Continuing the previous ~~elements~~ example

$$\vec{B}\vec{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \vec{A}$$

so for this example $\vec{B}\vec{A} = \vec{A}$, but $\vec{B} \neq \vec{I}$

SUMMARY!
Unlike arithmetic $\vec{A}\vec{B} = \vec{0}$ doesn't mean one of them is $\vec{0}$

$\vec{A}\vec{B} = \vec{A}$ doesn't mean $\vec{B}$ is $\vec{I}$

SOLVING LINEAR EQUATIONS – often we run into cases where we have N equations and N unknowns. Linear algebra provides a systematic method for determining if solutions exists, and if they do, what the solutions are.

EXAMPLE 1

$$x + 2y = 3 \quad (1)$$
$$4x + 5y = 6 \quad (2)$$

Two equations, two unknowns: x, y

The way you probably learned to solve this is something like

STEP 1: multiply both sides of (1) and subtract it from (2)

4 times (1) gives $4x + 8y = 12$

subtracting this from (2) gives $-3y = -6$

STEP 2: Solve for $y \Rightarrow y = 2$

STEP 3: Plug result into (1) and solve for x

$$x + 2(2) = 3 \Rightarrow x = -1$$

STEP 4: For good measure, plug x and y in (2) $\quad 4(-1) + 5(2) = 6$

GAUSSIAN ELIMINATION

We can also write this system of equations in matrix form

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}}_{\text{coefficients}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\text{unknowns}} = \underbrace{\begin{pmatrix} 3 \\ 6 \end{pmatrix}}_{\text{RIGHT HAND SIDES (RHS)}}$$

Let's repeat our steps and see what happens

subtract 4 times row 1 from row 2 gives

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

NOTE: ROW 1 DOESN'T CHANGE

NOTE: MUST DO SAME CALCULATIONS ON RHS

THIS IS CALLED AN UPPER TRIANGULAR MATRIX SINCE EVERYTHING BELOW THE DIAGONAL IS ZERO. OUR GOAL IN SOLVING THESE EQS. IS TO GET THE MATRIX INTO THIS UPPER TRIANGULAR FORM.

BACK SUBSTITUTION - THE REASON we put the matrix in upper triangular form is that now it becomes simple to solve for the unknowns using a process called Back Substitution. If we start at the bottom of the triangle, we know

$$\begin{pmatrix} 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -6$$

$$0 \cdot x - 3 \cdot y = -6 \Rightarrow y = 2$$

Moving up to the next row, we know

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ 2 \end{pmatrix} = 3$$

$$1 \cdot x + 2 \cdot 2 = 3 \Rightarrow x = -1$$

Let's revisit the first step to get to the upper triangular matrix. Intrinsically, we are multiplying both sides of our matrix equation by a 2x2 matrix

$$\begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

We know this is the same as

$$\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

So what are the elements of the unknown matrix?

Well the first row needs to be $[\boxed{1}\ \boxed{0}]$ since the first row is unchanged. The bottom row needs to be $[\boxed{-4}\ \boxed{1}]$ since this is equivalent to subtracting 4 times row1 from row2.

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

This matrix is in Lower Triangular Form since everything above diagonal is zero. It has the effect of subtracting a scaled version of one row from another row

EXAMPLE 2

$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 6 \end{aligned} \implies \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

Let's convert to triangular form

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \end{pmatrix}$$

Now we have a problem with back substitution since

NO SOLUTIONS

$$\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -6 \Rightarrow 0 \cdot x + 0 \cdot y \neq -6$$

Example 3

$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Again we have a problem with back substitution

$$\begin{pmatrix} 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow 0 \cdot x + 0 \cdot y = 0$$

