is just the Airy ~~disk~~ pattern. When aberrations are present, PSF spreads further. For a given F/#, the Airy ~~disk~~ pattern is the best we can do.

1.5.5 Sign and Coordinate System Conventions (Book 2.1.5)



Right handed coordinate systems

we usually look in negative z direction so positive values are

EXIT PUPIL PLANE

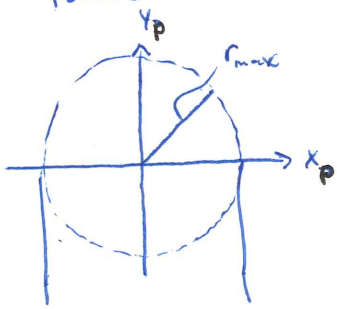
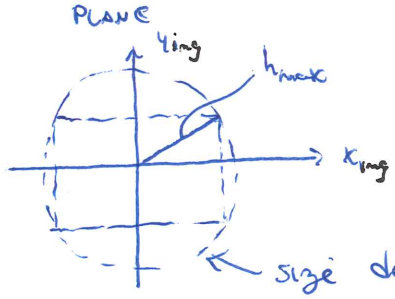


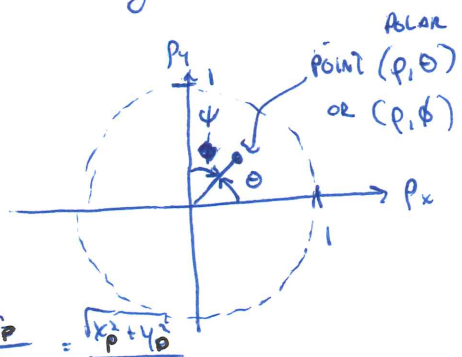
IMAGE PLANE



size determined by field stop

$D_{E'}$ = Diameter of exit pupil $= 2r_{max}$

Normalized Coordinates



POLAR: POINT (ρ, θ) or (ρ, ϕ)

CARTESIAN: (p_x, p_y)

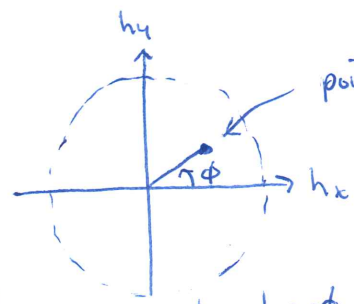
$\rho = \frac{r_p}{r_{max}} = \frac{\sqrt{x_p^2 + y_p^2}}{r_{max}}$

$\theta = \tan^{-1}\left[\frac{y_p}{x_p}\right]$

$\psi = \frac{\pi}{2} - \theta$

$p_x = \rho\cos\theta = \rho\sin\psi$

$p_y = \rho\sin\theta = \rho\cos\psi$



point (h_x, h_y) Cartesian

OR

(h, ϕ) Polar

$h_x = h\cos\phi$

$h_y = h\sin\phi$

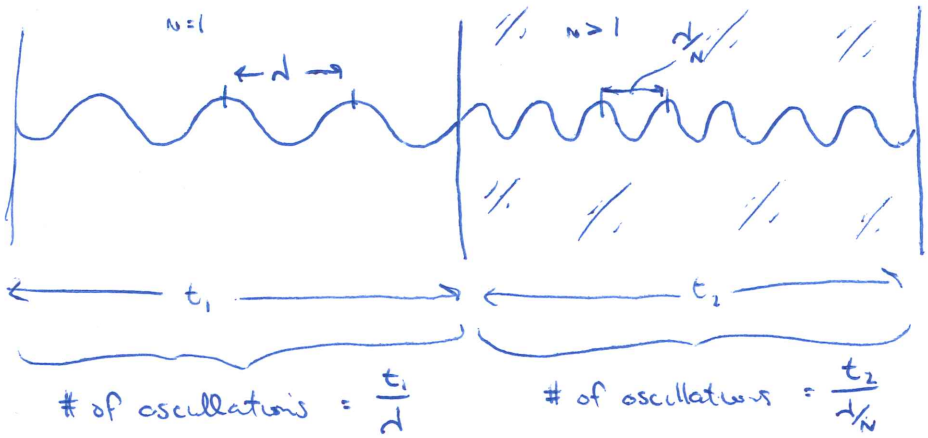
$h = \frac{r_{img}}{h_{max}} = \frac{\sqrt{x_{img}^2 + y_{img}^2}}{h_{max}}$

$\phi = \tan^{-1}\left[\frac{y_{img}}{x_{img}}\right]$

Care must be taken when dealing with pupil coordinates. Most texts on aberration theory use φ as angle (for historical reasons), while most lens design software uses θ. I will mainly stick with θ.

1.5.6 Optical Path Length (OPL), OPTICAL PATH DIFFERENCE (OPD), WAVEFRONT ERROR

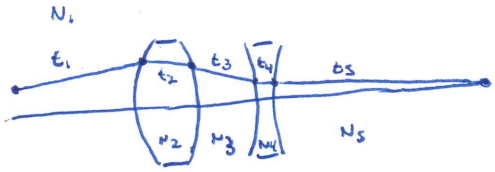
[Book 2.1.6]



$$\#\text{ of oscillations} = \frac{t_1}{\lambda} \qquad \#\text{ of oscillations} = \frac{t_2}{\lambda/N}$$

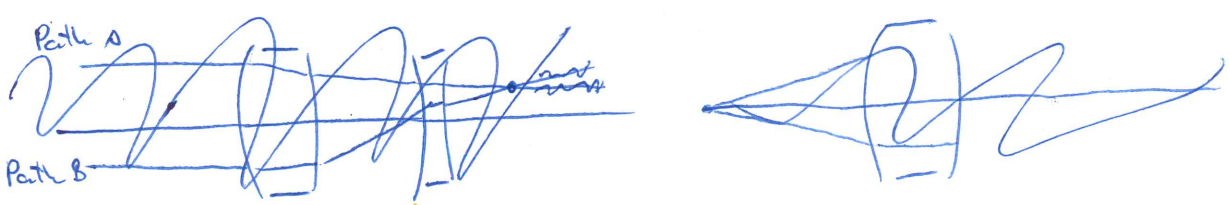
$$\text{Total } \#\text{ of oscillation} = \frac{t_1 + N t_2}{\lambda}$$

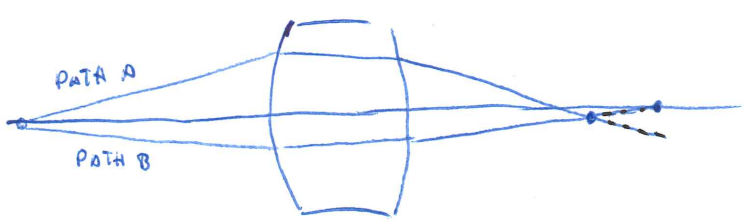
The numerator is the optical path length (OPL). It is related to the amount of time it takes a ray to propagate from one point to another. In the ray picture



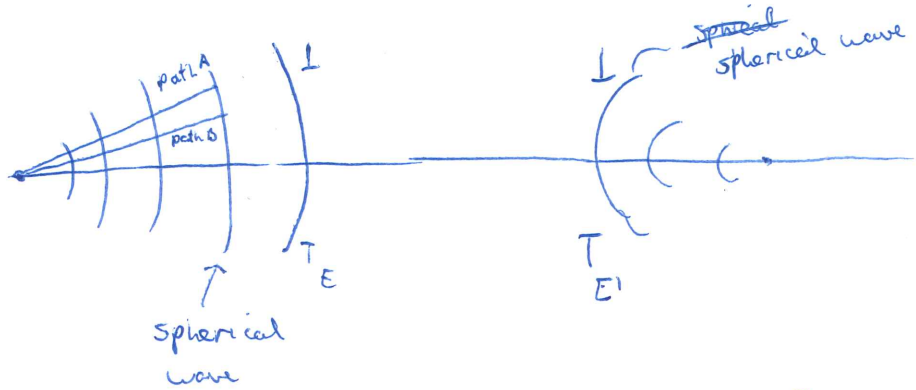
$$OPL = \sum_i N_i t_i$$

We are usually interested in the difference in OPLs between two different paths through a system. For path A and path B, the optical path difference (OPD) is $OPL_A - OPL_B$





Interested in OPD since it tells us how to combine the rays.

