Finally, divide each row by its respective pivot

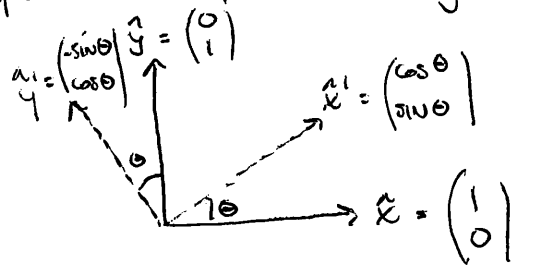
$$\begin{pmatrix} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix}$$

The first three columns now contain the identity matrix and the last three columns contain $\vec{A}^{-1}$.

$$\vec{A}^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{pmatrix}$$

$$\vec{A}^{-1}\vec{A} = \begin{pmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{5}{4} + \frac{3}{4} & \cdots & \\ \vdots & \ddots & \\ \vdots & & \ddots \end{pmatrix} = I$$

LINEAR TRANSFORMATIONS - For the matrix equation $\vec{A}\vec{x} = \vec{b}$, we can think of $\{\vec{x}\}$ having some shape and $\vec{A}$ transforms these vectors into some new set of vectors $\{\vec{b}\}$ with a different shape. Let's first think about keeping the shape the same, but rotating a pattern about the origin.



Suppose we want to convert a vector in the $\hat{x}$-$\hat{y}$ coordinate system to a vector rotated by some angle θ.

Simply trig shows that $\hat{x} \Rightarrow \hat{x}'$ and $\hat{y} \Rightarrow \hat{y}'$ following the rotation. These new vectors become the columns of a 2×2 rotation matrix $\vec{R}_\theta$ which can be applied to any vector in $\hat{x}$-$\hat{y}$ space.

$$\vec{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Suppose we have vectors representing the corners of a "house". We'll write these as the columns of a matrix

$$\vec{x} = \begin{pmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{pmatrix}$$

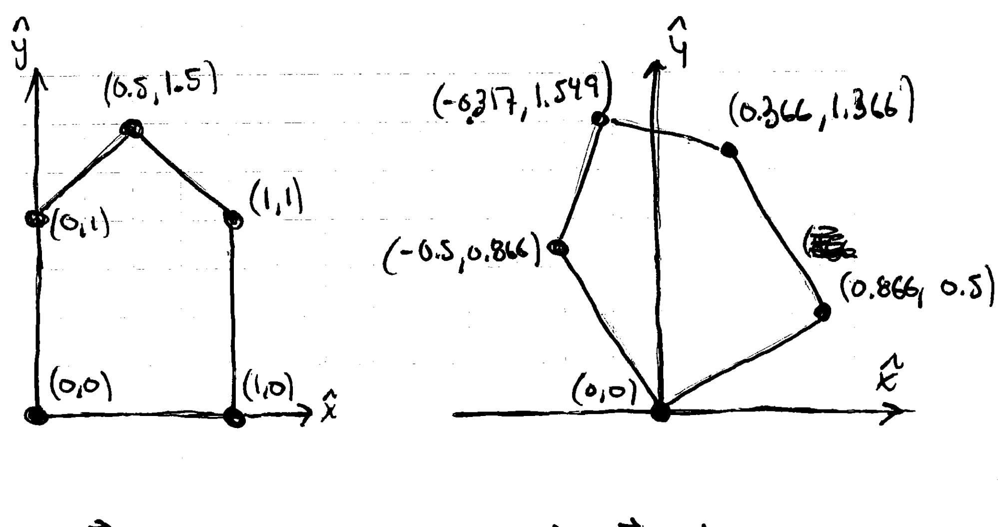
We can rotate each of these by θ by calculating

$$\vec{y} = \vec{R}_\theta \vec{x}$$

For example, a $30°$ rotation gives

$$\vec{y} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 1.5 & 1 \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} 0 & 0.866 & 0.366 & -0.317 & -0.5 \\ 0 & 0.5 & 1.366 & 1.549 & 0.866 \end{pmatrix}$$



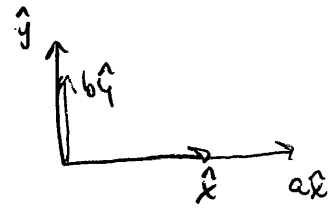
$\vec{x}$ $\qquad\qquad$ $\vec{y} = \vec{R}_{30°}\vec{x}$

There are a whole series of linear transformations that represent rotation, scaling, reflection, projection and shearing. Let's look at the general form for each of these transformations.

