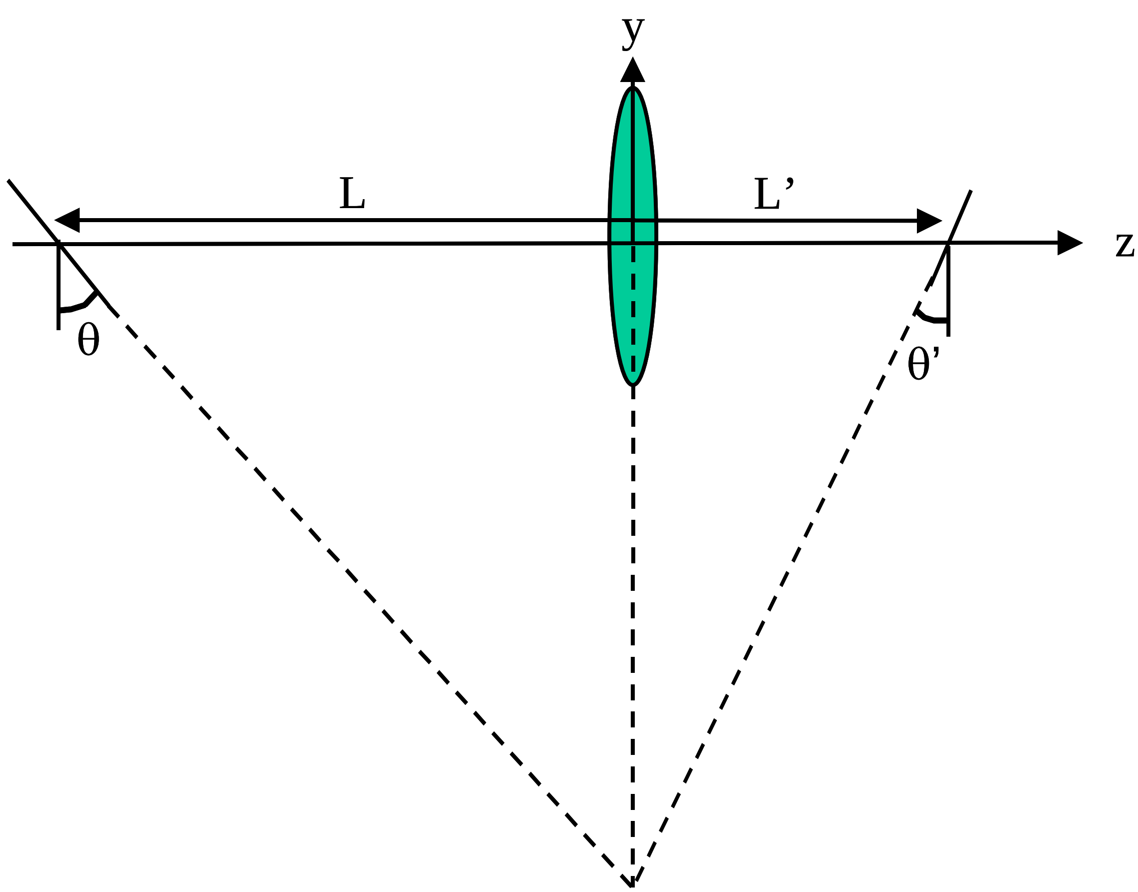

The imaging formula is given by

$$\frac{1}{z'} - \frac{1}{z} = \frac{1}{f}. \quad (1)$$

In conventional imaging, the object and image planes are parallel to one another with z = L (L is negative in the figure above) and z' = L'. If the object plane is tilted by an angle θ, then the Scheimpflug condition says the image plane is tilted as well. The tilted object and image planes become functions of y, so the Lensmaker's formula becomes

$$\frac{1}{z'(y)} - \frac{1}{z(y)} = \frac{1}{f}. \quad (2)$$

From the geometry in the image above, the object plane is described by a plane tilted about the x axis such that

$$z(y) = L - y \tan\theta, \quad (3)$$

where a counterclockwise rotation of the object plane corresponds to a positive value of θ. Plugging this expression (3) into equation (2) and solving for z'(y) leads to

$$z'(y) = \frac{f\left(L - y\tan\theta\right)}{f + L - y\tan\theta} \cong \frac{f\left(L - y\tan\theta\right)}{f + L} \quad (4)$$

where the assumption that L >> y tan θ has been made. Equation (4) also describes a plane tilted about the x axis.

Location of Image Plane

The location of the image plane can be found by evaluating z'(0).

$$z'(0) = \frac{fL}{f + L}$$
$$\frac{f + L}{fL} = \frac{1}{z'(0)} \quad (5)$$
$$\frac{1}{f} + \frac{1}{L} = \frac{1}{z'(0)}$$

Equation (5) is just a statement of the Lensmaker's formula, requiring z'(0) = L'.

Intersection of the Object and Image Planes

The object and image planes intersect when z(y) = z'(y). This intersection occurs when

$$y = \frac{L}{\tan\theta} \quad (6)$$

Plugging (6) back into the expressions for the object and image planes leads to

$$z\left(\frac{L}{\tan\theta}\right) = L - L = 0 \quad \text{and} \quad z'\left(\frac{L}{\tan\theta}\right) = \frac{fL}{L + f} - \frac{fL}{L + f} = 0 \qquad (7)$$

In other words, the object and image plane intersect at the plane of the lens.

Image Plane Tilt

Equation (4) can be rewritten as

$$z'(y) = \frac{fL}{f + L} - \frac{f\tan\theta}{f + L}y = L' - y\tan\theta' \quad (8)$$

where

$$\tan\theta' = \frac{f\tan\theta}{L + f}. \quad (9)$$

Magnification

The magnification m_o for the axial object and image points is given by

$$m_o = \frac{L'}{L} = \frac{f}{L+f} . \quad (10)$$

To calculate the magnification m as a function of y for the tilted system, equation (4) without the approximation L >> y tan θ needs to be used.

$$z'(y) = \frac{f(L - y\tan\theta)}{f + L - y\tan\theta} = \frac{f\ z(y)}{f + L - y\tan\theta} \Rightarrow m = \frac{z'(y)}{z(y)} = \frac{f}{(f+L)}\left[\frac{1}{1 - \frac{y\tan\theta}{f+L}}\right] \quad (11)$$

$$m = \frac{m_o}{1 - \frac{\tan\theta}{f+L}y} . \quad (12)$$

Using a binomial expansion on equation (12) leads to

$$m = m_o\left[1 + \frac{\tan\theta}{f+L}y + \ldots\right] . \quad (13)$$

In other words, the magnification is linear in y or there is keystone distortion in the system (at where the truncated binomial expansion closely approximates equation (12)).