Consider a ray $\begin{pmatrix}1\\0\end{pmatrix}$ at the front principal plane. It will leave the rear principal plane with angle $-\Phi$ so that it passes through rear focal point

$$\begin{pmatrix}1\\-\Phi\end{pmatrix} = \begin{pmatrix}1 & d'\\0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & t_1\\-\Phi & a_{22}\end{pmatrix}\begin{pmatrix}1 & -d\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

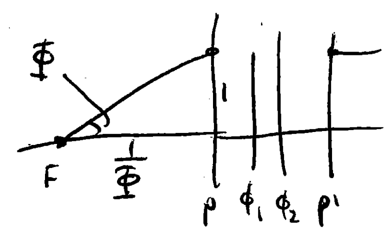
$$\begin{pmatrix}1\\-\Phi\end{pmatrix} = \begin{pmatrix}a_{11}-d'\Phi & t_1+a_{22}d'\\-\Phi & a_{22}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$a_{11}-d'\Phi = 1$$

$$\boxed{d' = \frac{a_{11}-1}{\Phi} = \frac{-t_1\phi_1}{\Phi}}$$

LOCATION OF REAR PRINCIPAL POINT RELATIVE TO LAST SURFACE

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1 & d'\\0 & 1\end{pmatrix}\begin{pmatrix}a_{11} & t_1\\-\Phi & a_{22}\end{pmatrix}\begin{pmatrix}1 & -d\\0 & 1\end{pmatrix}\begin{pmatrix}1\\\Phi\end{pmatrix}$$



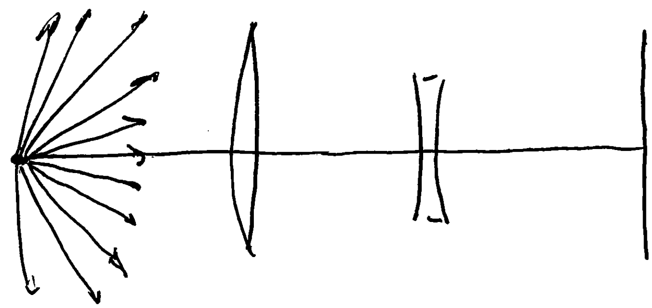
$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}-d'\Phi & t_1+a_{22}d'\\-\Phi & a_{22}\end{pmatrix}\begin{pmatrix}1-d\Phi\\\Phi\end{pmatrix}$$

$$-\cancel{\Phi} + d\Phi^{\cancel{2}} + a_{22}\cancel{\Phi} = 0 \qquad \text{second element}$$

$$\boxed{\begin{aligned} d &= \frac{-a_{22}+1}{\Phi} \\ d &= \frac{+t_1\phi_2}{\Phi}\end{aligned}}$$

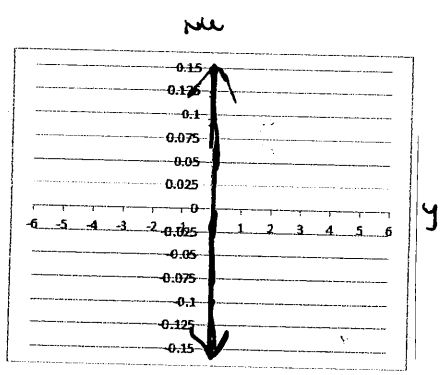
LOCATION OF FRONT PRINCIPAL POINT RELATIVE TO FIRST SURFACE

Now back to linear transformations. Both T and R are shear matrices. T shears in the x-direction and R shears in the y-direction. Consider the following for the thin lens system example we did previously.

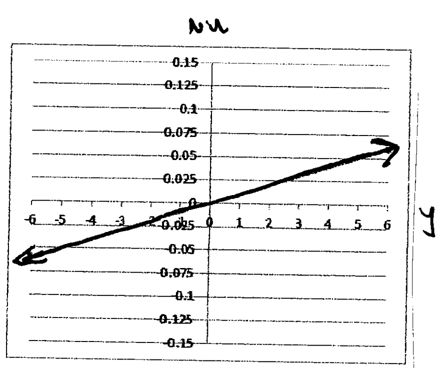


A point source on-axis will radiate rays for all values of u, but will only have one value of y ($y=0$). Only a subset of these rays will be captured by the first lens

<u>Phase Space</u> - Method of examining the effects of the optical system on many rays simultaneously. Plot y along the horizontal axis and nu along the vertical axis.