Scaling

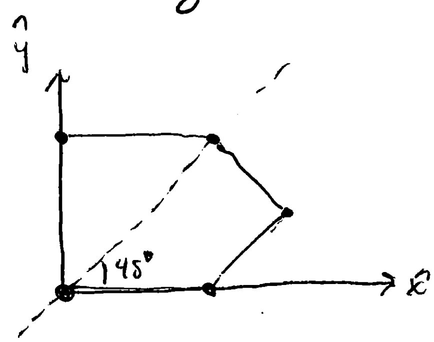
$$\vec{M} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

This matrix scales a vector by a factor a in the x direction and a factor b in the y direction



Reflection - Mirror reflection about a line at an angle Θ

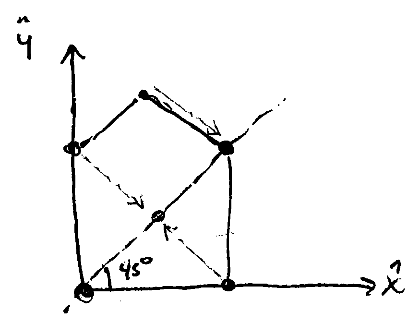
$$\vec{M}_\Theta = \begin{pmatrix} 2\cos^2\Theta - 1 & 2\sin\Theta\cos\Theta \\ 2\sin\Theta\cos\Theta & 2\sin^2\Theta - 1 \end{pmatrix}$$



For example, when $\Theta = 45°$, $\vec{M}_{45} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Projection - projects onto a line at an angle Θ

$$P_\Theta = \begin{pmatrix} \cos^2\Theta & \sin\Theta\cos\Theta \\ \sin\Theta\cos\Theta & \sin^2\Theta \end{pmatrix}$$



For example, when $\Theta = 45°$, $\vec{P}_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

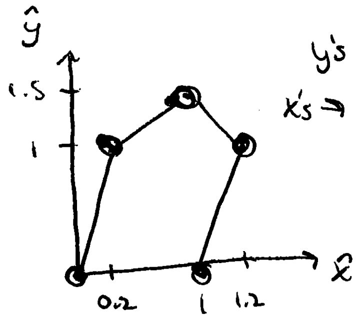
Note that $\det(P_\Theta) = 0$, which means its inverse does not exist. Geometrically, this means when we project onto a line, we can't recover the original shape.

Also, $P_\Theta P_\Theta = P_\Theta$ i.e. once we project onto a line, further projections don't change anything

<u>Shearing</u> - slants the vectors by a given amount

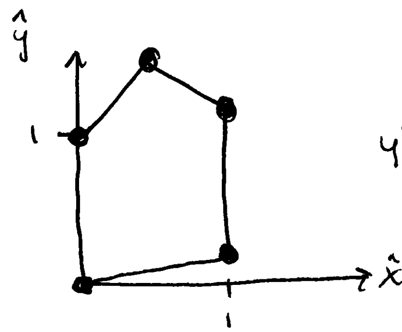
$$\vec{S}_x = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad \vec{S}_y = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

Shear (slant) in x direction — Shear (slant in y direction



y's stay same
x's → x + ay

for a = 0.2



x's stay same
y's → y + ax

for a = 2

This last transformation is especially relevant to paraxial optics

Paraxial Raytracing iteratively uses a set of linear equations to propagate a paraxial ray through an optical system. The two equations are

Transfer
k^{th} to $k{+}1^{th}$ surface

$$y_{k+1} = y_k + \frac{t_k}{N_k} N_k u_k$$

t_k distance between surfaces
N_k index between surfaces

Refract
trajectory after refraction at k^{th} surface

$$N_{k+1} u_{k+1} = N_k u_k - y_k \phi_k$$

These equations hold for thick lens systems where $\phi_k = \dfrac{N_{k+1} - N_k}{R_k}$ is the power of the k^{th} surface

These equations also hold for thin lens systems where ϕ_k = power of the k^{th} thin lens. N_k equals 1 typically in this case.