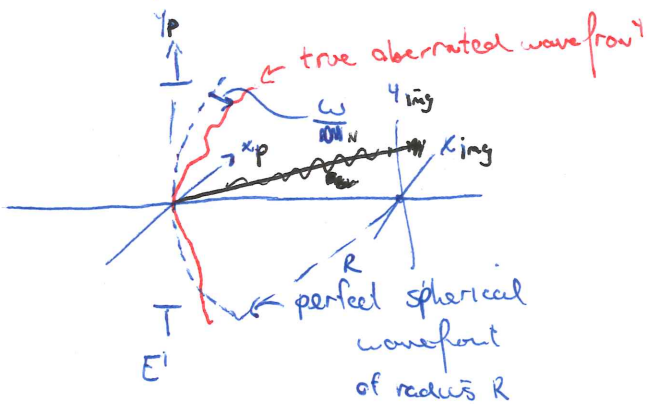


What is OPD between path A and path B? OPD = 0

Each path has a OPL = radius of wavefront

In fact we can turn this around and say a wavefront is a surface of constant OPL and any two points ~~of~~ on the wavefront have OPD = 0.

What happens when we have aberrations?



Some paths through the optical system are faster than others. wavefront is no longer spherical and will not converge to a point.

$$r_{oi} = \sqrt{(x_p - x_{img})^2 + (y_p - y_{img})^2}$$

Let's say our aberrated wave front is given by $\frac{r_{oi}^2}{2R} + w(x_p, y_p)$

↑ parabolic approximation to perfect spherical wave

~~[illegible]~~.

$w(x_p, y_p)$ is called the wavefront error

The electric field corresponding to the aberrated wavefront is

$$E(x_p, y_p) = E_0 e^{-\frac{i2\pi}{\lambda}\left[\frac{r_{oi}^2}{2R} + w(x_p, y_p)\right]}$$

If we plug this into the Fresnel diffraction integral to calculate $E(x_{img}, y_{img})$ at $z = R$

$$E(x_{img},y_{img}) = \frac{e^{ikR}}{i\lambda R}\, e^{\frac{i\pi}{\lambda R}(x_{img}^2+y_{img}^2)}\, \mathcal{F}\left\{ P(x_p,y_p)\, E_0\, e^{-\frac{i2\pi}{\lambda}\left[\frac{r_{0i}^2}{2R}+W(x_p,y_p)\right]}\, e^{\frac{i\pi}{\lambda R}(x_p^2+y_p^2)} \right\}$$

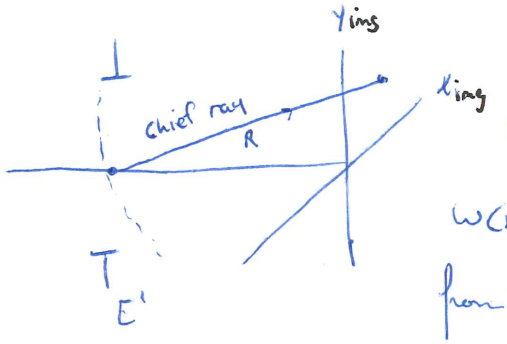
$$\xi = \frac{x_{img}}{\lambda R}$$

$$\eta = \frac{y_{img}}{\lambda R}$$

$$E(x_{img},y_{img}) = \frac{E_0 e^{ikR}}{i\lambda R}\, e^{\frac{i\pi}{\lambda R}(x_{img}^2+y_{img}^2)}\, \mathcal{F}\left\{ P(x_p,y_p)\, e^{-\frac{i2\pi}{\lambda}W(x_p,y_p)} \right\}_{\xi}$$

$$\boxed{I(x_{img},y_{img}) = |E(x_{img},y_{img})|^2 = PSF = \frac{|E_0|^2}{\lambda^2 R^2}\left|\mathcal{F}\left\{ P(x_p,y_p)\, e^{-\frac{i2\pi}{\lambda}W(x_p,y_p)} \right\}\right|^2}$$

How do we find R, the radius of the reference sphere?

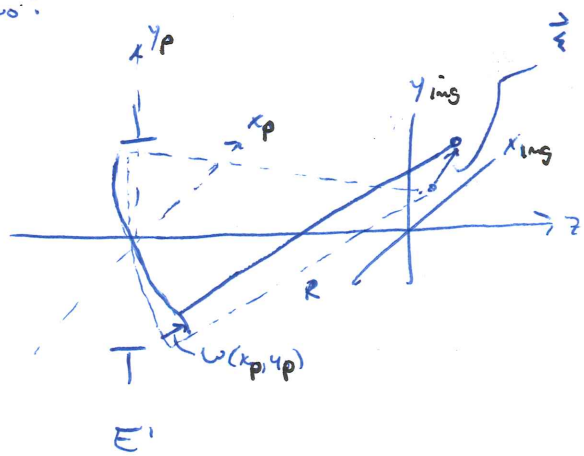


R = distance from intersection with (x_{img},y_{img}) plane and center of exit pupil.

$W(x_p,y_p)$ is OPD between a ray emerging from the exit pupil at (x_p,y_p) and the chief ray.

All rays normal to the Reference Sphere pass through ~~[illegible]~~ a single pt. in (x_{img},y_{img})

When wavefront error is present, rays (i.e. normals to the wavefront) do not meet at a single point. Instead, the ray intersection in the (x_{img},y_{img}) plane depends on the exit pupil coordinates (x_p,y_p). How do we connect the two?

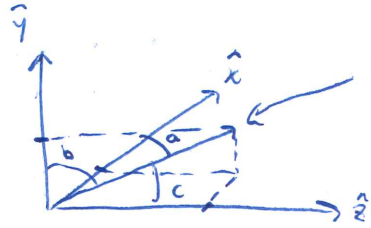


The intersection ~~[illegible]~~ is displaced by a vector $\vec{\xi}$ where $\vec{\xi} = \xi_x \hat{x} + \xi_y \hat{y}$

Our goal is to relate ξ_x, ξ_y to the wavefront error $W(x_p,y_p)$.

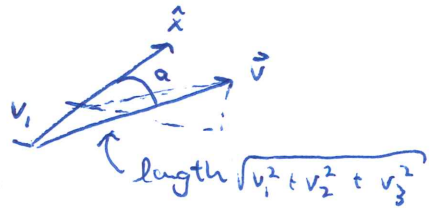
1.5.7 Transverse Ray Error and Spot Diagrams [Book 2.1.7]

Direction Cosines



vector $\vec{v} = v_1\hat{x} + v_2\hat{y} + v_3\hat{z} = (v_1, v_2, v_3)$

Look at $\hat{x}$ axis



$$\cos a = \frac{\vec{v}\cdot\hat{x}}{\sqrt{v_1^2+v_2^2+v_3^2}} = \alpha = \frac{v_1}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

Similarly

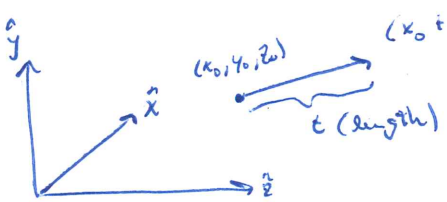
$$\cos b = \frac{\vec{v}\cdot\hat{y}}{\sqrt{v_1^2+v_2^2+v_3^2}} = \beta = \frac{v_2}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

$$\cos c \quad \frac{\vec{v}\cdot\hat{z}}{\sqrt{v_1^2+v_2^2+v_3^2}} = \gamma = \frac{v_3}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

