Similarly, for $\lambda_2 = 2$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 2\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$3x_2 + y_2 = 2x_2$ first row $\Rightarrow$ $x_2 + y_2 = 0$

$x_2 + 3y_2 = 2y_2$ second row $\Rightarrow$ $x_2 + y_2 = 0$

chose $x_2 = 1$, so $y_2 = -1$

$\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ with eigen value $\lambda_2 = 2$

What can we do with results?

Suppose we want to know

$$\vec{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

We can also rewrite $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of eigenvectors

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\vec{A}$ acting on eigenvector just gives eigenvalue

$$\vec{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\vec{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}\vec{A}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{4}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{2}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Sometimes this route is easier.

Let's generalize the process for finding the eigenvalues of a 2×2 matrix

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = 0 \quad \text{and} \quad \vec{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

Trace $(\vec{A}) = \mathrm{Tr}(\vec{A})$ is sum of diagonal elements

$$\lambda^2 - \mathrm{Tr}(\vec{A})\lambda + \det(\vec{A}) = 0$$

$$\lambda_1 = \frac{\mathrm{Tr}(\vec{A})}{2} + \sqrt{\frac{(\mathrm{Tr}(\vec{A}))^2}{4} - \det(\vec{A})}$$

$$\lambda_2 = \frac{\mathrm{Tr}(\vec{A})}{2} - \sqrt{\frac{(\mathrm{Tr}(\vec{A}))^2}{4} - \det(\vec{A})}$$

It can also be shown that the eigenvectors are:

$$\begin{pmatrix} \lambda_1 - a_{22} \\ a_{21} \end{pmatrix}, \begin{pmatrix} \lambda_2 - a_{22} \\ a_{21} \end{pmatrix} \quad \text{if} \quad a_{21} \neq 0$$

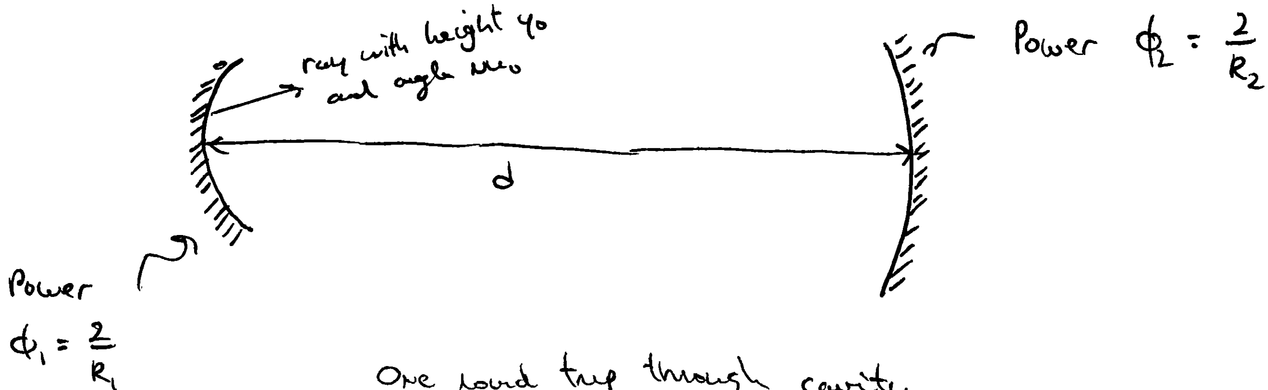
$$\begin{pmatrix} a_{12} \\ \lambda_1 - a_{11} \end{pmatrix}, \begin{pmatrix} a_{12} \\ \lambda_2 - a_{11} \end{pmatrix} \quad \text{if} \quad a_{12} \neq 0$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{if} \quad a_{12} = a_{21} = 0$$

Another good case for this is when we repeatedly apply the same matrix to a vector

$$\vec{A}^N \vec{x} = \vec{A} \cdot \vec{A} \cdot \vec{A} \cdots \vec{A}\,\vec{x} = \lambda^N \vec{x}$$

Consider a resonator



$$\begin{pmatrix} y_1 \\ nu_1 \end{pmatrix} = \underbrace{\overbrace{\begin{pmatrix} 1 & 0 \\ -\phi_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\phi_2 & 1 \end{pmatrix}\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}^{\text{One round trip through cavity}}}_{\vec{S}}\begin{pmatrix} y_0 \\ nu_0 \end{pmatrix}$$

$$\vec{S} = \begin{pmatrix} 1 - d\phi_2 & d(2 - d\phi_2) \\ -\phi_2 + \phi_1(-1 + d\phi_2) & 1 + d^2\phi_1\phi_2 - d(2\phi_1 + \phi_2) \end{pmatrix}$$