We can think of a paraxial ray as being defined by a vector $\begin{pmatrix} y \\ Nu \end{pmatrix}$

The two raytracing equations can be written in matrix form as

$$T_k = \begin{pmatrix} 1 & \frac{t_k}{N_k} \\ 0 & 1 \end{pmatrix}$$

Transfer ~~(drop k subscripts for convenience)~~

$$R_k = \begin{pmatrix} 1 & 0 \\ -\phi_k & 1 \end{pmatrix}$$

so that

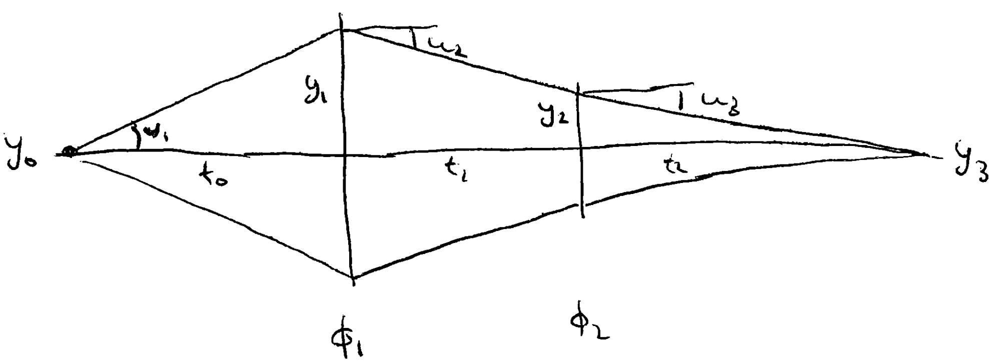
$$\begin{pmatrix} y_{k+1} \\ N_{k+1}u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{t_k}{N_k} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_k \\ N u_k \end{pmatrix}$$

y_k changes on transfer, but optical angle stays the same

$$\begin{pmatrix} y_{k+1} \\ N_{k+1}u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\phi_k & 1 \end{pmatrix} \begin{pmatrix} y_k \\ N u_k \end{pmatrix}$$

y_k stays the same upon refraction but optical angle changes.

For a system, we can string together multiple T's and R's



$$\begin{pmatrix} y_3 \\ u_3 \end{pmatrix} = \underbrace{T_2 R_2 T_1 R_1 T_0}_{\text{order is important here}} \begin{pmatrix} y_0 \\ u_0 \end{pmatrix}$$

The matrix product $T_2 R_2 T_1 R_1 T_0$ results in a new 2x2 matrix called the system matrix which describes the input/output relationship for the whole system

Let's do a quick example

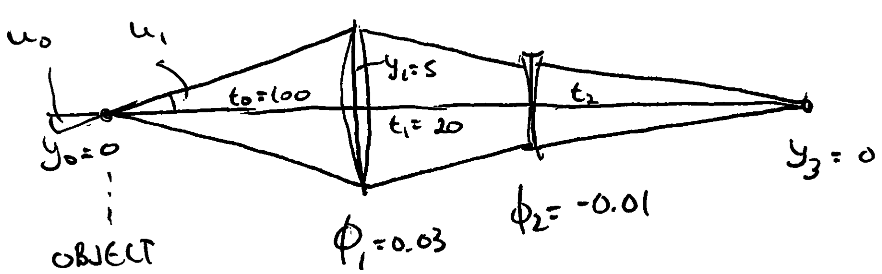
$\phi_1 = 0.03 \text{ mm}^{-1}$ Diameter = 10 mm Stop

