How about $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$? DOES THIS HAVE A UNIQUE SOLUTION?

ROW 3 = 2 x ROW 1 + ROW 2 So the answer is <u>NO</u>

COLUMN 1 + COLUMN 3 = 2 x COLUMN 2

THIS ROUTE IS OK FOR SMALL MATRICES, BUT QUICKLY BECOMES NON-OBVIOUS

GAUSSIAN ELIMINATION

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$$

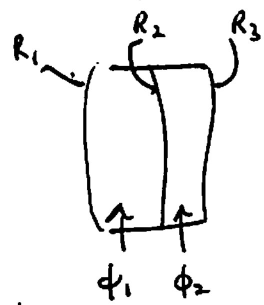
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \quad \begin{matrix} \text{ROW2} - 2\text{ROW1} \\ \text{ROW3} - 4\text{ROW1} \end{matrix}$$

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{ROW3} - \text{ROW2}$$

The number of NON-ZERO pivots will tell us the number of independent rows (columns)

Let's do some optics, finally

Suppose we want to create an achromatic doublet (thin). The powers of the two lenses at the F, d, C wavelengths are



1^{st} requirement $\boxed{\phi_{1d} + \phi_{2d} = \phi_d}$

$$\phi_{1F} + \phi_{2F} = \phi_F$$

$$\phi_{1C} + \phi_{2C} = \phi_C$$

We also want powers of ϕ_F and ϕ_C to be the same

2^{nd} requirement $\boxed{\phi_F - \phi_C = 0}$

Let's rewrite these into a system of equations that we can solve using our matrix techniques. Goal: Find values of ϕ_{1d} and ϕ_{2d} given ϕ_d and the material properties of each lens.

$$\phi_F - \phi_C = 0$$

$$\phi_{1F} + \phi_{2F} - \phi_{1C} - \phi_{2C} = 0$$

Lensmaker's Formula

$$\phi_{1F} - \phi_{1C} + \phi_{2F} - \phi_{2C} = 0$$

$$(N_{1F}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) - (N_{1C}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right) + (N_{2F}-1)\left(\frac{1}{R_2}-\frac{1}{R_3}\right) - (N_{2C}-1)\left(\frac{1}{R_2}-\frac{1}{R_3}\right) = 0$$

$$(N_{1F}-N_{1C})\left(\frac{1}{R_1}-\frac{1}{R_2}\right) + (N_{2F}-N_{2C})\left(\frac{1}{R_2}-\frac{1}{R_3}\right) = 0$$

But $\phi_{1d} = (N_{1d}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ Similar for ϕ_{2d}

$$\frac{N_{1F}-N_{1C}}{N_{1d}-1}\phi_{1d} + \frac{N_{2F}-N_{2C}}{N_{2d}-1}\phi_{2d} = 0$$

$$\frac{\phi_{1d}}{V_1} + \frac{\phi_{2d}}{V_2} = 0$$ where V_1, V_2 are the Abbe #'s of the two materials

Now we have two equations and two unknowns. Write these as a matrix equation

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{V_1} & \frac{1}{V_2} \end{pmatrix}\begin{pmatrix} \phi_{1d} \\ \phi_{2d} \end{pmatrix} = \begin{pmatrix} \phi_d \\ 0 \end{pmatrix}$$

Now we learned that the rows (columns) need to independent. This means we can't have $V_1 = V_2$. Gaussian elimination gives

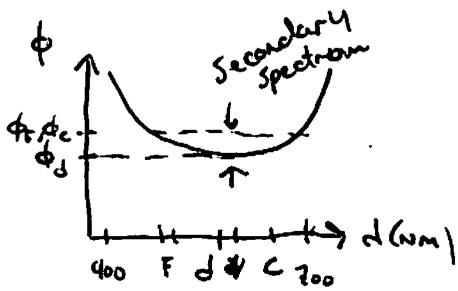
$$\begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{V_2}-\frac{1}{V_1} \end{pmatrix}\begin{pmatrix} \phi_{1d} \\ \phi_{2d} \end{pmatrix} = \begin{pmatrix} \phi_d \\ \frac{-\phi_d}{V_1} \end{pmatrix}$$ row 2 $- \frac{1}{V_1}$ row 1