Note that $\alpha^2 + \beta^2 + \gamma^2 = 1$

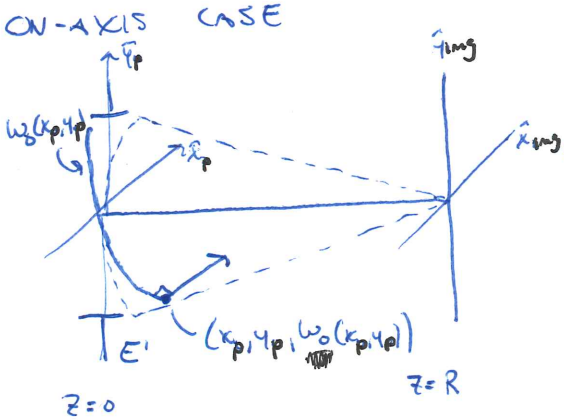
So vector (α, β, γ) is just a unit vector pointing in the same direction as $\vec{v}$

Rays Rays are extensions of the concept of a vector. They are described by two vectors: (x_0, y_0, z_0) denoting the ray origin and (α, β, γ) denoting the ray's trajectory



Since (α, β, γ) a unit vector, t is a scalar that extends the ray in the direction of the trajectory

ON-AXIS CASE



Denote the true aberrated wavefront as $W_0(x_p, y_p)$

where

$$W_0(x_p, y_p) = \underbrace{\frac{x_p^2 + y_p^2}{2R}}_{\text{Reference Sphere}} + \underbrace{W(x_p, y_p)}_{\text{Wavefront Error}}$$

FOR ON-AXIS CASE

a ray on this wavefront has it's origin at $(x_p, y_p, w_0(x_p, y_p))$ and its trajectory is defined by the unit normal at point (x_p, y_p)

$$\vec{N} : \left(-\frac{\partial w_0}{\partial x_p}, -\frac{\partial w_0}{\partial y_p}, 1\right) \quad \text{wavefront normal}$$

$$\vec{N} = \left(-\frac{x_p}{R} - \frac{\partial w}{\partial x_p}, -\frac{y_p}{R} - \frac{\partial w}{\partial y_p}, 1\right)$$

$$\hat{N} : \left(\frac{\overbrace{-\frac{x_p}{R} - \frac{\partial w}{\partial x_0}}^{\alpha}}{\sqrt{}}, \frac{\overbrace{-\frac{y_p}{R} - \frac{\partial w}{\partial y_p}}^{\beta}}{\sqrt{}}, \frac{\overbrace{1}^{\gamma}}{\sqrt{}}\right) \quad \text{unit normal}$$

where $\sqrt{} = \sqrt{\left(\frac{x_p}{R} + \frac{\partial w}{\partial x_p}\right)^2 + \left(\frac{y_p}{R} + \frac{\partial w}{\partial y_p}\right)^2 + 1}$

Ray is given by

$$(x_p, y_p, w_0(x_p, y_p)) + t\hat{N}$$

look at z component in image plane $(z = R)$

$$w_0(x_p, y_p) + \frac{t}{\sqrt{\left(\frac{x_p}{R} + \frac{\partial w}{\partial x_p}\right)^2 + \left(\frac{y_p}{R} + \frac{\partial w}{\partial y_p}\right)^2 + 1}} = R$$

$$t : \left[R - w_0(x_p, y_p)\right]\sqrt{}$$

Approximation: $R - w_0(x_p, y_p) \simeq R$

so $t \simeq R\sqrt{}$

and ray $(x_0, y_0, w_0(x_p, y_p)) + \left(-x_p - \frac{R\,\partial w}{\partial x_p}, -y_p - \frac{R\,\partial w}{\partial y_p}, R\right)$

$$\simeq \left(\underbrace{-\frac{R\,\partial w}{\partial x_p}}_{\xi_x}, \underbrace{-\frac{R\,\partial w}{\partial y_p}}_{\xi_y}, R\right)$$

when $w(x_p, y_p) = 0$, ray strikes at $(0, 0, R)$

For off-axis case, reference sphere is $\frac{(x_p - x_{img})^2 + (y_p - y_{img})^2}{2R}$. Similar derivation with same results.

$$\boxed{\varepsilon_x = \frac{-R}{N}\frac{\partial W(x_p,y_p)}{\partial x_p} \qquad \varepsilon_y = \frac{-R}{N}\frac{\partial W(x_p,y_p)}{\partial y_p}}$$ Transverse Ray Error (in Air)

If we use normalized pupil coordinates instead

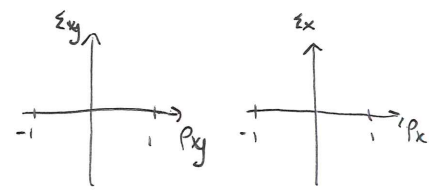
$$\rho_x = \frac{x_p}{r_{max}} \qquad \rho_y = \frac{y_p}{r_{max}}$$

$$\frac{\partial \rho_x}{\partial x_p} = \frac{1}{r_{max}} \qquad \frac{\partial \rho_y}{\partial y_p} = \frac{1}{r_{max}}$$

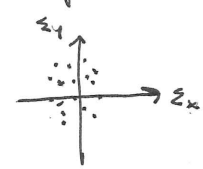
From chain rule

$$\varepsilon_x = -R\frac{\partial W(\rho_x,\rho_y)}{\partial \rho_x}\frac{\partial \rho_x}{\partial x_p} \qquad \varepsilon_y = -R\frac{\partial W(\rho_x,\rho_y)}{\partial \rho_y}\frac{\partial \rho_y}{\partial y_p}$$

$$\boxed{\varepsilon_x = \frac{-R}{N r_{max}}\frac{\partial W(\rho_x,\rho_y)}{\partial \rho_x} \qquad \varepsilon_y = \frac{-R}{N r_{max}}\frac{\partial W(\rho_x,\rho_y)}{\partial \rho_y}}$$

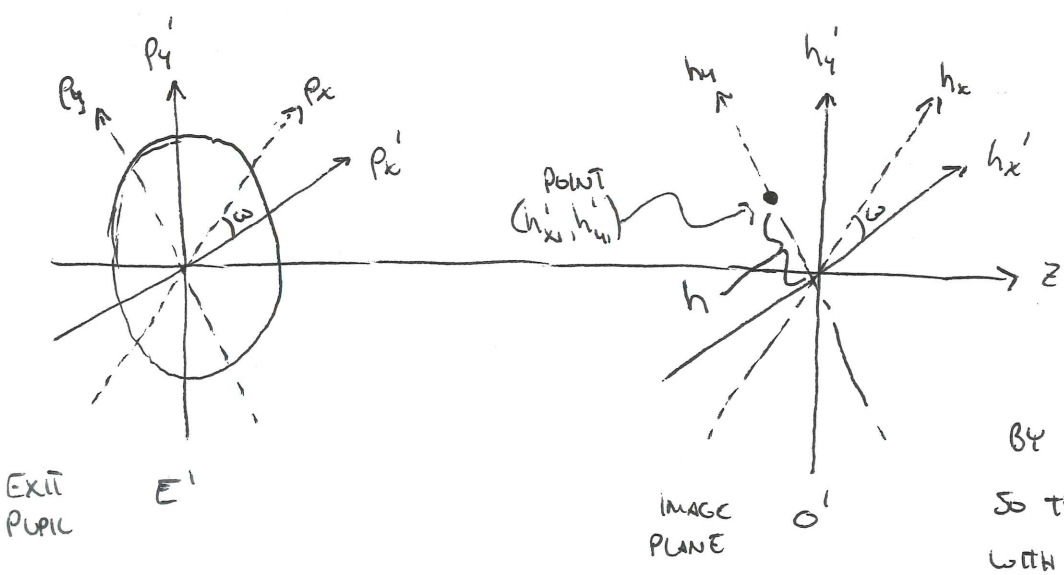


<u>SPOT DIAGRAM</u> - A scatterplot of ε_x vs. ε_y. Values of ρ_x and ρ_y are chosen to determine $(\varepsilon_x, \varepsilon_y)$. This point is marked and the process repeated. Cartesian, polar and random values of ρ_x, ρ_y typical.



[Book 2.2]

1.5.8 ABERRATIONS OF ROTATIONALLY SYMMETRIC OPTICAL SYSTEMS.



POINT OF INTEREST IN IMAGE PLANE IS (h_{x1}', h_{y1}').

