1.3 First Order Properties of an optical system

1.3.1 Gaussian Imaging with multiple surfaces

\[ \frac{1}{Z_1'} - \frac{1}{Z_1} = \frac{N_1' - N_1}{R_1} \]

Solve for \( Z_1' \)

\[ \frac{1}{Z_2'} - \frac{1}{Z_2} = \frac{N_2' - N_2}{R_2} \]

but \( Z_2 = Z_1' - \xi_1 \); \( N_2 = N_1' \)

Solve for \( Z_2' \)

The output (image location) for the 1st surface becomes the input (object location) for the next surface. The process is repeated until the final image plane. Doable, but tedious.

1.3.2 Basic raytrace - Method for tracing rays through a paraxial system. We need two pieces of information for ray tracing. First, how does the ray change direction (reflect) at the interface of two materials. Second, how does the ray change height when crossing between interfaces.

Counter-clockwise \( \alpha > 0 \)

Clockwise \( \alpha < 0 \)
Paraxial Transfer Equation

\[ \tan u_2 = \frac{y_2 - y_1}{t} \]

\[ y_2 = y_1 + u_2 t = y_1 + N_2 u_2 \left( \frac{t}{N_2} \right) \]

To trace rays through an optical system repeat the refraction and transfer process for each surface and intervening space.

**NOTE:** For REFLECTIVE SURFACES \( N' = -N \)

We can use a spreadsheet to perform the paraxial raytracing

**SHOW EXCEL SPREADSHEET EXAMPLE**

Example lens is achromatic doublet from Edmund Optics 45793

N-LAK22

\[ R_3 = -84.13 \]

\[ R_2 = 15.56 \]

\[ R_1 = 4.5 \]

\[ N = 880 \text{ nm} \]

For N-LAK22 \( n = 1.6438 \)

For N-SFG6 \( n = 1.7801 \)

at design wavelength

To determine where image is formed in the example

\[ y_{\text{Image}} = y_3 + N_3 u_3 \left( \frac{t_3'}{N_3} \right) \]

Want \( y_{\text{Image}} = 0 \)

\[ t_3' = \frac{-N_3 y_3}{N_3 u_3} \]

\[ t_3' = \frac{-y_3}{u_3} \text{ in AIR} \]
From the drawing:

\[ i_1 = n_1 \alpha \quad i_2 = n_2 \alpha \Rightarrow i_2 = n_2 \alpha \]

In paraxial approximation, we assume \( u_1, u_2, i_1, i_2, \alpha \) all \( \ll 1 \)

So \( \sin i = i - \frac{i^3}{3!} + \ldots \approx i \)

Snell's Law \( n_1 \sin i_1 = n_2 \sin i_2 \)

becomes \( n_1 i_1 = n_2 i_2 \)

Note: When we derived transverse magnification, we used \( n_1 u_1 = n_2 u_2 \).

The ray in this case passed through the apex vertex of the surface.
So \( \alpha = 0 \) and \( i_1 = u_1 \) at \( i_2 = u_2 \).

Plugging in the expression for \( i_1 \) and \( i_2 \):

\[ n_1 u_1 - n_1 \alpha = n_2 u_2 - n_2 \alpha \Rightarrow n_2 u_2 = n_1 u_1 + (n_2 - n_1) \alpha \]

Again from our drawing and using the small angle approximation:

\[ \tan(\alpha) \approx \alpha = \frac{y}{R} \]

So \( \alpha = \frac{-y}{R} \)

\[ n_2 u_2 = n_1 u_1 - \frac{n_2 - n_1}{R} y = \frac{d(n_1 u_1 y)}{d(y)} \]

\[ n_2 u_2 = n_1 u_1 - \phi y \] Paraxial refraction equation
In our example, what are the two rays corresponding to \( y_a, y_b \) and \( y_{1a}, y_{1b} \)?

- If the aperture stop is at surface 1 and the diameter of the aperture is 2 mm, then \( y_{1a} \) = marginal ray and \( y_{1b} \) = chief ray.

What if stop located somewhere else?

