LINEAR ALGEBRA FOR OPTICS

What is an algebra? Take two members from a set and combine them with an operator to get a new member of the set.

e.g. Consider the set of integers $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$ and the operator "+"

$$a + b = c$$

if a and b are integers and "+" means our usual definition of arithmetic addition, then c is also an integer.

$$4 + 1 = 5$$

Let's look at a different operator "x"

$$a \times b = c$$

if a and b are integers, then c is an integer when "x" denotes arithmetic multiplication.

$$4 \times 3 = 12$$

There are several properties of an algebra that are useful for solving more complex combinations of the set members and the operators.

(1) Identity Elements - combining ~~one~~ a member of the set with with an identity element just gives back the original member.

For addition $a + 0 = a$, 0 is identity element

For multiplication $a \times 1 = a$, 1 is identity element.

(2) Inverse elements - combining a member of the set with the inverse element leads the identity element

For addition $a + (-a) = 0$

For multiplication $a \times (1/a) = a \times a^{-1} = 1$

at first glance, everything appears ok here, but we need to be careful. First, we are using a new operator "−". In basic algebra, "−" means to negate e.g. $-(3) = -3$, $-(-3) = 3$. Second, we also introduced the division operator "/", but this operator doesn't result in a member of the set.

$$a/b = c$$

For a and b in the set of integers, sometimes c is in the set of integers

$$2/2 = 1 \quad \text{or} \quad 8/2 = 4$$

and sometimes c is not in the set of integers

$$2/3 = c \quad \text{or} \quad 8/3 = c$$

So the inverse element doesn't necessarily exist.

What if a and b come from the set of real numbers?

$$2/2 = 1 \quad 8/2 = 4 \quad 2/3 = 0.6\bar{6} \quad 8/3 = 2.6\bar{6}$$

So all of these examples are now OK. However

$2/0 = c$ is undefined or at the very least not finite.

So inverse elements don't necessarily always exist and they are different depending upon what sets you are interested in.

(3) ASSOCIATIVITY - THE grouping of the set members doesn't matter when an operator is used repeatedly.

$$(a + b) + c = a + (b + c)$$
$$(a \times b) \times c = a \times (b \times c)$$
are associative

How about subtraction?

$$(a - b) - c \neq a - (b - c) \quad \text{NOT ASSOCIATIVE!}$$

$$(3-2) - 1 \neq 3 - (2-1)$$

$$0 \neq 2$$

(4) COMMUTATIVITY - The order of operations doesn't matter

$$a + b = b + a$$
$$a \times b = b \times a$$
are commutative

How about division?

$$a/b \neq b/a \quad \text{NOT COMMUTATIVE!}$$

$$2/1 \neq 1/2$$

So, in summary, an algebra is a set of rules for combining two members of a set with an operator. We want to understand the implications of these rules and use properties such as identity elements, inverse elements, associativity and commutativity to solve more complex problems. We also need to know when any of these properties break down so that we are not tempted to use them in these situations.

Linear algebra is an algebra associated with a set containing matrices.

Matrices are two-dimensional arrays of numbers. These numbers can in general be real or complex. We can use a letter as short hand to represent the whole array of numbers.

$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

where a_{ij} are the numbers of elements of the matrix

$\vec{A}$ represents a 3×2 matrix (rows x columns). The subscript "i" on the elements denotes the row number and the subscript "j" on the elements denotes the column number.

The term "vector" is often used for a matrix with either one row or one column.

3×1 matrix

$$\vec{x} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \end{pmatrix}$$

Column Vector

1×3 matrix

$$\vec{b} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \end{pmatrix}$$

Row Vector

By convention capital letters are often used for matrices with more than one row and one column. Lower case letters are used to represent vectors. Often in texts both cases are in bold to distinguish them from scalar variables. Here, we'll put an arrow over the letter $\vec{A}$ and $\vec{x}$ to distinguish the matrix (vector) from scalar values

TRANSPOSE - Matrix transposition is a basic operation where the transpose of a matrix A (denoted by A^T) is a new matrix where the rows of A^T are the columns of A.

3x2 matrix

$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad \vec{A}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{pmatrix}$$

2x3 matrix

e.g.
$$\vec{B} = \begin{pmatrix} 2 & 4 & 6 & 8 \\ 1 & 3 & 5 & 7 \end{pmatrix} \qquad \vec{B}^T = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \\ 8 & 7 \end{pmatrix}$$

2x4 matrix   4x2 matrix

NOTE: $(\vec{B}^T)^T = \vec{B}$

MATRIX ADDITION - The elements of each matrix are added elementwise. ~~Each~~ Matrices must have same dimensions.

$$\vec{A} + \vec{B} = \vec{C}$$

2x2 case

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

e.g.