ROTATE COORDINATE SYSTEM BY ANGLE ω ABOUT Z-AXIS SO THIS POINT IS ALIGNED WITH h_y-axis. NEW POINT IS $(0, h)$

$$\rho_x = \rho_x' \cos\omega + \rho_y' \sin\omega \qquad h_x = h_x' \cos\omega + h_y' \sin\omega$$

$$\rho_y = \rho_y' \cos\omega - \rho_x' \sin\omega \qquad h_y = h_y' \cos\omega - h_x' \sin\omega$$

Relationship between rotated and non-rotated coordinates

Choose ω so that

$$0 = h_{x1}' \cos\omega + h_{y1}' \sin\omega \Rightarrow h_{x1}' = -h_{y1}' \tan\omega$$

$$h = h_{y1}' \cos\omega - h_x' \sin\omega \Rightarrow h = h_{y1}' \cos\omega + h_{y1}' \frac{\sin^2\omega}{\cos\omega}$$

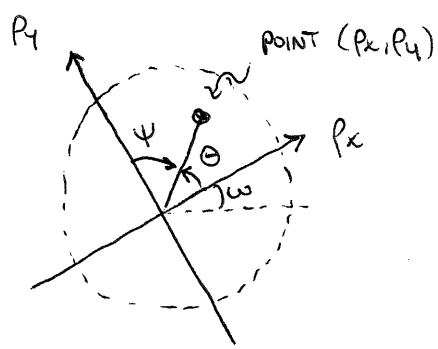
$$h \cos\omega = h_{y1}'$$

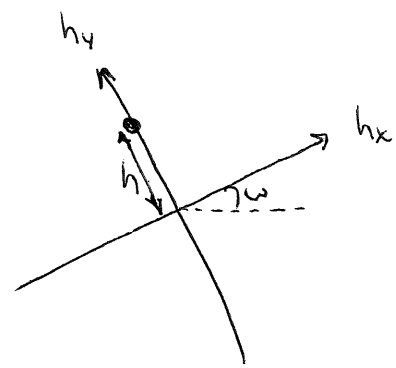
$$\boxed{h = \frac{h_{y1}'}{\cos\omega}}$$

$$\tan\omega = -\frac{h_{x1}'}{h_{y1}'}$$

$$\boxed{\omega = \tan^{-1}\left(\frac{-h_{x1}'}{h_{y1}'}\right) = -\tan^{-1}\left(\frac{h_{x1}'}{h_{y1}'}\right)}$$

Note that pupil rotates with image plane





POINT (ρ_x, ρ_y) IN CARTESIAN

IS POINT (ρ, θ) IN POLAR

where $\rho^2 = \rho_x^2 + \rho_y^2 \qquad \rho_x = \rho\cos\theta$

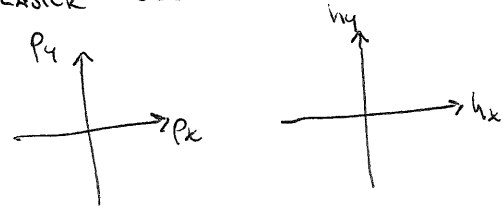
$\theta = \tan^{-1}\left[\frac{\rho_y}{\rho_x}\right] \qquad \rho_y = \rho\sin\theta$

$\psi = \frac{\pi}{2} - \theta$ FROM ABERRATION THEORY

$\rho_x = \rho\sin\psi \qquad \rho^2 = \rho_x^2 + \rho_y^2$

$\rho_y = \rho\cos\psi \qquad \psi = \tan^{-1}\left[\frac{\rho_x}{\rho_y}\right]$

EASIER JUST TO LOOK AT



IDEAL IMAGE POINT LOCATED AT $(0, h)$

In general wavefront error is a function of ρ_x, ρ_y, h_x, h_y. For rotationally symmetric system

$$W(\rho_x, \rho_y, h_x, h_y) = W(\rho_x', \rho_y', h_x', h_y')$$

In other words, shape of wavefront independent of how we rotate coordinate system

Look for combinations of ρ_x, ρ_y, h_x, h_y that satisfy this requirements

$$\rho_x^2 + \rho_y^2 = \rho_x'^2 \cos^2\omega - 2\rho_x'\rho_y' \cancel{\sin\omega\cos\omega} + \rho_y'^2 \sin^2\omega$$
$$\rho_x'^2 \sin^2\omega + 2\rho_x'\rho_y' \cancel{\sin\omega\cos\omega} + \rho_y'^2 \cos^2\omega = \rho_x'^2 + \rho_y'^2 \quad \text{i.e independent of } \omega$$

Same for $h_x^2 + h_y^2$

$$\rho_x h_x + \rho_y h_y = \rho_x' h_x' \cos^2\omega - \rho_x' h_y' \cancel{\sin\omega\cos\omega} - \rho_y' h_x' \cancel{\sin\omega\cos\omega} + \rho_y' h_y' \sin^2\omega$$
$$\rho_x' h_x' \sin^2\omega + \rho_x' h_y' \cancel{\sin\omega\cos\omega} + \rho_y' h_x' \cancel{\sin\omega\cos\omega} + \rho_y' h_y' \cos^2\omega$$

$$= \rho_x' h_x' + \rho_y' h_y'$$

For a given point in the image plane (h_x', h_y'), we can always rotate our coordinate system such that the point is $(0, h)$. We represent our wavefront as

$$W(\rho_x, \rho_y, h_x, h_y) = W(\rho^2, h^2, h\rho\cos\psi)$$

where $\rho^2 = \rho_x^2 + \rho_y^2 \quad h^2 = h_x^2 + h_y^2 = h_y^2 \quad h\rho\cos\psi = \rho_y h_y$

NOTE ψ measured clockwise from ρ_y axis

<u>POWER SERIES</u>

1-D $f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 \cdot 1 + a_1 x + a_2 x^2 + \cdots$ converges if $f(x)$ "nice", smooth

2-D $f(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots$

want to represent $W(\rho^2, h^2, h\rho\cos\psi)$ as power series.

$$W(\rho^2, h^2, h\rho\cos\psi) = \sum_{i,j,k=0}^{\infty} a_{ijk} (h^2)^i (\rho^2)^j (h\rho\cos\psi)^k$$

Let's modify the numbering scheme on a_{ijk}

Define $W_{(2i+k),(2j+k),k} = a_{ijk}$

Regrouping like terms

$$W(\rho^2, h^2, h\rho\cos\psi) = \sum_{i,j,k=0}^{\infty} W_{(2i+k),(2j+k),k}\, h^{(2i+k)}\, \rho^{(2j+k)} \cos^k\psi$$

i	j	k	$W_{(2i+k),(2j+k),k}$	Term
0	0	0	W_{000}	1
1	0	0	W_{200}	h^2
0	1	0	W_{020}	ρ^2
0	0	1	W_{111}	$h\rho\cos\psi$
1	1	0	W_{220}	$h^2\rho^2$
1	0	1	W_{311}	$h^3\rho\cos\psi$
0	1	1	W_{131}	$h\rho^3\cos\psi$
2	0	0	W_{400}	h^4
0	2	0	W_{040}	ρ^4
0	0	2	W_{222}	$h^2\rho^2\cos^2\psi$

⋮

1.5.8.1 PISTON, TILT AND DEFOCUS

Piston is a constant offset between the reference sphere and the wavefront

we can just think of this as the wavefront having propagated a little further, ~~but not changing shape~~

Typically, we define $W(0, h^2, 0) = 0$ i.e. no wavefront error in center of pupil.