**Ray Scaling** - One two independent rays are traced through the system, a new ray can be formed as a linear combination of the two rays. In other words:

\[
\begin{align*}
y_c &= A y_a + B y_b \\
n_{uc} &= A n_{ua} + B n_{ub}
\end{align*}
\]

\[
\begin{pmatrix} y_a & y_b \\ n_{ua} & n_{ub} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} y_c \\ n_{uc} \end{pmatrix}
\]

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

\[
A^{-1} = \frac{1}{\text{Det} A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

\[
\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{y_a n_{ub} - y_b n_{ua}} \begin{pmatrix} n_{ub} & -y_b \\ -n_{ua} & y_a \end{pmatrix} \begin{pmatrix} y_c \\ n_{uc} \end{pmatrix}
\]

\[
A = \frac{y_c n_{ub} - y_b n_{uc}}{y_a n_{ub} - y_b n_{ua}} \quad B = \frac{-y_c n_{ua} + y_a n_{uc}}{y_a n_{ub} - y_b n_{ua}}
\]
In our doublet example, suppose $n_{c} = 0.1$ and $n_{b} = 1.48$.

First, we determine $y_0 = 0$ at surface 3. At each point, we don't have enough information to determine the required equation. Let's just assume $y_0 = 0$ for now.

The ray scaling coefficients are

\[
\begin{align*}
A &= \frac{-0.1y_3}{n_{a3} - n_{b3}} & B &= \frac{0.1y_3}{n_{a3} - n_{b3}} \\
&= \frac{n_{b3} - n_{a3}}{n_{a3} - n_{b3}} & &\text{Calculate } \frac{n_{b3} - n_{a3}}{n_{a3} - n_{b3}} \text{ on spreadsheet}
\end{align*}
\]

\[
y_{a3} n_{b3} - y_{b3} n_{a3} = 0.1
\]

\[
A = -y_{b3} = -0.4186 & & B = y_{a3} = 0.841838
\]

\[
\text{Calculate } y_{c}, n_{c} \text{ on spreadsheet}
\]

### 1.3.3 Cardinal Points

We are typically interested in finding 6 cardinal points of an optical system: Front and Rear Focal Points; Front and Rear Nodal Points; Front and Rear Principal Points. Knowledge of the locations of these points allows us to simplify multielement complex optical systems down to a black box where the principal points (and their corresponding planes) allow us to determine the imaging properties of the system.

**Definitions**

- **Rear Focal Point**
  - Optical System
  - Where a collimated ray in object space intersects the optical axis in image space.

- **Front Focal Point**
  - Optical System
  - A ray passing through the Front Focal Point in object space, emerges parallel to optical axis in image space.
**FRONT / REAR NODAL POINTS**

![Optical System Diagram]

A ray appearing to pass through the front nodal point at an angle $\theta$, appears to emerge from the rear nodal point at the same angle $\theta$.

**FRONT / REAR PRINCIPAL PLANES (POINTS)**

![Optical System Diagram]

The principal points define a pair of planes perpendicular to the optical axis. A ray appearing to strike the front principal plane at a height $y$ gets mapped to the rear principal plane at the same height for the emerging ray. **Note:** The angle of the emerging ray is typically different than the incident ray.

How do we find the Conical Points from an actual raytrace?

**Rear Focal Point** - Trace a ray with $y_0 = 0$ through the system. The incident ray height is arbitrary ($y_i = 1$ is a nice choice). The rear focal point is the point where the ray emerging from the last surface crosses the optical axis. **Note:** You may need to project this ray backwards to find an intersection.

In our doublet example, we have already traced this ray. With $y_0 = 1$ at $y_i = 0$, we also calculate $y' = 21.04498\text{mm}$ to get $y_{\text{image}} : 0$. This means $F'$ is located $21.04498\text{mm}$ behind the last surface. **Note:** Edmund's website calls this Back Focal Length (BFL). Don't confuse with Rear Focal Length we defined in a previous class. Later we will call the Back Focal Distance...
FRONT Focal Point - Trace two independent rays through the system. Use ray scaling to create a ray with \( \nu_{c1} = 0 \). Again, \( \gamma_{c1} \) is arbitrary so choose \( \gamma_{c1} = 1.0 \).