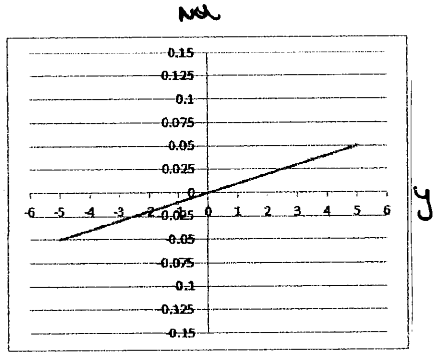


At the object an axial point source will have $y=0$, but all values of u.

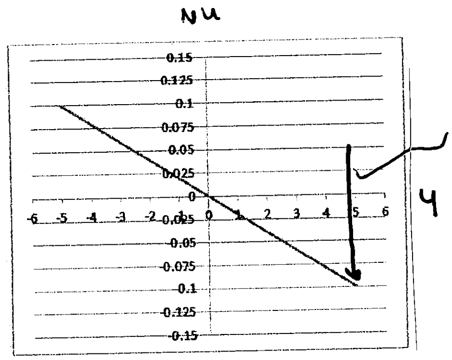


$T_0 = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}$ is a shear in x-direction

The slope of this line is now $\frac{1}{100}$

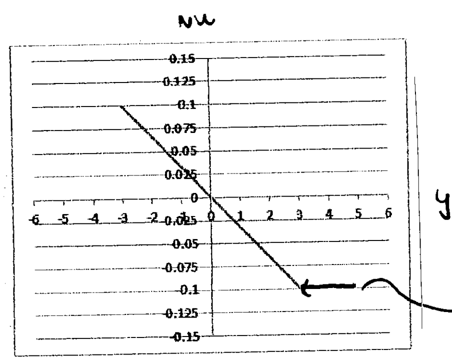


The first lens clips the incident rays. For the example, the first lens has a diameter of 10 mm, so only rays with $-5 \le y \le 5$ continue



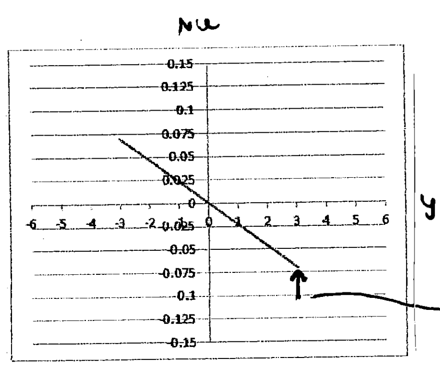
$5(-0.03)$

$$R_1 = \begin{pmatrix} 1 & 0 \\ -0.03 & 1 \end{pmatrix}$$ is a shear in y-direction



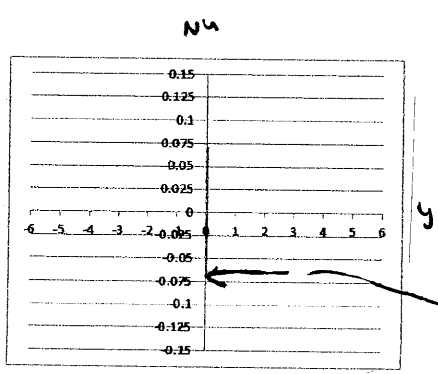
$$T_1 = \begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix}$$ is a shear in x-direction

$-0.1(20)$



$$R_2 = \begin{pmatrix} 1 & 0 \\ 0.01 & 1 \end{pmatrix}$$ shear in y-direction

$3(0.01)$



$$T_2 = \begin{pmatrix} 1 & 42.857 \\ 0 & 1 \end{pmatrix}$$ shear in x-direction

$-0.07(42.857)$

Vertical line means all rays at same point (i.e. image point)

AFFINE TRANSFORMATION - with our 2x2 linear transformation, we were able to do Scaling, Rotation, Reflection and Shearing. One obvious exception to our capabilities is translation. Linear transformations cannot do translation because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Basically, the point at the origin is always stuck at the origin. Fortunately, we can do a little trick to incorporate translation into our toolbox. Here, we will use homogeneous coordinates.

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$ we still have our 2D coordinates x, y, but will write them as a 3D vector with the last element = 1

Our transformation matrix is now 3D as well

$$\begin{pmatrix} a & b & \Delta x \\ c & d & \Delta y \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is our conventional 2x2 linear transformation matrix and Δx and Δy are the translations in the x and y directions.