$$\det(\vec{S}) = 1$$

Eigenvalues are

$$\lambda_1 = \frac{Tr(\vec{S})}{2} + \sqrt{\frac{(Tr(\vec{S}))^2}{4} - 1}$$

$$\lambda_2 = \frac{Tr(\vec{S})}{2} - \sqrt{\frac{(Tr(\vec{S}))^2}{4} - 1}$$

If $\frac{(\mathrm{Tr}(\vec{S}))^2}{4} > 1$ then both λ_1 and λ_2 are real which means that we can rewrite the eigenvalues as an exponential

$$\lambda_1 = e^{\alpha_1} \quad \text{where} \quad \alpha_1 = \ln\left[\frac{\mathrm{Tr}(\vec{S})}{2} + \sqrt{\frac{(\mathrm{Tr}(\vec{S}))^2}{4} - 1}\right]$$

$$\lambda_2 = e^{\alpha_2} \quad \text{where} \quad \alpha_2 = \ln\left[\frac{\mathrm{Tr}(\vec{S})}{2} - \sqrt{\frac{(\mathrm{Tr}(\vec{S}))^2}{4} - 1}\right]$$

If we look at a large number N passes through the resonator, the ray properties are

$$\begin{pmatrix} y_N \\ nu_N \end{pmatrix} = \vec{S}^N \begin{pmatrix} y_0 \\ nu_0 \end{pmatrix} = \lambda^N \begin{pmatrix} y_0 \\ nu_0 \end{pmatrix}$$

In general α_1 and/or α_2 will be > 0 so λ^N becomes large $\Rightarrow \infty$ or α_1 and/or α_2 will be < 0 so λ^N goes to zero for N large. No solutions in the case and we say the resonator is unstable.

If $\frac{(\mathrm{Tr}(\vec{S}))^2}{4} \leq 1$ then both λ_1 and λ_2 are complex numbers which means that we can rewrite the eigenvalues as a complex exponential

$$\lambda_1 = e^{i\alpha_1} \quad \text{where} \quad \alpha_1 = \tan^{-1}\left[\frac{\sqrt{1 - \frac{\mathrm{Tr}(\vec{S})}{4}}}{\frac{\mathrm{Tr}(\vec{S})}{2}}\right]$$

$$\lambda_2 = e^{i\alpha_2} \qquad \alpha_2 = \tan^{-1}\left[\frac{-\sqrt{1 - \frac{\mathrm{Tr}(\vec{S})}{4}}}{\frac{\mathrm{Tr}(\vec{S})}{2}}\right]$$

Here the eigenvalues just oscillate since they are just sines + cosines. This means for large N, we still get good values for y_N and nu_N and the resonator is stable

$$\frac{Tr(\vec{S})^2}{4} \leq 1 \quad \Rightarrow -1 \leq \frac{Tr(\vec{S})}{2} \leq 1$$

$$0 \leq \frac{Tr(\vec{S})+2}{2} \leq 2$$

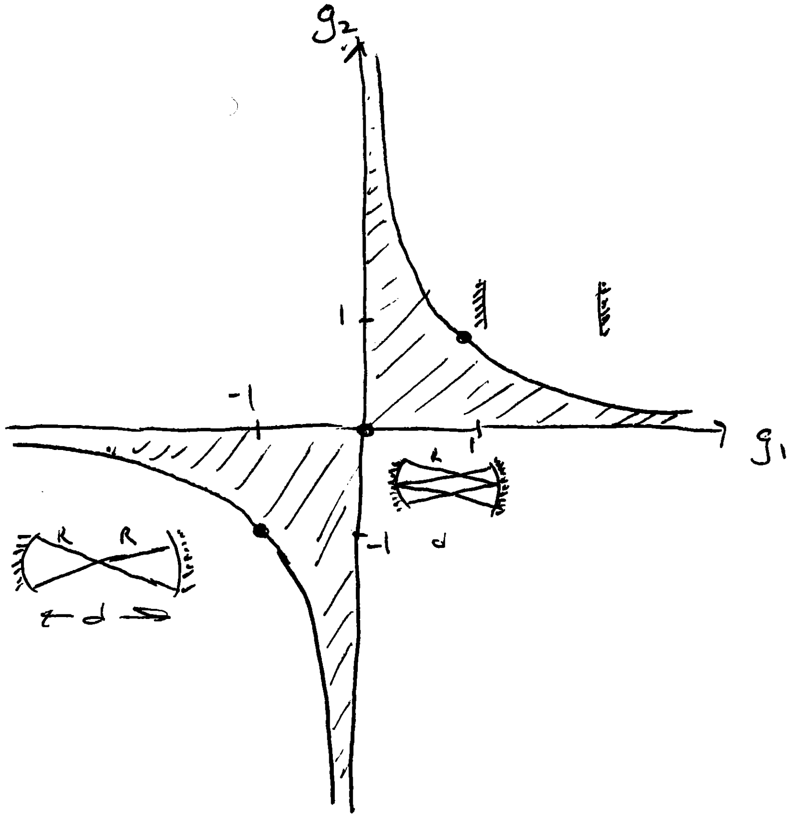
$$0 \leq \frac{Tr(\vec{S})+2}{4} \leq 1$$