For the doublet example, the ray scaling equations are:

\[
A = \frac{\nu_{c1} \sin \alpha}{0.1} \quad B = \frac{-\nu_{c1} \sin \alpha}{0.1}
\]

Note: Denominator is the same as previous example.

These values correspond to a ray with \( y_{c1} = 0.98897 \) \( \nu_{c1} = 0.040002 \). Where does this cross the optical axis?

\[
d = \frac{0.98897}{0.040002}
\]

REAR PRINCIPAL Point (Plane) - Trace ray with \( \nu_{c} = 0 \) (e.g., the ray we used to find the rear focal point). Determine location of the intersection the object space ray and its corresponding image space ray.

In the doublet example, use similar triangles.

\[
\frac{y_3}{\ell_3} = \frac{1}{\ell_3'} \\
\frac{21.04498}{0.841838} = 24.999 \text{ mm}
\]
**Front Principal Plane (Point)** - Trace ray where $\mu \Delta s = 0$ (e.g., the ray we used to find the front focal point). Determine where the object and image positions of the ray intersect.

In the doublet example, use similar triangles.

\[
\frac{0.98897}{24.72311} = \frac{1}{PF}
\]

\[
PF = -24.999 \text{ mm}
\]

$PF$ is front focal length $f_F$

**Front/Rear Nodal Points**

In general, the nodal points are shifted relative to the principal points. The amount of shift is given by

\[
\overline{PN} = \overline{PN'} = f_F + f_R' = (n' - n) f_F
\]

where $n'$ is image space index, $n$ is object space index.

In many cases $n' = n$, so $\overline{PN} = \overline{PN'} = 0$.

In our doublet example, the lens is $n$-air, so the Nodal Points coincide with the Principal Points.
1.3.4entrance and exit pupils

The entrance pupil is the image of the aperture stop formed in object space by all of the optical surfaces preceding it. The exit pupil is the image of the aperture stop in image space formed by all of the optical surfaces following the aperture stop. Finally, the entrance pupil is conjugate to the exit pupil meaning that if an object is placed at the entrance pupil, then the image is at the exit pupil.

The positions of the entrance and exit pupils can be determined by where the chief ray appears to cross the optical axis in object and image space. The height of the marginal ray at these crossings determines the size of the pupil.

Show slides (1-3)

For our doubled example, entrance pupil is at the first surface and has a diameter of 2 mm.

\[ Y_e = 0.095897 \]

\[ d = \frac{0.4186}{0.095897} = 4.233 \text{ mm} \]

Left of surface 3

\[ Y_e = 0.841838 + 0.04(4.233) \]

\[ Y_e = 1.011 \text{ mm} \]

Show Excel spreadsheet
Cascading Optical Systems

Typically, you want to match the entrance and exit pupils of a cascaded system:

\[ I \quad I \quad I \]
\[ E \quad E \quad E \quad E \]

The entrance pupil can be thought of as a port that captures light from the object scene. The larger the port, the more light that gets through the system.

The exit pupil (for a well-corrected optical system) is just a 1:1 mapping from the entrance pupil.

1.3.5 Extension of Gaussian Imaging to Thick Systems

Knowledge of the Cardinal Points allows us to extend the Gaussian imaging equation to thick lenses and multilenset systems.

\[ \frac{1}{p_0} - \frac{1}{p'} = \phi = \frac{n'}{n} = \frac{1}{p'f'} \]
\[ f' = p'f \]
\[ f = pf \]
1.3.6 Transverse and Longitudinal Magnification

We already showed for a single surface that the transverse magnification \( m \) is
\[
m = \frac{N^'z'}{N'z}
\]

For thick systems \( z = \rho_0 \) and \( z^' = \rho_0' \) and the same definition holds
\[
m = \frac{N\rho_0}{N'\rho_0'}
\]

For longitudinal magnification, we are interested in how far the image plane shifts when we shift the object plane.

