Linear Algebra for Describing Polarization and Polarizing Elements

Alex Miles, September 2, 2014

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1 Matrices as Operators

So far we've been using matrices to describe linear systems of simultaneous equations, however they can also be used to describe maps or operations. A mapping will take an input vector, here denoted $\vec{x}$, and yield an output vector, denoted $\vec{x}'$.

$$M\vec{x} = \vec{x}'$$

A simple example of this sort of action would be a matrix that "scales the x-component by 5, while scaling the y component by 3", which would look like this:

$$M = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

A more complicated mapping, such as "rotate the input vector by $\pi/2$" is similarly easy once we know the general form of the 2D rotation matrix:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

As has been discussed, it is important to note that matrix operations do not, in general, commute. For example (left) rotating by 90 degrees then scaling the components separately is **not** equivalent to applying the scaling then rotating.

$$\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} R(\pi/2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad R(\pi/2)\begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

2 Polarization

Recalling from electromagnetism that energy flows in the direction of the Poyting vector, $\vec{S} \propto \vec{E} \times \vec{B}$, meaning that (in linear media) the electric field must oscillate perpendicular to the direction of propagation. This still permits two of freedom in the perpendicular plane, and materials both on the macroscopic level (reflection coefficients) and the microscopic level (atomic transitions) are sensitive to how the field is configured. It is useful to be able to describe the field using a vector, and the mappings or operations that various optical elements cause as matrices. It is useful here to describe the function of several common polarization elements that we will shortly express as matrices.

- Linear polarizers pass a given linear polarization while attenuating the orthogonal component.
- Circular polarizers pass a given circular polarization while attenuating the opposite component.
- Linear retarders impart a phase difference between the two orthogonal linear polarizations states.
  - $\Delta\phi = \pi$, (half-wave) will rotate the polarization incident on it by 2θ where θ is the angle between the fast axis of the plate and the incident linear polarization.
  - $\Delta\phi = \pi/2$, (quarter-wave) converts ± 45 degree linear polarized light to left and right hand circularly polarized light.

3 Jones Calculus

First formulated in 1941, Jones calculus utilizes a 2x1 vector to describe the polarization of light, and 2x2 matrices to describe the action of an element upon that light. It is important to note that Jones calculus can only be used to describe light that is **fully polarized** and **coherent**.

Jones calculus uses $\hat{x}$ and $\hat{y}$ as its basis states, and this means that all Jones vectors and matrices must be defined with respect to some axes. Linear polarization of any angle can be described as a super-position of these two basis states.

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{p}_{+45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \hat{p}_{-45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The most useful feature of Jones vectors are that they describe not only the amplitude of the two components, but also the relative phase of the two components. The direction and frequency of the wave are factored out.

$$\begin{pmatrix} E(t)_x \\ E(t)_y \end{pmatrix} = \begin{pmatrix} E_{0x} e^{i(kz-\omega t+\phi_x)} \\ E_{0y} e^{i(kz-\omega t+\phi_y)} \end{pmatrix} = \begin{pmatrix} E_{0x} e^{i\phi_x} \\ E_{0y} e^{i\phi_y} \end{pmatrix} e^{i(kz-\omega t)}$$

This allows us to describe right and left circular polarizations easily.

$$\hat{R} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y}), \quad \hat{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$$

Similarly, it can occasionally be useful to phrase $\hat{x}$ and $\hat{y}$ in terms of circular polarizations (e.g. when your system has circular diattenuation and retardance, but you input linear light).

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{R} + \hat{L}), \quad \hat{y} = \frac{i}{\sqrt{2}}(\hat{R} - \hat{L})$$

There are two distinct classes of elements we must be able to describe: diattenuators, elements that preferentially block one polarization and pass another, and retarders, elements that yield a phase difference between two polarization components. First let us look at the most common components, linear polarizers. Note the scaling factor on the ± 45 degree case.

$$LP_h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad LP_v = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad LP_{\pm 45} = \frac{1}{2} \begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

Similarly, we can construct right and left circular polarizers by utilizing those states (and their conjugates) as columns:

$$CP_r = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad CP_l = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

Retarder elements that have no diattenuation take on the following general form, where the relative phase $\Delta\phi = \phi_x - \phi_y$. By plugging in values we obtain the matrices for quarter and half-wave plates.

$$LR = \begin{pmatrix} e^{i\phi_x} & 0 \\ 0 & e^{i\phi_y} \end{pmatrix}, \quad LR_{\lambda/4,vertical} = e^{i\pi/4} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \quad LR_{\lambda/2}(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

To find the matrix for any rotated element, a standard rotation matrix can be applied. To find the rotated version of a matrix the rotation matrix must be applied on both sides with a negated argument on the right.