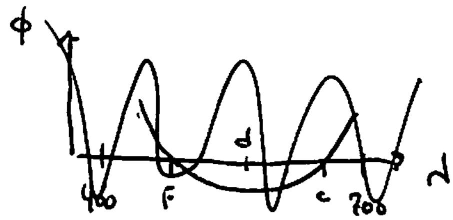
Now use back substitution

$$\phi_{2d}\left(\frac{1}{V_2}-\frac{1}{V_1}\right) = \frac{-\phi_d}{V_1} \Rightarrow \boxed{\phi_{2d} = \frac{-\phi_d V_2}{V_1 - V_2}}$$ 2nd row

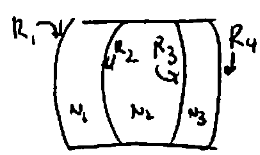
$$\phi_{1d} + \phi_{2d} = \phi_d$$ 1st row

$$\boxed{\phi_{1d} = \phi_d + \frac{\phi_d V_2}{V_1 - V_2} = \frac{\phi_d V_1}{V_1 - V_2}}$$





Ideally, we would like to reduce the secondary spectrum too. The strategy is the same. Make powers at three wavelengths equal. Since we have three requirements, (i.e. total lens power; power at first λ equals power at second λ; power at first λ equals power at third λ) we need a minimum of three elements to achieve.



ϕ_d = TOTAL POWER at d wavelength

ϕ_{1d} = power of 1st element at d wavelength

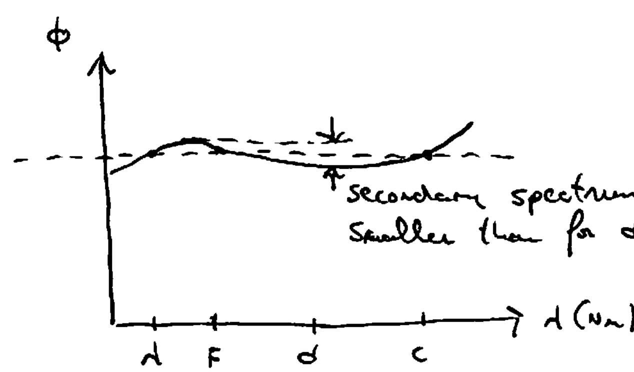
Similar for of elements and wavelengths

$$\left(\phi_{1d} + \phi_{2d} + \phi_{3d} = \phi_d\right)$$

$$\phi_F - \phi_C = 0$$

$$\phi_\lambda - \phi_F = 0$$

LENSES WITH THREE EQUAL POWERS AT DIFFERENT WAVELENGTHS ARE CALLED APOCHROMATS



Second and third eqs. have similar forms. For eq. 2 we can follow a similar procedure as before

$$(N_{1F}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)-(N_{1C}-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+(N_{2F}-1)\left(\frac{1}{R_2}-\frac{1}{R_3}\right)-(N_{2C}-1)\left(\frac{1}{R_2}-\frac{1}{R_3}\right)+(N_{3F}-1)\left(\frac{1}{R_3}-\frac{1}{R_4}\right)-(N_{3C}-1)\left(\frac{1}{R_3}-\frac{1}{R_4}\right)$$
$$=0$$

$$(N_{1F}-N_{1C})\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+(N_{2F}-N_{2C})\left(\frac{1}{R_2}-\frac{1}{R_3}\right)+(N_{3F}-N_{3C})\left(\frac{1}{R_3}-\frac{1}{R_4}\right)=0$$

$$\frac{N_{1F}-N_{1C}}{N_{1d}-1}\phi_{1d}+\frac{N_{2F}-N_{2C}}{N_{2d}-1}\phi_{2d}+\frac{N_{3F}-N_{3C}}{N_{3d}-1}\phi_{3d}=0$$

$$\boxed{\frac{\phi_{1d}}{V_1}+\frac{\phi_{2d}}{V_2}+\frac{\phi_{3d}}{V_3}=0}$$

For eq. 3, a slight change is needed

$$(N_{1d}-N_{1F})\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+(N_{2d}-N_{2F})\left(\frac{1}{R_2}-\frac{1}{R_3}\right)+(N_{3d}-N_{3F})\left(\frac{1}{R_3}-\frac{1}{R_4}\right)=0$$