This definition means

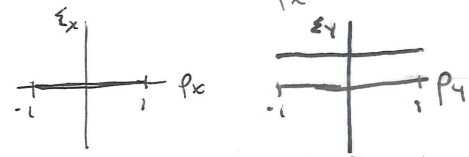
$$W_{000} = W_{200} = W_{400} = 0$$

<u>Tilt</u> is the portion of the wavefront error given by the $W_{111} h \rho \cos\psi$ term.

$$W(\rho_x, \rho_y) = W_{111} h \rho_y \qquad \text{(we'll drop the primes on } \rho_y\text{)}$$

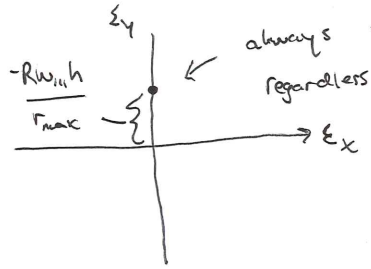
linear in ρ_y i.e. a plane rotated about ρ_x axis

Transverse Ray Error (in air)



$$\varepsilon_x = \frac{-R}{r_{max}} \frac{\partial W}{\partial \rho_x} = 0 \qquad \varepsilon_y = \frac{-R}{r_{max}} \frac{\partial W}{\partial \rho_y} = \frac{-R W_{111} h}{r_{max}} = \text{constant for given } h$$

Spot Diagram



always forms point regardless of where ray is in exit pupil.

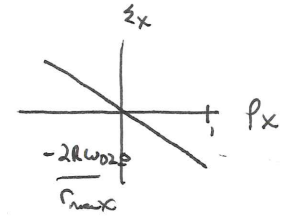
<u>Defocus</u> is the portion of the wavefront error given by the $W_{020}\rho^2$ term

$$W(\rho_x, \rho_y) = W_{020}\rho^2 = W_{020}(\rho_x^2 + \rho_y^2)$$

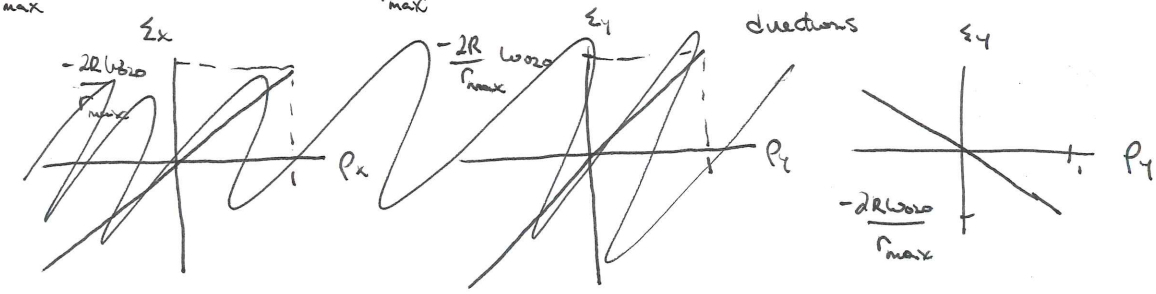
(in air)

$$\varepsilon_x = \frac{-R}{r_{max}} 2 W_{020} \rho_x \qquad \varepsilon_y = \frac{-R}{r_{max}} 2 W_{020} \rho_y$$

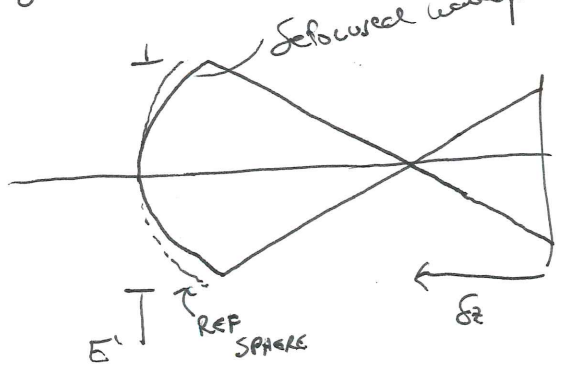
linear in both directions

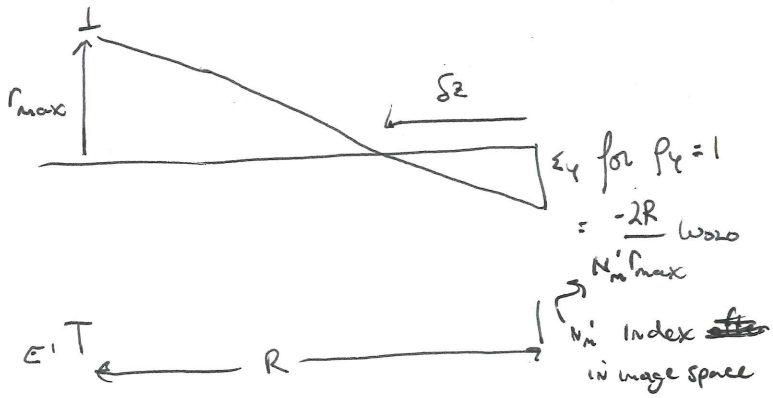


Transverse Ray Error



Longitudinal Aberration



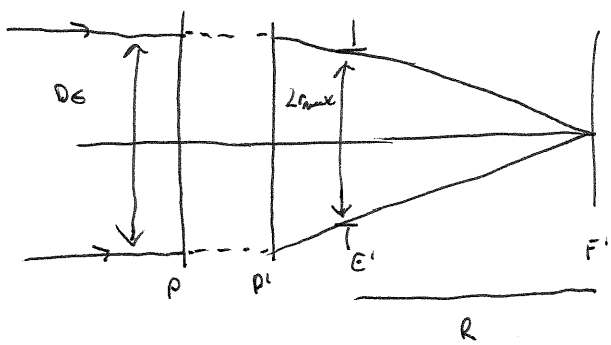


From ~~sol~~ similar triangles

$$\frac{r_{max}}{R+\delta z} = \frac{\dfrac{2R}{N_m' r_{max}} W_{020}}{-\delta z}$$

NOTE $\delta z < 0$ for $W_{020} > 0$

$$\delta z = -\frac{2R W_{020}}{N_m' r_{max}^2}(R+\delta z) \simeq -\frac{2R^2 W_{020}}{N_m' r_{max}^2} = -\frac{8}{N_m'}\left(\frac{R}{2r_{max}}\right)^2 W_{020}$$



$$\frac{R}{2r_{max}} = \frac{\overline{P'F'}}{D_E} = f/\#$$

$$\boxed{\delta z = -\frac{8}{N_m'}(f/\#)^2 W_{020}}$$

For $W_{020} > 0$ shift is towards the exit pupil

This relationship will come up repeatedly when we do non-interferometric testing

[Book 2.2.2]

$\boxed{1.5.8.2}$ Seidel Aberrations - Seidel defined aberrations much in the way we have previously. They are often labeled S_I, S_{II}, S_{III}, S_{IV}, S_V and are proportional to our $W_{(2i+k),(2j+k),k}$ coefficients (called wavefront coefficients from here on out).

$S_I = 8 W_{040}$ = SPHERICAL ABERRATION

$S_{II} = 2 W_{131}$ = COMA

$S_{III} = 2 W_{222}$ = ASTIGMATISM

$S_{IV} = 4 W_{220}$ = FIELD CURVATURE

$S_V = 2 W_{311}$ = DISTORTION

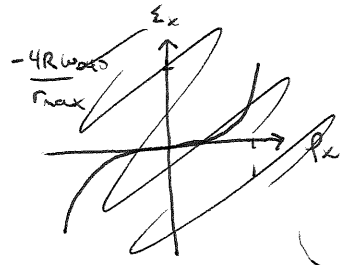
Spherical Aberration

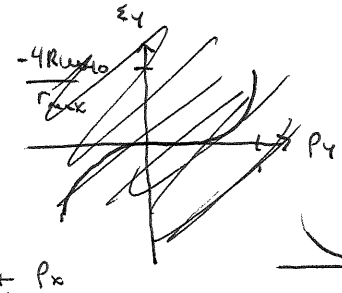
$$W(\rho_x, \rho_y) = W_{040}\rho^4 = W_{040}(\rho_x^2+\rho_y^2)^2$$

Transverse Ray Errors (in air)

$$\varepsilon_x = \frac{-R}{r_{max}} 2W_{040}(\rho_x^2+\rho_y^2)(2\rho_x) = \frac{-4R}{r_{max}}W_{040}(\rho_x^3+\rho_x\rho_y^2) = \frac{-4R}{r_{max}}W_{040}\rho^3 \sin\psi$$

$$\varepsilon_y = \frac{-R}{r_{max}} 2W_{040}(\rho_x^2+\rho_y^2)(2\rho_y) = \frac{-4RW_{040}}{r_{max}}(\rho_x^2\rho_y+\rho_y^3) = \frac{-4R}{r_{max}}W_{040}\rho^3\cos\psi$$