From $\vec{S}$

$$\frac{Tr(\vec{S})+2}{4} = \left(1 - \frac{d\phi_1}{2}\right)\left(1 - \frac{d\phi_2}{2}\right) = \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right)$$

Define $g_i = 1 - \frac{d}{R_i}$

For stable cavity we need $0 \leq g_1 g_2 \leq 1$



Fitting Data

Suppose we want to fit data to a line. We can define the line as

$$y = \underbrace{a_0}_{\text{y-intercept}} + \underbrace{a_1}_{\text{slope}} x$$

For two points we can get an exact solution

x	y
1	1.2
3	1.6

As a matrix equation

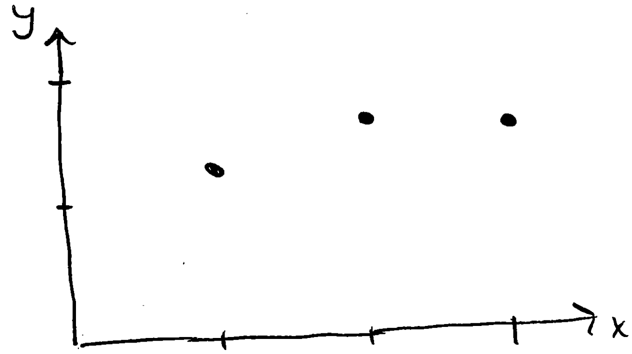
$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.6 \end{pmatrix}$$

We know multiple ways to solve this

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix}$$

What happens if we have 3 points now and the don't lie on the same line?

x	y
1	1.2
2	1.6
3	1.6



As a matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1.2 \\ 1.6 \\ 1.6 \end{pmatrix}$$

$$\vec{A}\,\vec{x} = \vec{b}$$

rows (cols) are linearly independent, which means the three points don't lie on the same point.

3 eqs. two unknowns.
Can't find $\vec{A}^{-1}$

Try multiplying both sides by $\vec{A}^T$

$$\vec{A}^T \vec{A}\, \vec{x} = \vec{A}^T \vec{b} \qquad \text{NORMAL EQUATION}$$

$\vec{A}^T\vec{A}$ is a square matrix meaning we can find its inverse as long as $\det[\vec{A}^T\vec{A}] \neq 0$. Multiply both sides of the eq. by $[\vec{A}^T\vec{A}]^{-1}$ gives

$$\vec{x} = [\vec{A}^T\vec{A}]^{-1}\vec{A}^T\vec{b}$$

Note: if we generalize this to complex matrices transpose → ~~complex~~ conjugate transpose (denoted by †) and $[\vec{A}^\dagger\vec{A}]^{-1}\vec{A}^\dagger$ is known as the Moore-Penrose pseudoinverse

For our example

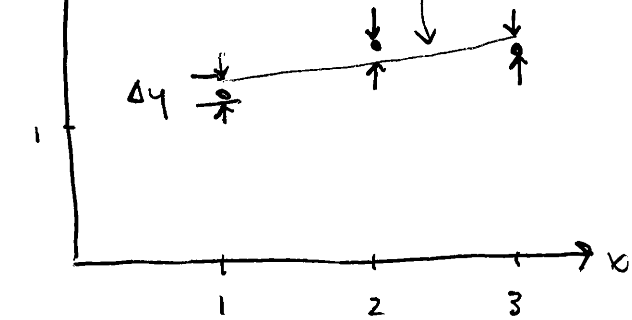
$$\vec{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \vec{A}^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1.2 \\ 1.6 \\ 1.6 \end{pmatrix}$$

$$\vec{A}^T\vec{A} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \qquad \vec{A}^T\vec{b} = \begin{pmatrix} 4.4 \\ 9.2 \end{pmatrix}$$

$$(\vec{A}^T\vec{A})^{-1} = \begin{pmatrix} 7/3 & -1 \\ -1 & \frac{1}{2} \end{pmatrix} \qquad \vec{x} = [\vec{A}^T\vec{A}]^{-1}\vec{A}^T\vec{b} = \begin{pmatrix} 1.066\bar{6} \\ 0.2 \end{pmatrix}$$

x	y	y_{fit}	Δy
1	1.2	$1.26\bar{6}$	$0.06\bar{6}$
2	1.6	$1.46\bar{6}$	$-0.13\bar{3}$
3	1.6	$1.66\bar{6}$	$0.06\bar{6}$

$y = 1.066\bar{6} + 0.2x$



This solution minimizes $\sum_i \Delta y_i^2$