$$\left(\frac{N_{1d}-N_{1F}}{N_{1F}-N_{1C}}\right)\left(\frac{N_{1F}-N_{1C}}{N_{1d}-1}\right)\phi_{1d}+\left(\frac{N_{2d}-N_{2F}}{N_{2F}-N_{2C}}\right)\left(\frac{N_{2F}-N_{2C}}{N_{2d}-1}\right)\phi_{2d}+\left(\frac{N_{3d}-N_{3F}}{N_{3F}-N_{3C}}\right)\left(\frac{N_{3F}-N_{3C}}{N_{3d}-1}\right)\phi_{3d}=0$$

$$\boxed{\frac{P_{1dF}\phi_{1d}}{V_1}+\frac{P_{2dF}}{V_2}\phi_{2d}+\frac{P_{3dF}}{V_3}\phi_{3d}=0}$$

where $P_{dF}=\dfrac{N_d-N_F}{N_F-N_C}$ $\Rightarrow$ RELATIVE PARTIAL DISPERSION

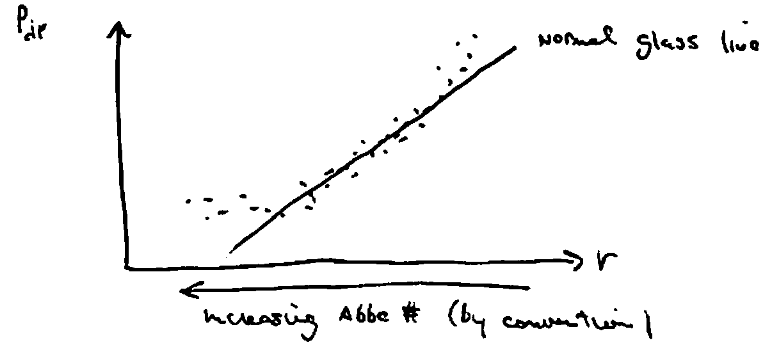
Now let's write our eqs. in matrix form

$$\begin{pmatrix} 1 & 1 & 1 \\ 1/V_1 & 1/V_2 & 1/V_3 \\ \frac{P_{1dF}}{V_1} & \frac{P_{2dF}}{V_2} & \frac{P_{3dF}}{V_3} \end{pmatrix}\begin{pmatrix}\phi_{1d}\\ \phi_{2d}\\ \phi_{3d}\end{pmatrix}=\begin{pmatrix}\phi_d\\ 0\\ 0\end{pmatrix}$$

What requirements can we immediately see from the matrix?

First, three glasses are needed. If we only used two glasses, then two of the columns would match and not be linearly independent.

Second, we want the ratio $\frac{P_{idF}}{V_i}$ to be different for each of the elements. Otherwise, row 3 is just a scaled version of row 1.



If we plot P_{dF} vs. Abbe #, most glasses lie along the "normal" glass line. This means $\frac{P_{dF}}{V}$ (i.e. the slope of the normal line) is approximately constant for most glasses. Typically we try to pick glasses off the normal glass line to correct for secondary spectrum.

Solving the matrix equation gives

$$\phi_{1d} = \frac{(P_{2dF} - P_{3dF})\, v_1 \phi_d}{P_{3dF}(-v_1 + v_2) + P_{2dF}(v_1 - v_3) + P_{1dF}(-v_2 + v_3)}$$

Glasses far from the normal glass line are said to have anomalous dispersion

$$\phi_{2d} = \frac{(P_{1dF} - P_{3dF})\, v_2 \phi_d}{P_{3dF}(v_1 - v_2) + P_{2dF}(-v_1 + v_3) + P_{1dF}(v_2 - v_3)}$$

$$\phi_{3d} = \frac{(P_{1dF} - P_{2dF})\, v_3 \phi_d}{P_{3dF}(-v_1 + v_2) + P_2(v_1 - v_3) + P_1(-v_2 + v_3)}$$

MATRIX INVERSE AND DETERMINANTS

Suppose we have the matrix equation

$$\vec{A}\vec{x} = \vec{b}$$

where $\hat{A}$ is a square matrix. The inverse of $\vec{A}$ is denoted by $\vec{A}^{-1}$ and satisfies

$$\vec{A}^{-1}\vec{b} = \vec{x}.$$

From this relationship, we can get some properties of the inverse. First, multiply both sides of the first equation by $\hat{A}^{-1}$

$$\vec{A}^{-1}\vec{A}\,\vec{x} = \vec{A}^{-1}\vec{b} = \vec{x} \leftarrow \text{from second equation}$$

This means $\boxed{\vec{A}^{-1}\vec{A} = \vec{I}}$

The inverse of a matrix doesn't necessarily exist. Two conditions are necessary in order for the inverse to exist.

