

Undergraduate and Graduate Students do

all three problems.

1. Given the wavefront

$$W(\rho, \theta) = W_{11}\rho \cos \theta + W_{31}\rho^3 \cos \theta :$$

(a) What is the wavefront

variance? Note $\int_0^{2\pi} \cos^2 \theta d\theta = \pi$

Zernike Polynomials: Table in Polar Coordinates

j	n	m	$Z_n^m(\rho, \theta)$
0	0	0	1
1	1	-1	$2 \rho \sin \theta$
2	1	1	$2 \rho \cos \theta$
3	2	-2	$\sqrt{6} \rho^2 \sin 2\theta$
4	2	0	$\sqrt{3} (2\rho^2 - 1)$
5	2	2	$\sqrt{6} \rho^2 \cos 2\theta$
6	3	-3	$\sqrt{8} \rho^3 \sin 3\theta$
7	3	-1	$\sqrt{8} (3\rho^3 - 2\rho) \sin \theta$
8	3	1	$\sqrt{8} (3\rho^3 - 2\rho) \cos \theta$
9	3	3	$\sqrt{8} \rho^3 \cos 3\theta$

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W[ρ_, θ_] = W11 * ρ * Cos[θ] + W31 * ρ^3 * Cos[θ]
: W11 ρ Cos[θ] + W31 ρ^3 Cos[θ]

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The average wavefront is given by

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: Wavg = Integrate[Integrate[W[ρ, θ] * ρ dρ dθ, {ρ, 0, 1}], {θ, 0, 2*π}]
: 0

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The second moment is given by

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: W2 = Distribute[Integrate[Integrate[W[ρ, θ]^2 * ρ dρ dθ, {ρ, 0, 1}], {θ, 0, 2*π}]]
: π W11^2 / 4 + π W11 W31 / 3 + π W31^2 / 8

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The wavefront variance is given by

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: σw2 = Distribute[Simplify[(1/π) * (W2 - Wavg^2)]]
: W11^2 / 4 + W11 W31 / 3 + W31^2 / 8

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(b) What value of W_{11} (in terms of W_{31}) minimizes the wavefront variance?

Differentiate the wavefront variance and set it equal to zero to find the minimum.

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Solve[D[σw2, W11] == 0, W11]

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{{W11 -> -2 W31 / 3}}

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(c) Rewrite $W(\rho, \theta) = W_{11}\rho \cos \theta + W_{31}\rho^3 \cos \theta$ in terms of Zernike polynomials. What are the non-zero expansion coefficients a_{nm} in terms of W_{11} and W_{31} ?

We should expect a $Z_1^1(\rho, \theta)$ term because of the $\rho \cos \theta$ in the first term of the wavefront. We should also expect a $Z_3^1(\rho, \theta)$ because of the $\rho^3 \cos \theta$ in the second term of the wavefront, so $W(\rho, \theta) = a_{11} * 2\rho \cos \theta + a_{31} * \sqrt{8} (3\rho^3 - 2\rho) \cos \theta$. Comparing like terms gives $W_{11} = 2 a_{11} - 2 a_{31}\sqrt{8}$ and $W_{31} = 3 a_{31}\sqrt{8}$. Solving for the expansion coefficients gives

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Solve[{W11 = 2 * a11 - 2 * a31 * Sqrt[8], W31 = 3 * a31 * Sqrt[8]}, {a11, a31}]
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$$\left\{ \left\{ a_{11} \rightarrow \frac{1}{6} (3 W_{11} + 2 W_{31}), a_{31} \rightarrow \frac{W_{31}}{6\sqrt{2}} \right\} \right\}$$

(d) Calculate the sum of the squares of the Zernike expansion coefficients. How does this compare to part a?