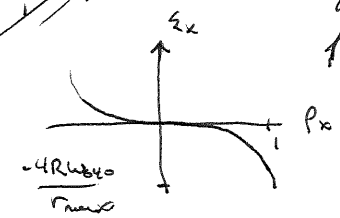


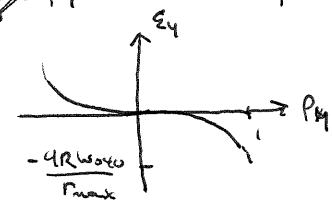


cubic in ρ_x and ρ_y

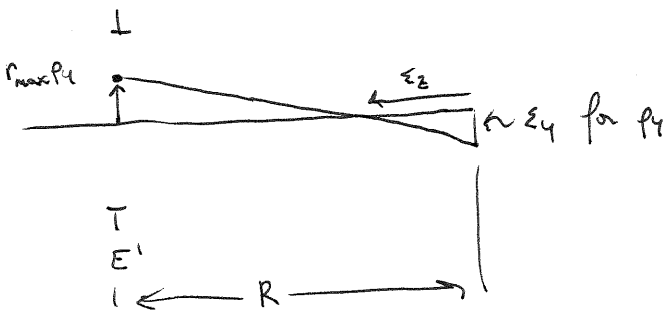
same for both cases

Longitudinal Errors





Essentially same derivation as defocus case



$$\frac{r_{max}\rho_y}{R+\varepsilon_z} = \frac{\frac{4R}{r_{max}}W_{040}\rho_y^3}{-\varepsilon_z}$$

but $R+\varepsilon_z \simeq R$

NOTE: $\varepsilon_z < 0$ for $W_{040} > 0$

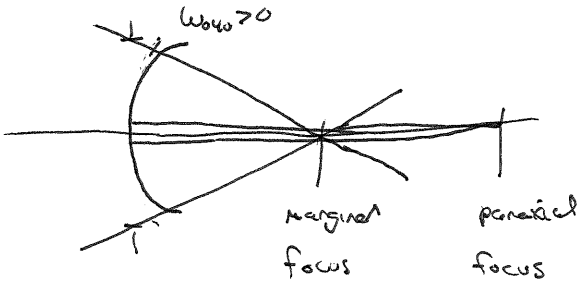
So $$\varepsilon_z \simeq -\frac{4R^2}{r_{max}^2}W_{040}\rho_y^2 \simeq -16(f/\#)^2 W_{040}\rho_y^2$$

Can think of spherical aberration as defocus which depends on ρ_y^2

Definition of Marginal Focus ε_z when $\rho_y = 1 \Rightarrow \varepsilon_z = 16(f/\#)^2 W_{040}$

Paraxial Focus ε_z when $\rho_y = 0$ i.e. gaussian image plane

For $W_{040} > 0$ marginal focus is before paraxial focus.



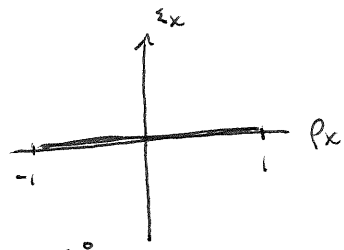
Coma

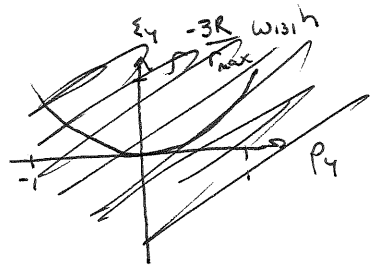
$$W(\rho_x, \rho_y) = W_{131} h \rho^3 \cos\psi = W_{131} h (\rho_x^2 \rho_y + \rho_y^3)$$

$$\varepsilon_x = \frac{-R}{r_{max}} W_{131} h (2 \rho_x \rho_y) = \frac{-R}{r_{max}} W_{131} h \rho^2 \sin 2\psi$$

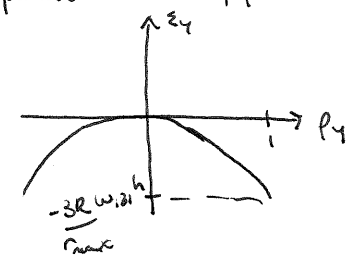
$$\varepsilon_y = \frac{-R}{r_{max}} W_{131} h (\rho_x^2 + 3\rho_y^2) = \frac{-R}{r_{max}} W_{131} h \rho^2 (2 + \cos 2\psi)$$

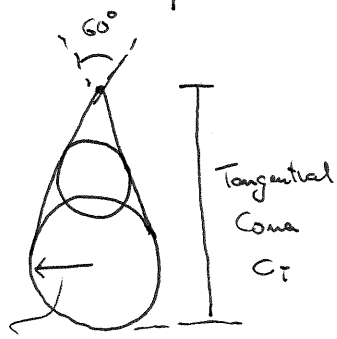
zero in ρ_x





parabolic in ρ_y





C_S
Sagittal Coma

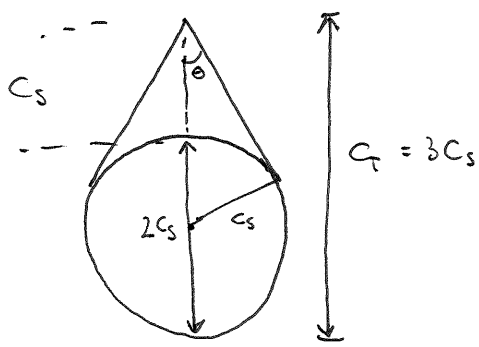
$C_S = \varepsilon_x$ for $h = 1$ and $\rho_x = \cancel{\rho_y} = \pm \frac{1}{\sqrt{2}}$

$\rho_y = \pm \frac{1}{\sqrt{2}}$

$$C_S = \frac{-R W_{131}}{r_{max}}$$

$C_T = \varepsilon_y$ for $h = 1$ and $\rho_x = 0$, $\rho_y = 1$

$$C_T = \frac{-3 W_{131}}{r_{max}} = 3 C_S$$



$$\sin\theta = \frac{C_S}{2C_S} = \frac{1}{2}$$

$$\theta = 30°$$

$$2\theta = 60°$$

ASTIGMATISM

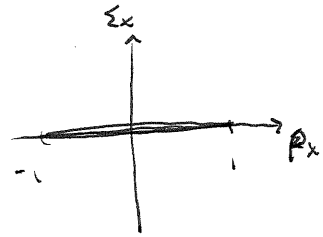
$$W(\rho_x, \rho_y) = W_{222} h^2 \rho^2 \cos^2\psi = W_{222} h^2 \rho_y^2$$

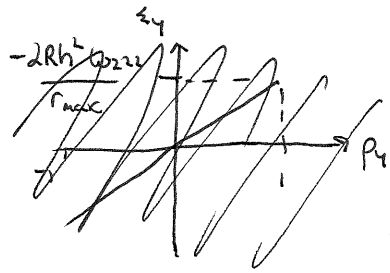
Transverse Ray Error (in air)

$$\varepsilon_x = 0$$

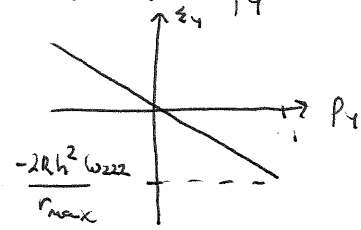
$$\varepsilon_y = \frac{-R}{r_{max}} \left(2 W_{222} h^2 \rho_y \right)$$

Zero in ρ_x

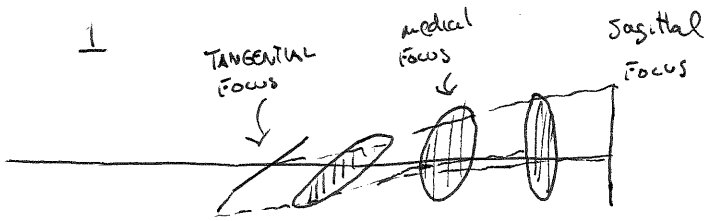




linear in ρ_y



1



"Circle of Least Confusion" at medial focus. Line foci and sagittal and tangential foci.