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$M_{rotated} = R(\theta) M_{original} R(-\theta)$$

For example, a horizontal polarizer becomes a vertical polarizer if rotated 90 degrees.

$$R\left(\frac{\pi}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R\left(-\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

3.1 Examples: Jones Calculus

Imagine we have input light that is horizontally polarized (along $\hat{x}$). If we multiply it by the matrix for a vertical polarizer we see that it is entirely attenuated.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

However, if we insert a 45-degree polarizer one would expect that we should get some light through, as the input is not orthogonal to the first polarizer, nor is that resulting state perpendicular to the final polarizer.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \boxed{\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

It seems that we get 1/4th of the light through (as must square the field to find the intensity). Now, let us be entirely general. What happens when we rotate a linear polarizer arbitrarily?

$$LP(\theta) = \underbrace{R(\theta) \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}^{\text{Horizontal LP}} R(-\theta)}_{\text{Rotated LP}} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$LP(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} \cos^2(\theta) & \cos(\theta)\sin(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix}}$$

We see that if we put in a horizontally polarized input, we obtain the expected cosine-squared dependence of Malus's Law.

$$LP(\theta) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) \\ \cos(\theta)\sin(\theta) \end{pmatrix}$$

It can easily be seen that Jones calculus permits one to take an arbitrarily large sequence of elements, rotated at arbitrary angles or otherwise, and reduce them to a single 2x2 matrix with predictable action on any expressible input polarization.

4 Mueller Calculus

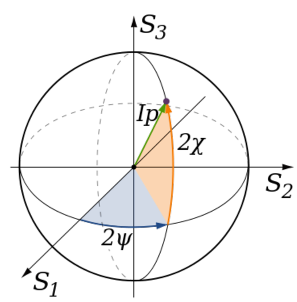
First proposed in 1943, Mueller calculus allows us to analyze incoherent and partially-polarized light. The vector that describes the light is now 4 elements long. While the elements are less intuitive than those of the Jones vector, they are experimentally convenient. Any element of the stokes vector can be determined by taking two measurements with either a linear or circular polarizers. It is important to note that the Mueller approach **neglects phase** and works directly with **intensities**.

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} |E_x|^2 + |E_y|^2 \\ |E_x|^2 - |E_y|^2 \\ 2\Re\left(E_x E_y^*\right) \\ -2\Im\left(E_x E_y^*\right) \end{pmatrix} = \begin{pmatrix} I \\ pI\cos(2\psi)\cos(2\chi)) \\ pI\cos(2\psi)\sin(2\chi) \\ pI\sin(2\chi) \end{pmatrix} \Rightarrow \begin{cases} \text{Intensity of the light} \\ \text{Linear-axial component} \\ \text{Linear 45-degree component} \\ \text{Circular component} \end{cases}$$

The above definition requires understanding of two concepts, the first being **degree of polarization**, denoted p. A state that is only partially polarized may have $p < 1$. We can relate this quantity using the relation below. It can be seen that if the length of the lower 3-element vector is equal to this element then we have perfect polarization, $p = 1$.

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}$$

Secondly we need to introduce the Poincaré sphere (pronounced "Pwahn-car-ay"). This unit sphere exists in a space where the x, y, z coordinate axes are instead S_1, S_2, S_3. The polarization state is represented by a vector with one end rooted at the origin and with a length p. This means that perfectly polarized states lie on the surface of the sphere, while the interior represents partially polarized states. The parameter ψ tracks the azimuth as the state varies between $S_2 = -1$ to $S_2 = +1$ corresponding to ± 45 degree linear while χ tracks the altitude as the state varies from $S_3 = -1$ to $S_3 = +1$ corresponding to variation between left-hand circular and right-hand circular.



The equator contains all linear polarizations, while circular polarizations lie at the two poles. All other states are elliptical polarized. In this illustration one can imagine the action of a half-wave plate to be traversal in the azimuthal direction (moving along the equatorial direction at whichever latitude you start at) while the action of a quarter-wave plate is to alter the altitude

Common polarization states must be expressed differently, but also that we can express unpolarized light:

$$\hat{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{y} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{p}_{\pm 45} = \begin{pmatrix} 1 \\ 0 \\ \pm 1 \\ 0 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \text{unpolarized} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Similarly, optical elements must be represented by significantly different matrices in Mueller calculus.

$$LP_h = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad LP_v = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad LP_{\pm 45} = \begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 \\ \pm 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Quarter-wave and half-wave plates with the fast axis oriented vertically can be expressed similarly:

$$LR_{\lambda/4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad LR_{\lambda/2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$