\[
\Delta z = z_2 - z_1
\]
\[
\Delta z^' = z_2^' - z_1^'
\]

Already showed the gaussian imaging eq. can be written in terms of transverse magnification

\[
\Delta z = -f_e \left( \frac{1 - m_1}{m_1} \right) + f_e \left( 1 - m_2 \right)
\]
\[
\Delta z^' = f_e \left( 1 - m_2 \right) - f_e \left( 1 - m_1 \right)
\]
\[
\Delta z = -f_e \left( \frac{1 - m_2}{m_2} - \frac{1 - m_1}{m_1} \right)
\]
\[
\Delta z^' = f_e \left( 1 - m_2 - (1 - m_1) \right)
\]

Similarly

\[
\Delta z = \frac{m_1 - m_2}{m_1m_2}
\]
\[
\Delta z^' = \frac{m_1m_2 - m_2^2}{m_1m_2}
\]
\[ \Delta z' = \frac{-f_e}{n e' m_2} \left( \frac{\Delta z}{n_1 m_2} \right) \]

Recall \( f_e = \frac{f_e'}{n^2} = -\frac{f_e}{n} \)

Also for small \( \Delta z \), \( m_1 = m_2 \Rightarrow \Delta z' = \frac{n_1}{n} \Delta z \)

Ideal longitudinal magnification
\[ \bar{n} = \left( \frac{n_1}{n} \right)^2 \]

[1.3.7] Lagrange invariant, Etendue, Throughput, DA Product

The lagrange invariant \([\text{Expression}]\) is a quantity that is constant throughout the optical system.

Paraxial Refraction Eq.
\[ n u' = n u - \phi y \]
\[ n v' = n v - \phi y \]
\[ \phi = \frac{n u - n u'}{y} = \frac{n v - n v'}{y} \]
\[ n u y - n u' y = n v y - n v' y \]
\[ n u y - n v y = n u' y - n v' y \]

Paraxial Transfer Eq.
\[ y' = y + n u \left( \frac{t}{n} \right) \]
\[ y' = y + n u \left( \frac{t}{n} \right) \]
\[ t = \frac{y' - y}{n u} = \frac{y' - y}{n u} \]
\[ n u y - n u y = n u y' - n u y' \]

Quantity is same after transfer.
Define Lagrange deviencit \( \mathbf{H} \) (in Cyrillic "Мати")
\[
\mathbf{H} = \mathbf{N} - \mathbf{W}
\]

Special cases

1. **shape plane** \( y = 0 \) \( \mathbf{H} = \mathbf{N} \)
2. **pupil plane** \( \bar{y} = 0 \) \( \mathbf{H} = \mathbf{N} \)

We already saw this in our ray tracing

Show Excel spreadsheet and propagate \( y_{mn} \)-\( y_{nm} \) forward and backward

Handy way of verifying your chief spreadsheet correctly.

The optical devirint \( \mathbf{I} \) is a generalization when rays besides the


Recall our ray scaling coefficients

\[
y_c = A y_a + B y_b \quad u_c = A u_a + B u_b
\]

\[
A = \frac{I_{ac}}{I_{ab}} \quad B = \frac{I_{bc}}{I_{ab}}
\]

The throughput, etendue, of the product are related to the square of

The Lagrange deviencit
Area of Entrance Pupil $A = \pi y^2$

Solid angle Of Source As Seen From Entrance Pupil

$$\Omega = \frac{\pi y^2}{r^2}$$

$$d\Omega = \frac{\pi y^2 d^2}{r^2}$$

$$\Delta \Sigma = \frac{\pi y^2}{r^2}$$

$$\Delta \Sigma = \pi y \cdot u y$$

$$\Delta \Sigma = \pi \left(\frac{y}{r}\right) \left(-\frac{y}{r}\right)$$

$$n^2 d\Omega = \pi \left(\frac{y^2}{r^2}\right) (nu\bar{y})$$

$$n^2 d\Omega = \pi \left(\frac{y^2}{r^2}\right) (nu\bar{y})$$

but $H_E = H_S$

$$\left(n^2 d\Omega = \pi \left(\frac{y^2}{r^2}\right) \left(\frac{nu\bar{y}}{y^2}\right)\right)$$

Increasing $H_E$ increases the amount of light getting into the entrance pupil.