T_E ← INTERVAL OF STURM →

Since $\varepsilon_x = 0$, pure astigmatism puts the sagittal focus on the Gaussian image plane. We need to add defocus to get to other foci

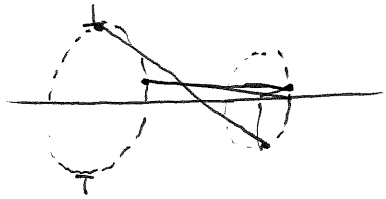
$$W(\rho_x, \rho_y) = W_{020} \rho^2 + W_{222} h^2 \rho_y^2$$

$$\varepsilon_x = \frac{-2R}{r_{max}} W_{020} \rho_x$$

$$\varepsilon_y = \frac{-2R}{r_{max}} W_{020} \rho_y - \frac{2R}{r_{max}} W_{222} h^2 \rho_y = \frac{-2R}{r_{max}} \rho_y \left(W_{020} + W_{222} h^2 \right)$$

when does ε_x (for $\rho_x = 1$) equal $-\varepsilon_y$ (for $\rho_y = 1$)?

$$\frac{-2R}{r_{max}} W_{020} = \frac{+2R}{r_{max}} \left(W_{020} + W_{222} h^2 \right)$$



$$W_{020} = -\frac{1}{2} W_{222} h^2$$

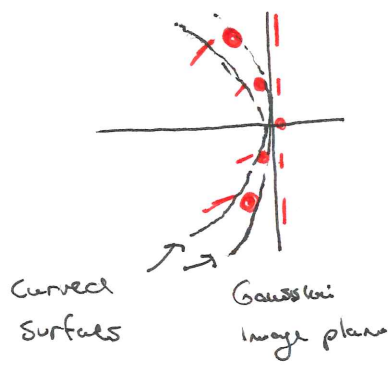
$$\boxed{\delta z = +4 (f/\#)^2 h^2 W_{222}}$$ medial focus

When does $\varepsilon_y = 0$?

$$\varepsilon_y = 0 \Rightarrow W_{020} = -W_{222} h^2$$

$$\boxed{\delta z = +8 (f/\#)^2 h^2 W_{222}}$$ Tangential focus

The sagittal focus is on gaussian image plane. The medial and tangential focus on spherical (parabolic approx.) surfaces



$$\delta z = -4 (f/\#)^2 W_{222} h^2 = \frac{h^2}{2 R_m}$$

medial surface radius of curvature $R_m = \dfrac{-1}{8 (f/\#)^2 W_{222}}$

Similarly $R_T = \dfrac{-1}{16 (f/\#)^2 W_{222}}$

tangential surface radius of curvature

Dimensions length of line foci $L = 2|\varepsilon_x|$ at tangential focus for $\rho_x = 1$

$$L = 2|\varepsilon_x| = \left| \frac{2R}{r_{max}} W_{020} \right|$$ but $W_{020} = -W_{222} h^2$ at tangential focus

$$L = 2|\varepsilon_x| = \left| \frac{2R}{r_{max}} W_{222} h^2 \right| = \frac{2R h^2}{r_{max}} |W_{222}|$$

diameter of circle of least confusion $D = 2|\varepsilon_x|$ at medial focus for $\rho_x = 1$

$$D = 2|\varepsilon_x| = \left| \frac{2R}{r_{max}} W_{020} \right|$$ but $W_{020} = \dfrac{-W_{222} h^2}{2}$ at medial focus

$$D = \frac{R h^2}{r_{max}} |W_{222}| = \frac{L}{2}$$

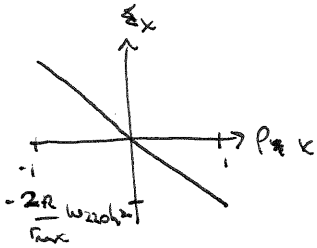
Field Curvature

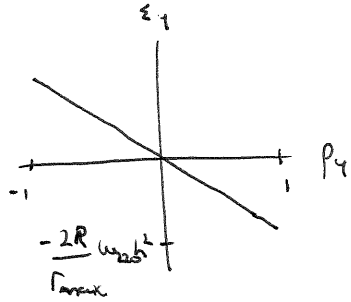
$$w(\rho_x, \rho_y) = w_{220} h^2 \rho^2 = w_{220} h^2 (\rho_x^2 + \rho_y^2)$$

$$\xi_x = \frac{-R}{r_{max}} \left(2 w_{220} h^2 \rho_x\right)$$

(in air)

$$\xi_y = \frac{-R}{r_{max}} \left(2 w_{220} h^2 \rho_y\right)$$





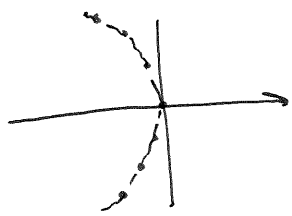
linear in both ρ_x and ρ_y

Looks like defocus, but with h^2 dependence, can calculate δz in same manner

$$\delta z = 8 (f/\#)^2 w_{220} h^2 = \frac{h^2}{2R_p}$$

Again, looks like spherical surface

$$R_p = \frac{1}{16 (f/\#)^2 w_{220}}$$



Gaussian image surface

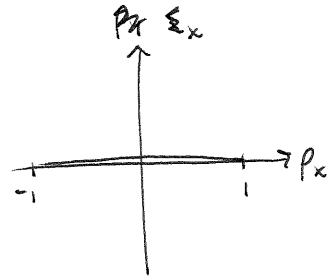
DISTORTION

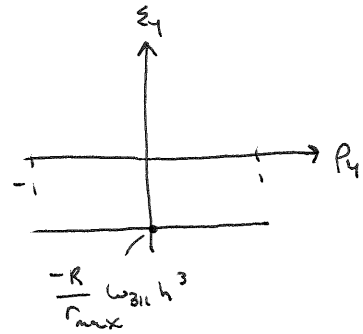
$$w(\rho_x, \rho_y) = w_{311} h^3 \rho \cos\psi = w_{311} h^3 \rho_y$$

$$\varepsilon_x = 0$$

(in air)

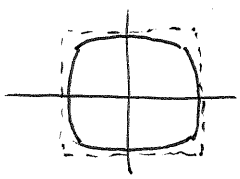
$$\varepsilon_y = \frac{-R}{r_{max}} w_{311} h^3$$





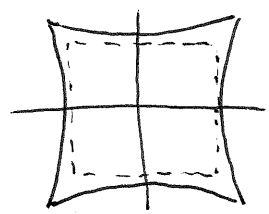
<u>Barrel Distortion</u>

$w_{311} > 0$; $\varepsilon_y < 0$ for $h > 0$



<u>Pincushion Distortion</u>

$w_{311} < 0$: $\varepsilon_y > 0$ for $h > 0$



looks like tilt with h^3 dependence instead of h dependence

SHOW ORDERS OF POLYNOMIALS SLIDES

1.5.10 THROUGH-FOCUS PSF AND STAR TEST [Book 3.2]

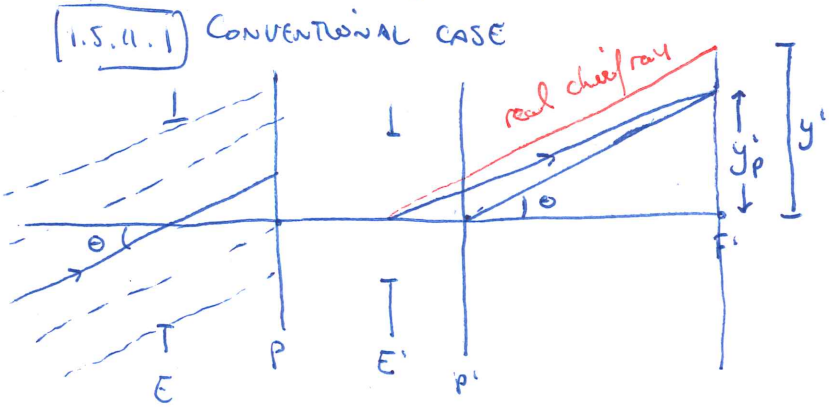
SHOW SLIDES

The star test is a qualitative test for examining the quality of ~~your~~ an optical system. A travelling microscope or camera is used to look at the image of a point source through focus. Different aberrations have different characteristic patterns.