1. Matrix must be square
2. The determinant of the matrix must be NON-ZERO

Matrices where the inverse does not exist are called singular matrices.

I'm not going to spend much time on the determinant. For the 2x2 matrix $\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$ (usually denoted as $\det(\hat{A})$ or $|\hat{A}|$). For larger matrices, the math quickly becomes ugly. Let the programs calculate this for you and just remember that if $\det(\hat{A}) = 0$, then $\hat{A}$ is singular; otherwise $\vec{A}^{-1}$ exists.

The inverse of a matrix can be found by the Gauss-Jordan method. The inverse allows us to solve $\vec{A}\vec{x} = \vec{b}$ for $\vec{x}$.

For $\hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \vec{A}^{-1} = \dfrac{1}{\det(\vec{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\quad$ 2x2 is easy

Gauss - Jordan Method

This method is similar to the Gaussian elimination technique we used to solve the matrix equation $\vec{A}\vec{x} = \vec{b}$. An example is the easiest way to demonstrate the technique. Suppose we have the 3×3 matrix

$$\vec{A} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

First, create a 3×6 matrix with $\vec{A}$ in the first 3 columns and the identity matrix in the second three columns

$$\left(\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right)$$

Next, we're going to do forward elimination and get zeros below the diagonal of the $\bar{A}$ portion of this matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \text{Row 2} - 2\,\text{Row 1} \\ \\ \text{Row 3} + \text{Row 1} \end{array}$$

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \qquad \text{Row 3} + \text{Row 2}$$

Next, we want to get zeros above the diagonal for the $\bar{A}$ portion of the matrix. Start with the third column and then go to the second.

$$\begin{pmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \qquad \begin{array}{l} \text{Row 1} - \text{Row 3} \\ \text{Row 2} + 2\,\text{Row 3} \end{array}$$

$$\begin{pmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \qquad \text{Row 1} + \tfrac{1}{8}\,\text{Row 2}$$

Finally, divide each row by its respective pivot

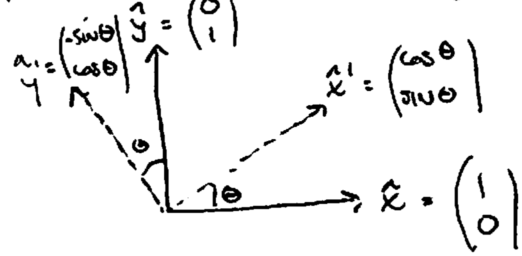
$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

The first three columns now contain the identity matrix and the last three columns contain $\vec{A}^{-1}$.

$$\vec{A}^{-1} = \begin{pmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{pmatrix}$$

$$\vec{A}^{-1}\vec{A} = \begin{pmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{5}{4} + \frac{3}{4} & \cdots & \\ \vdots & \ddots & \\ \vdots & & \ddots \end{pmatrix} = I$$

LINEAR TRANSFORMATIONS – For the matrix equation $\vec{A}\vec{x} = \vec{b}$, we can think of $\{\vec{x}\}$ having some shape and $\vec{A}$ transforms these vectors into some new set of vectors $\{\vec{b}\}$ with a different shape. Let's first think about keeping the shape the same, but rotating a pattern about the origin.



Suppose we want to convert a vector in the $\hat{x}$-$\hat{y}$ coordinate system to a vector rotated by some angle θ.

Simple trig shows that $\hat{x} \Rightarrow \hat{x}'$ and $\hat{y} \Rightarrow \hat{y}'$ following the rotation. These new vectors become the columns of a 2×2 rotation matrix $\vec{R}_\theta$ which can be applied to any vector in $\hat{x}$-$\hat{y}$ space.

$$\vec{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$