13.8 F-Number, Working F-Number and Numerical Aperture

In the paraxial picture

In the non-paraxial picture

$$u = \frac{y_0}{E_0}$$

so $u = \sin U$ relates paraxial angle $u$ to real marginal ray angle $U$

Numerical aperture $= \left|\sin U\right| = \left|\sin u\right|$ where prime denotes image space

We can also define Numerical Aperture of object space.
What does high NA mean? Large angle \( u \Rightarrow \) bigger pupil = more light
What is max value of NA? \( \text{NA}_{\text{max}} = N' \) since |\( \sin u \)|\( \text{max} = 1 \)

ASIDE: MICROSCOPE OBJECTIVES

Show Slides: Comment on Tube Length / \( \text{NA} \)

F-Number
\[
F/\# = \frac{f_e}{D_{\text{ep}}}
\]
where \( D_{\text{ep}} \) is characteristic of entrance pupil
This value describes the cone of light in image space for an object at infinity.

\[
\begin{align*}
F/\# & \quad \text{Decrease } f_e \quad \text{Decrease } F/\# \\
D_{\text{ep}} & \quad \text{Decrease } D_{\text{ep}} \quad \text{Increase } F/\#
\end{align*}
\]

Show Slide of Camera Lens

Sequence of F/#s
\[
\begin{align*}
F/\# & \quad 22 & 16 & 11 & 8 & 5.6 & 4 & 2.8 & 2 \\
D_{\text{ep}}(\mu m) & \quad 1.5 & 2.1 & 3.6 & 4.6 & 6.2 & 8.7 & 12.5 & 17.5 \\
T_D & \quad 7.94 & 15.07 & 31.77 & 60.26 & \\
\end{align*}
\]

In general for fixed \( f_e \), halving \( F/\# \), quadruples entrance pupil area.
If we assume a thin lens with stop at the lens then
\[
F/\# \sim \frac{1}{2 \text{NA}} \quad \text{NA in image space}
\]
Often we are working with finite conjugates, so working f/# was inappropriate.

1.3.9 Depth of Field/Depth of Focus

Depth of Focus DOF is an image space concept. Suppose we can tolerate some blur since \( B' \).

Then we say \( \text{DOF} = \pm b' \).

\[ \text{DOF} = \pm b' \Rightarrow \frac{f/\#}{2NA} \]

\( B' \) might be the dimension of a pixel or some other value which is application specific.

For the microscope objectives, the higher the NA, the shallower the DOF.
For the camera lens, the higher the f/\#, the larger the DOF.

Depth of Field is just the DOF mapped to object space.
### 1.3.10 Field of View

Field of View (FOV) (Sometimes FFov for full field of view) is the maximum angular size of the object as seen from the entrance pupil.

HFov is half field of view. Sometimes FOV is used loosely, so really FFov or HFov

\[
\tan \Theta_v = \frac{h}{L}
\]

\[
\Theta_v = \frac{h}{L}
\]

### 1.3.11 Front or Back Focal Distances

Distance from respective focal point to the vertex of the first/last optical surface.

\[
f = \text{front focal distance}
\]

\[
f' = \text{back focal distance}
\]

**Note:** EDMUND page called this back focal length in achromatic example. Do not confuse with rear focal length \( f' \)

### 1.3.11.1 Standard Flange Distances for Cameras

- Comment: originate with TV and Movie Cameras

1.7526 mm

CS-mount same threads as C-mount but flange focal distance is 12.526 mm. Go to PIGLEY and EDMUND websites.
T-MOUNT

- M42 x 0.75 Thread
- Metric 42 mm Diameter
- 0.75 mm pitch for threads

F-MOUNT (Nikon Lens Family)

- Bayonet Mount
- Flange distance 46.5 mm

- 44 mm throat diameter

EF-MOUNT (Canon Lens Family)

- Throat diameter 54 mm
- Flange distance 44 mm