- NOTE FROM slides. For spherical aberration and astigmatism, ~~defocus~~ Zernike terms incorporate defocus to bring mid-focus ~~to~~ to paraxial focus. For coma, tilt is incorporated to center coma pattern.

1.5.11 Measurement of Distortion [Book 3.3]

1.5.11.1 CONVENTIONAL CASE



Height of paraxial chief ray

$$y'_p = \overline{P'F'} \tan\theta$$

If height of real chief ray is y', then the percent distortion is given by

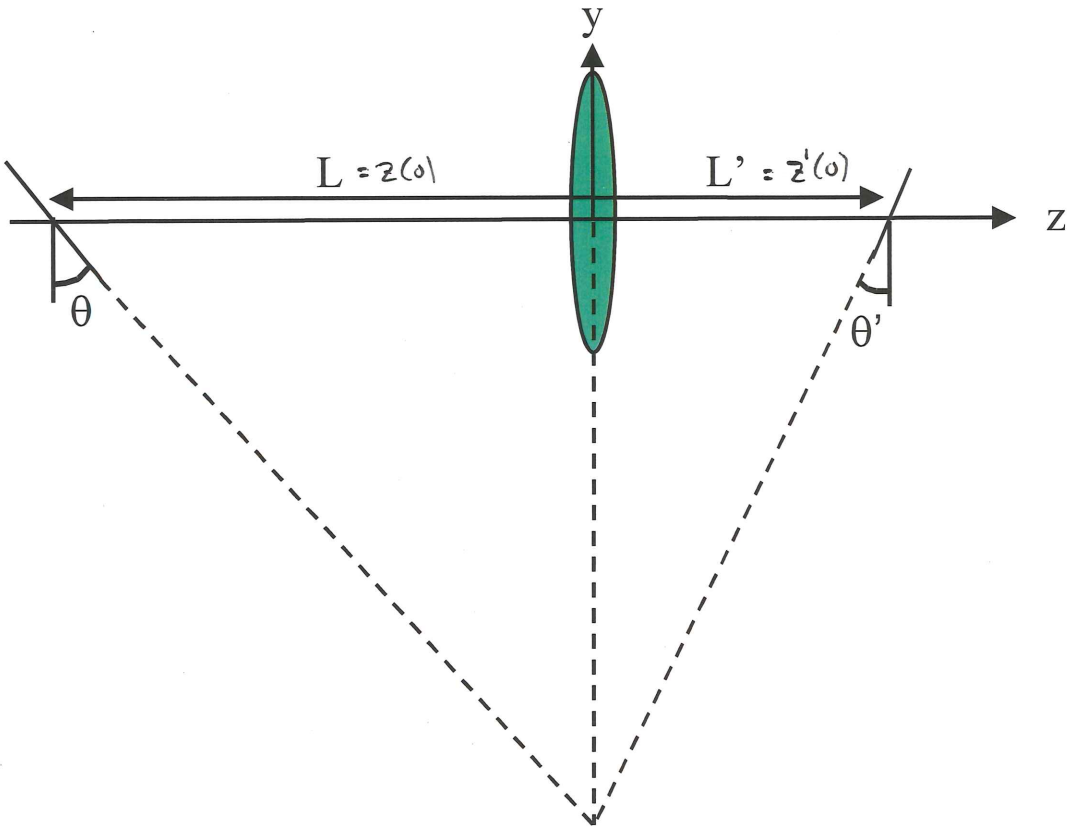
$$\boxed{\%\,\text{Distortion} = 100\,\frac{y' - y'_p}{y'_p}}$$

SHOW DISTORTION SLIDES

1.5.11.2 Special Cases: Anamorphic, fθ lens, Scheimpflug

fθ lens incorporate distortion such that $y' = \overline{P'F'} \cdot \theta$ (i.e. linear in θ)

Scheimpflug imaging [Book 3.3.2]



The imaging formula is given by

$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f_E}. \quad (1)$$

In conventional imaging, the object and image planes are parallel to one another with z = L (L is negative in the figure above) and z' = L'. If the object plane is tilted by an angle θ, then the Scheimpflug condition says the image plane is tilted as well. The tilted object and image planes become functions of y, so the Lensmaker's formula becomes

$$\frac{1}{z'(y)} - \frac{1}{z(y)} = \frac{1}{f_E}. \quad (2)$$

From the geometry in the image above, the object plane is described by a plane tilted about the x axis such that

$$z(y) = L - y \tan\theta, \quad (3) \qquad \text{where } L = z(0)$$

where a counterclockwise rotation of the object plane corresponds to a positive value of θ. Plugging this expression (3) into equation (2) and solving for z'(y) leads to

$$z'(y) = \frac{f_\epsilon(L - y\tan\theta)}{f_\epsilon + L - y\tan\theta} \cong \frac{f_\epsilon(L - y\tan\theta)}{f_\epsilon + L} \quad (4)$$

where the assumption that L >> y tan θ has been made. Equation (4) also describes a plane tilted about the x axis.

Location of Image Plane

The location of the image plane can be found by evaluating z'(0).

$$z'(0) = \frac{f_\epsilon L}{f_\epsilon + L}$$

$$\frac{f + L}{f_\epsilon L} = \frac{1}{z'(0)} \quad (5)$$

$$\frac{1}{f_\epsilon} + \frac{1}{L} = \frac{1}{z'(0)}$$

Equation (5) is just a statement of the Lensmaker's formula, requiring z'(0) = L'.

Intersection of the Object and Image Planes

The object and image planes intersect when z(y) = z'(y). This intersection occurs when

$$y = \frac{L}{\tan\theta} \quad (6)$$

Plugging (6) back into the expressions for the object and image planes leads to

$$z\left(\frac{L}{\tan\theta}\right) = L - L = 0 \quad \text{and} \quad z'\left(\frac{L}{\tan\theta}\right) = \frac{f_\epsilon L}{L + f_\epsilon} - \frac{f_\epsilon L}{L + f_\epsilon} = 0 \quad (7)$$

In other words, the object and image plane intersect at the plane of the lens.

Image Plane Tilt

Equation (4) can be rewritten as

$$z'(y) = \frac{f_\epsilon L}{f_\epsilon + L} - \frac{f_\epsilon \tan\theta}{f_\epsilon + L} y = L' - y\tan\theta' \quad (8)$$

where

$$\tan\theta' = \frac{f_\epsilon \tan\theta}{L + f_\epsilon}. \quad (9)$$

Magnification

The magnification m_o for the axial object and image points is given by

$$m_o = \frac{L'}{L} = \frac{f_\varepsilon}{L + f_\varepsilon}. \quad (10)$$

To calculate the magnification m as a function of y for the tilted system, equation (4) without the approximation L >> y tan θ needs to be used.

$$z'(y) = \frac{f_\varepsilon(L - y\tan\theta)}{f_\varepsilon + L - y\tan\theta} = \frac{f_\varepsilon z(y)}{f_\varepsilon + L - y\tan\theta} \Rightarrow m = \frac{z'(y)}{z(y)} = \frac{f_\varepsilon}{(f_\varepsilon + L)}\left[\frac{1}{1 - \frac{y\tan\theta}{f_\varepsilon + L}}\right] \quad (11)$$

$$m = \frac{m_o}{1 - \frac{\tan\theta}{f_\varepsilon + L}y}. \quad (12)$$

Using a binomial expansion on equation (12) leads to

$$m = m_o\left[1 + \frac{\tan\theta}{f_\varepsilon + L}y + \ldots\right]. \quad (13)$$

In other words, the magnification is linear in y or there is keystone distortion in the system (at where the truncated binomial expansion closely approximates equation (12)).

SHOW KEYSTONE AND SCHEIMPFLUG SLIDES

Measurement of Distortion

Place a target with known levels of distortions and examine image