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a11 = 1/6 (3 W11 + 2 W31);
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$$a_{31} = \frac{W_{31}}{6\sqrt{2}};$$

```
Distribute[Simplify[a11^2 + a31^2]]
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$$\frac{W_{11}^2}{4} + \frac{W_{11} W_{31}}{3} + \frac{W_{31}^2}{8}$$

This is the same as part (a) since wavefront variance is the sum of the squares of the Zernike coefficients

(e) Using the result from part b, what are the Zernike expansion coefficients?

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= a11 = 1/6 (3 W11 + 2 W31) /. W11 -> (-2/3) * W31
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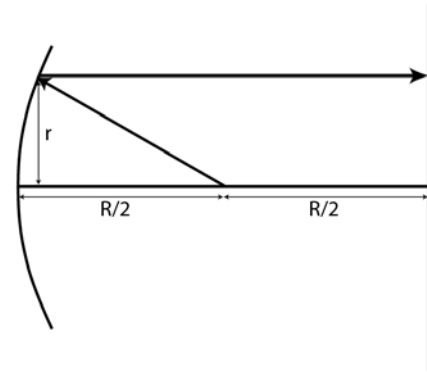
$$a_{31} = \frac{W_{31}}{6\sqrt{2}} /. W_{11} \rightarrow (-2/3) * W_{31}$$

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= 0
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$$= \frac{W_{31}}{6\sqrt{2}}$$

The Zernike coma term already has the minimum variance, so the Zernike tilt term must be zero.

2. A parabolic reflector with apical radius R is separated from a plane by a distance R . A point source is located on-axis, halfway between the reflector and the plane. A ray leaving the point source strikes the reflector at a height r above the optical axis, reflects and then continues on to the plane. Answer the following:



- (a) What is the sag of the reflector?

The sag of the reflector is $\text{sag} = \frac{r^2}{2R}$.

- (b) What is the optical path length of the ray starting at the point source and ending at the plane?

The distance from the source to the reflector is the hypotenuse of a triangle with sides r and $R/2 - r^2/2R$, so the length of that portion of the ray is

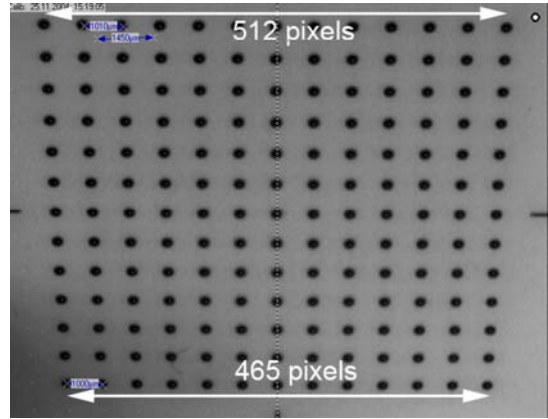
$$\sqrt{\left(\frac{R}{2} - \frac{r^2}{2R}\right)^2 + r^2} = \frac{R}{2} + \frac{r^2}{2R}.$$

The distance from the reflector to the plane is $R - r^2/2R$, so the total optical path length is $3R/2$.

- (c) How does the optical path length depend upon the height r ?

The optical path length is independent of r , which is basically a statement that a source located at the first foci is perfectly imaged to the second foci (at infinity).

3. The following image was taken with a Scheimpflug imaging system and exhibits keystone distortion. Assume the sensor has 10 micron square pixels. The object is a 12 mm x 12 mm target of dots with each dots separated by 1 mm. You can assume the linear model



for magnification $m = m_0 \left[1 + \frac{\tan \theta}{f + L} y \right]$, where m_0 is the magnification on axis, f is the focal length of the lens, L is the object distance, y is the object height and θ is the tilt of the object plane. Answer the following:

- (a) What is the width of the middle row of the image?

Since the distortion is linear, the width of the middle row will be the average of the top and bottom rows or 488.5 pixels. Since each pixel is 10 microns, this corresponds to 4.885 mm on the sensor.

- (b) What is m_0 ? (Don't forget the minus sign)

This middle row on the object corresponds to the case where $y = 0$. The middle row on the object is 12 mm wide and on the image it is 4.885 mm from part (a). Therefore,

$$m_0 = -\frac{4.885}{12} = -0.407$$

- (c) If the object distance is -150 mm, what is the image distance L' and focal length of the lens f ?

The image distance is $L' = m_0 L = 61.05$ mm. The focal length of the lens is then obtained from the Gaussian imaging equation:

$$\frac{1}{61.05} - \frac{1}{-150} = \frac{1}{f} \Rightarrow f = 43.49 \text{ mm} .$$

(d) What is θ , the object plane tilt?

The bottom row of dots corresponds to the case where $y = 6 \text{ mm}$ in the object. The width of the bottom row on the sensor is $465 \text{ pixels} \times 10 \text{ microns} = 4.65 \text{ mm}$. The magnification of the bottom row $m = -4.65 / 12 = -0.3875$. So

$$m = m_0 \left[1 + \frac{\tan \theta}{f + L} y \right] \Rightarrow -0.3875 = -0.407 \left[1 + \frac{\tan \theta}{43.49 - 150} 6 \right] \Rightarrow \theta = 40.408^\circ .$$

(e) What is θ' , the image plane tilt?

$$\tan \theta' = \frac{\tan \theta}{43.49 - 150} 43.49 \Rightarrow \theta' = -19.1103^\circ$$

(f) Draw a sketch of the system