Let's look at our "house" example

$$\underbrace{\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ & 0.5 \\ \sin 30^\circ & \cos 30^\circ & 0.5 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{Rotate } 30^\circ}\begin{pmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

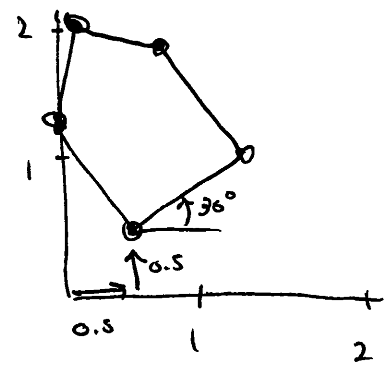
"house" x, y coordinates

last row = 1's to make homogeneous coordinates

translate by 0.5 in x
0.5 in y

The resultant matrix is

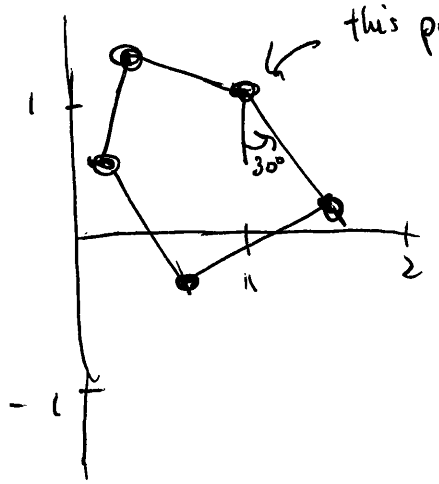
$$\begin{pmatrix} 0.5 & 1.366 & 0.866 & 0.183 & 0 \\ 0.5 & 1 & 1.866 & 2.049 & 1.366 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



Our 2×2 Rotation was a rotation about the origin. The affine transformation now makes it easy to rotate about any point. Suppose we want to rotate by 30° the "house" about the point (1,1). We translate the point (1,1) to the origin, do the 30° rotation and then translate back.

$$\begin{pmatrix} \overbrace{1 \quad 0}^{I} & \overbrace{1}^{\text{translate }(1,1)} \\ 0 \quad 1 & 1 \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} \overbrace{\cos 30^\circ \quad -\sin 30^\circ}^{\text{Rotate }30^\circ} & \overbrace{0}^{\text{NO translation}} \\ \sin 30^\circ \quad \cos 30^\circ & 0 \\ 0 \quad\quad 0 & 1 \end{pmatrix} \begin{pmatrix} \overbrace{1 \quad 0}^{I} & \overbrace{-1}^{\text{translate }(-1,-1)} \\ 0 \quad 1 & -1 \\ 0 \quad 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.634 & 1.5 & 1 & 0.317 & 0.134 \\ -0.366 & 0.134 & 1 & 1.183 & 0.5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$



this point didn't change

Find inverse of our translation matrix

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \Delta x & 1 & 0 & 0 \\ 0 & 1 & \Delta y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Takes one step with Gauss-Jordan

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\Delta x \\ 0 & 1 & 0 & 0 & 1 & -\Delta y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Row 1 $\to$ Row 1 $- \Delta x$ Row 3

Row 2 $\to$ Row 2 $- \Delta y \cdot$ Row 3

<u>Eigenvectors and Eigenvalues</u> - Up to now we have been solving eqs. that look like $\vec{A}\vec{x} = \vec{b}$. Now let's try something a little different.

$$\vec{A}\vec{x} = \lambda \vec{x}$$ where λ is some scalar, $\vec{A}$ is square

We can rewrite this as

$$(\vec{A} - \lambda \vec{I})\vec{x} = \vec{0}$$

This has a trivial solution $\vec{x} = \vec{0}$. To have a non-trivial solution, we need

$$\det(\vec{A} - \lambda \vec{I}) = 0$$

Let's look at a 2x2 example. Suppose

$$\vec{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

We want to solve

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \vec{x} = \lambda \vec{x}$$

Goal find the $\vec{x}$'s (eigenvectors) and associated λ's (eigenvalues) that satisfy this eq.

So we know, we need

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = 0$$

$$(3-\lambda)^2 - 1 = 0$$

$$9 - 6\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$\lambda_1 = 4 \quad \lambda_2 = 2$ Typically order eigenvalues from highest to lowest

Let's look at λ_1 case

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \vec{x}_1 = 4 \vec{x}_1$$ let $\vec{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 4 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$3x_1 + y_1 = 4x_1$ first row

$-x_1 + y_1 = 0$

$x_1 + 3y_1 = 4y_1$ second row

$x_1 - y_1 = 0 \Rightarrow -x_1 + y_1 = 0$ Same thing. Need to choose something easy for x_1 and find y_1. How about $x_1 = 1$?

So if $x_1 = 1$, then $y_1 = 1$

$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector of $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ with eigenvalue $\lambda_1 = 4$