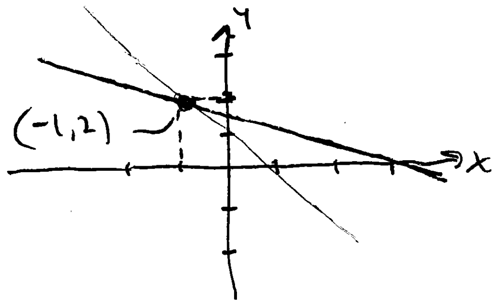
INFINITE # OF SOLUTIONS

x and y can be any value

Let's look at geometrical picture of what is going on

For Example 1 $\quad x+2y=3 \Rightarrow y=-\frac{x}{2}+\frac{3}{2}$

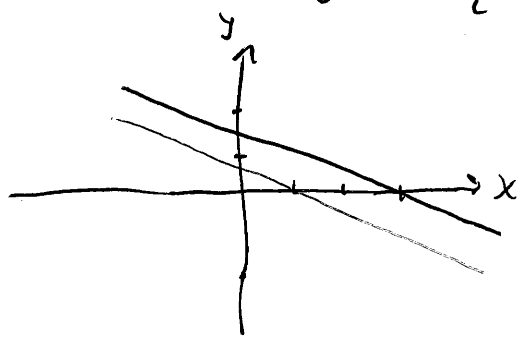
$\qquad\qquad\qquad\quad 4x+5y=6 \Rightarrow y=-\frac{4x}{5}+\frac{6}{5}$



1 point of intersection

For Example 2 $\quad x+2y=3 \Rightarrow y=-\frac{x}{2}+\frac{3}{2}$

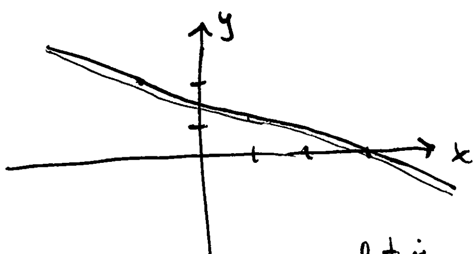
$\qquad\qquad\qquad\quad 4x+8y=6 \Rightarrow y=-\frac{x}{2}+\frac{3}{4}$



parallel lines, no points of intersection

For Example 3 $\quad x+2y=3 \Rightarrow y=-\frac{x}{2}+\frac{3}{2}$

$\qquad\qquad\qquad\quad 4x+8y=12 \Rightarrow y=-\frac{x}{2}+\frac{3}{2}$



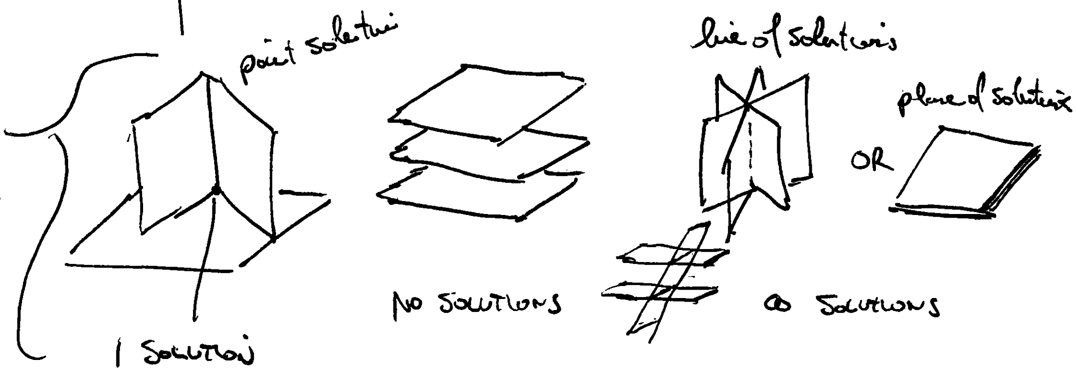
Same line, any point on the line is valid.