$\phi_2 = -0.01 \text{ mm}^{-1}$

$t_0 = 100$ mm

$t_1 = 20$ mm

Find t_2 so that we are one image plane



$$T_0 = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} 1 & 0 \\ -0.03 & 1 \end{pmatrix} \quad T_1 = \begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1 & 0 \\ 0.01 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix}$$

NOTE THAT $u_0 = u_1$

At first lens

$$\begin{pmatrix} y_1 \\ u_1 \end{pmatrix} = T_0 \begin{pmatrix} y_0 \\ u_0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 \\ u_1 \end{pmatrix} = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ u_0 \end{pmatrix}$$

Need to find u_1 that satisfies this

$$100 u_0 = 5 \Rightarrow u_0 = 0.05$$

$$u_1 = u_0 \qquad u_1 = 0.05$$

Refraction by first lens

$$\begin{pmatrix} y_1' \\ u_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -0.03 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ u_1 \end{pmatrix}$$

Note $y_1' = y_1$

$$\begin{pmatrix} y_1' \\ u_1' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -0.03 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0.05 \end{pmatrix} = \begin{pmatrix} 5 \\ -0.1 \end{pmatrix}$$

← new trajectory

At lens 2 (NOTE $u_1' = u_2$)

$$\begin{pmatrix} y_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -0.1 \end{pmatrix} = \begin{pmatrix} 3 \\ -0.1 \end{pmatrix}$$

new height

Refraction at lens 2 $y_2' = y_2$

$$\begin{pmatrix} y_2' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0.01 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -0.1 \end{pmatrix} = \begin{pmatrix} 3 \\ -0.07 \end{pmatrix}$$

At image plane $u_3 = u_2'$

$$\begin{pmatrix} y_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & t_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -0.07 \end{pmatrix} = \begin{pmatrix} 3 - 0.07 t_2 \\ -0.07 \end{pmatrix}$$

want $y_3 = 0$ at image plane so

$$3 - 0.07 t_2 = 0$$

$$t_2 = 42.857 \text{ mm}$$

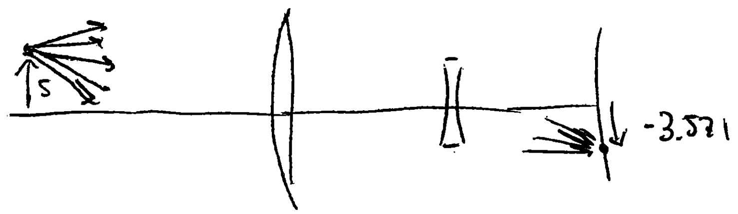
System Matrix $S = T_2 R_2 T_1 R_1 T_0$

$$S = \begin{pmatrix} -0.714 & 0 \\ -0.026 & -1.4 \end{pmatrix} \Rightarrow \begin{pmatrix} y_3 \\ u_3 \end{pmatrix} = S \begin{pmatrix} y_0 \\ u_0 \end{pmatrix}$$

Now let's trace a different ray with $y_0 = 5$ and $u_0 = 0$

$$\begin{pmatrix} y_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} -0.714 & 0 \\ -0.026 & -1.4 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} -3.571 \\ -0.13 \end{pmatrix}$$

In fact we can use any angle u_1 since 0 in upper right of system matrix and still get same answer. Basically says any ray leaving the object point goes to same image point.



Let's just look at lens portion $R_2 T_1 R_1$

$$R_2T_1R_1 = \begin{pmatrix} 1 & 0 \\ -\phi_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\phi_1 & 1 \end{pmatrix} = \begin{pmatrix} 1-t_1\phi_1 & t_1 \\ -(\phi_1+\phi_2-t_1\phi_1\phi_2) & 1-t_1\phi_2 \end{pmatrix}$$

- TOTAL POWER $= \Phi$

$$R_2T_1R_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$a_{11} = 1 - t_1\phi_1$; $a_{12} = t_1$; $a_{21} = -\Phi$; $a_{22} = 1 - t_1\phi_2$

Let's assume that the front principal plane P is a distance d from the front surface and the rear principal plane P' is a distance d' from the last surface