For 2x2 we have lines

For 3x3 we have planes

For NxN we have hyperplanes

Difficult to visualize, but same idea of 0, 1, or ∞ solutions

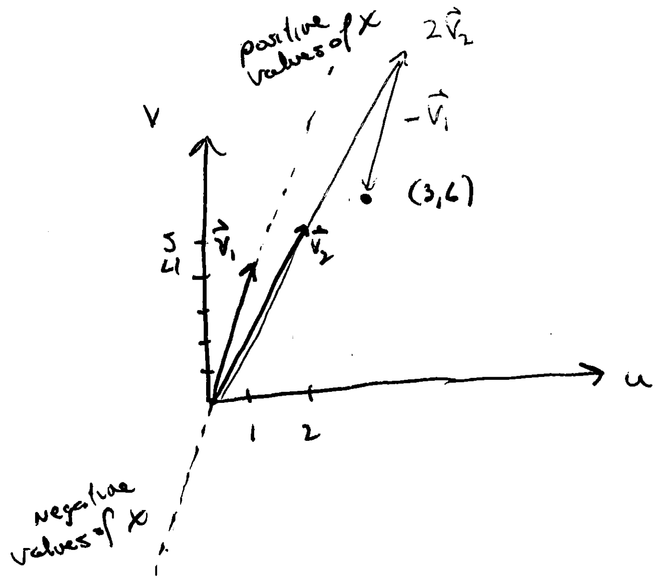


A different geometrical picture can be obtained by rewriting the equations as combinations of column vectors

For <u>Example 1</u>

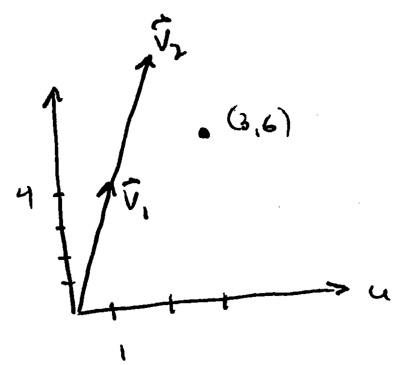
$$\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$x\underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\vec{v}_1} + y\underbrace{\begin{pmatrix} 2 \\ 5 \end{pmatrix}}_{\vec{v}_2} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$



For <u>Example 2</u>

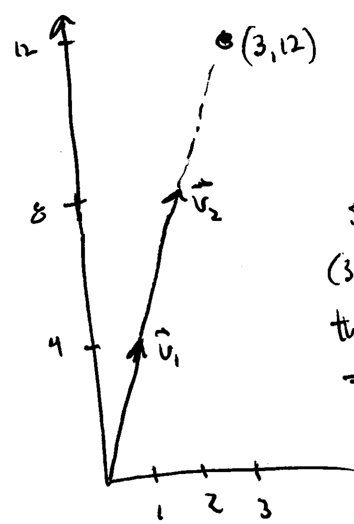
$$x\underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\vec{v}_1} + y\underbrace{\begin{pmatrix} 2 \\ 8 \end{pmatrix}}_{\vec{v}_2} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$



$\vec{v}_1$ and $\vec{v}_2$ point in same direction and no combination can get to (3,6)

For <u>Example 3</u>

$$x\underbrace{\begin{pmatrix} 1 \\ 4 \end{pmatrix}}_{\vec{v}_1} + y\underbrace{\begin{pmatrix} 2 \\ 8 \end{pmatrix}}_{\vec{v}_2} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$$



$\vec{v}_1$ and $\vec{v}_2$ point in same direction, but (3,12) happens to lie along this direction so infinite # of combinations of $\vec{v}_1$ and $\vec{v}_2$ can get there.

Let's do a 3×3 example

$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}$$

SUBTRACT 2 times Row 1 from Row 2

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 9 \end{pmatrix}$$

~~SUBTRACT~~ ADD Row 1 TO Row 3 (SUBTRACT −1 times Row 1)

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 14 \end{pmatrix}$$

GIVES ZEROS IN COLUMN 1 BELOW THE DIAGONAL

ADD Row 2 TO Row 3 (SUBTRACT −1 times Row 2)

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \\ 2 \end{pmatrix}$$

GIVES ZEROS IN COLUMN 2 BELOW THE DIAGONAL

NOW USE BACK SUBSTITUTION TO FIND UNKNOWNS

$$\boxed{w = 2}$$ BOTTOM ROW

$$-8v - 2w = -12$$ MIDDLE ROW

$$-8v - 2(2) = -12$$

$$\boxed{v = 1}$$

$$2u + v + w = 5$$

$$2u + 1 + 2 = 5$$

$$\boxed{u = 1}$$

The values along the diagonal ([2, -8, 1] in this example) are called PIVOTS. If all the PIVOTS are non-zero then a single solution exists. If one or more of the PIVOTS is zero, a solution may not exist. We may need to reorder equations to get all non-zero pivots.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \qquad \begin{array}{l} \text{Row2} - 2\,\text{Row1} \\ \text{Row3} - 4\,\text{Row1} \end{array}$$

REORDER ROWS 2 AND 3 TO GET RID OF "0" PIVOT

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} \leftarrow \text{REMEMBER TO REORDER THIS TOO}$$

NOW USE BACK SUBSTITUTION

$$3w = 0 \Rightarrow \boxed{w = 0}$$

$$2v + 4w = -4 \Rightarrow \boxed{v = -2}$$

$$u + v + w = 1 \Rightarrow \boxed{u = 3}$$

Let's look at one more case where this trick doesn't work

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Problems now because $4w = 0$ and $3w = 2$ NO SOLUTION

Why do we have problems and can we figure out when they occur?

To have a unique solution, each of the rows of the matrix must be linearly independent (i.e. we can't make one of the rows from some weighted combination of the other rows. Similarly, each of the columns needs to be linearly independent. The number of rows that ~~are~~ are not linearly independent will equal the number of colums that are not linearly independent. Let's look at our problem cases

Example 2 for 2x2 case on page 12

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$ row2 is just $4 \times$ row 1
column 2 is just $2 \times$ column 1

Same for Example 3 for ~~3x3~~ 2x2 case on page 13

3x3 example on previous page

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{pmatrix}$$

~~Row 2 equals 2x Row 1~~

column 1 equals column 2

$\frac{4}{3}$ row 1 + $\frac{4}{3}$ row 2 = row 3

How about $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$? DOES THIS HAVE A UNIQUE SOLUTION?

ROW 3 = 2 x ROW 1 + ROW 2 So the answer is <u>NO</u>

COLUMN 1 + COLUMN 3 = 2 x COLUMN 2

THIS ROUTE IS OK FOR SMALL MATRICES, BUT QUICKLY BECOMES NON-OBVIOUS

GAUSSIAN ELIMINATION

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$$

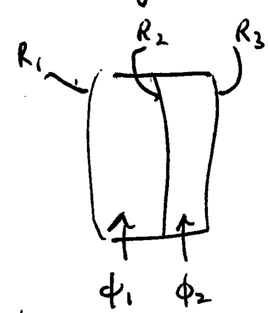
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} \text{ROW2} - 2\,\text{ROW1} \\ \text{ROW3} - 4\,\text{ROW1} \end{array}$$

$$\begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{ROW3} - \text{ROW2}$$

The number of NON-ZERO pivots will tell us the number of independent rows (columns)

Let's do some optics, finally

Suppose we want to create an achromatic doublet (thin). The ~~requirements~~ powers of the two lenses at the F, d, C wavelengths are



1^{st} requirement $\boxed{\phi_{1d} + \phi_{2d} = \phi_d}$

$$\phi_{1F} + \phi_{2F} = \phi_F$$

$$\phi_{1C} + \phi_{2C} = \phi_C$$

We also want powers of ϕ_F and ϕ_C to be the same

2^{nd} requirement $\boxed{\phi_F - \phi_C = 0}$