3x2 case

$$\begin{pmatrix} 3 & 0 \\ 1 & 5 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 1 & 1 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 2 & 6 \\ 0 & 6 \end{pmatrix}$$

ZERO MATRIX - matrix of all zeros

$$\vec{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & - & - & 0 \end{pmatrix}$$

The zero matrix is the identity element for matrix addition.

Scalar Multiplication

If we multiply a matrix by a scalar, each element of the matrix is multiplied by the scalar

Commutes

$$c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} c$$

Matrix Multiplication

Requires that the number of columns of the 1st matrix is equal to the number of rows in the second matrix.

$$\begin{array}{c c c c} & \text{K columns} & \text{m columns} & \text{m columns} \\ \text{N rows} & \begin{pmatrix} & \\ & \end{pmatrix} & \overset{\text{K rows}}{\begin{pmatrix} & \\ & \end{pmatrix}} & = \ {}^{\text{N rows}}\begin{pmatrix} & \\ & \end{pmatrix} \end{array}$$

Multiplying a $N \times k$ matrix by a $k \times m$ matrix gives a $N \times m$ matrix. There are a variety of ways to look at matrix multiplication. The formal definition for $\vec{A}\vec{B}$

$$\vec{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1K} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{N1} & - & - & a_{NK} \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ b_{K1} & - & - & b_{Km} \end{pmatrix}$$

$$\vec{A}\vec{B} = \begin{pmatrix} (AB)_{11} & (AB)_{12} & \cdots & (AB_{1m}) \\ (AB)_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ (AB)_{N1} & \cdots & & (AB)_{Nm} \end{pmatrix}. \qquad \text{where} \quad (AB)_{ij} = \sum_{\ell=1}^{K} a_{i\ell} b_{\ell j}$$

e.g. $\vec{A} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 4 \end{pmatrix}$ $\vec{B} = \begin{pmatrix} 5 & 4 & 1 & 2 \\ 2 & 2 & 0 & 3 \end{pmatrix}$

$\vec{A}$ is a 3×2 matrix and $\vec{B}$ is a 2×4 matrix. We expect the result to be a 3×4 matrix. Based on the definition of matrix multiplication:

$$\vec{A}\vec{B} = \begin{pmatrix} 16 & 14 & 2 & 13 \\ 7 & 6 & 1 & 5 \\ 8 & 8 & 0 & 12 \end{pmatrix} = \vec{C}$$

You probably learned to do this multiplication as something like the following

$$\underset{\vec{A}}{\begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 4 \end{pmatrix}} \underset{\vec{B}}{\begin{pmatrix} 5 & 4 & 1 & 2 \\ 2 & 2 & 0 & 3 \end{pmatrix}} = \underset{\vec{C}}{\begin{pmatrix} 16 & 14 & 2 & 13 \\ 7 & 6 & 1 & 5 \\ 8 & 8 & 0 & 12 \end{pmatrix}}$$

The first row of $\vec{A}$ as a row vector multiplied by the first column of ~~the~~ $\vec{B}$ as a column vector gives the element c_{11}.

$$\begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} = 2 \cdot 5 + 3 \cdot 2 = 16$$

In general, the i^{th} row of $\vec{A}$ multiplied by the j^{th} column of $\vec{B}$ gives c_{ij}

$$\begin{pmatrix} 0 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 8$$

Let's look at another way of multiplying when we have a matrix times a vector

$$\vec{A}\,\vec{x} = \vec{b}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{pmatrix}$$ From the definition

Alternatively, we can write this as

THESE ARE WEIGHTS ON EACH OF THE COLUMN VECTORS

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

EQUALS

Scalar multiply x's then add three vectors

THESE ARE THE COLUMNS OF $\hat{A}$

The form $\vec{A}\,\vec{x} = \vec{b}$ will arise repeatedly. This interpretation of matrix multiplication is useful for visualizing the meaning of the mathematics.

In general, matrix multiplication does not commute

$$\vec{A}\,\vec{B} \neq \vec{B}\,\vec{A}$$

From the earlier example

$$\begin{pmatrix} 5 & 4 & 1 & 2 \\ 2 & 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 4 \end{pmatrix}$$

CAN'T EVEN CALCULATE SINCE # of colums of the 1^{st} matrix is not equal to the # of rows in the second matrix.

Ok, what if we take two different matrices that we can multiply both ways

$$\vec{C} = \begin{pmatrix} 5 & 4 & 1 \\ 2 & 2 & 0 \end{pmatrix} \qquad \vec{D} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \\ 0 & 4 \end{pmatrix}$$

$$\vec{C}\vec{D} = \begin{pmatrix} 14 & 23 \\ 6 & 8 \end{pmatrix} \qquad \vec{D}\vec{C} = \begin{pmatrix} 16 & 14 & 2 \\ 7 & 6 & 1 \\ 8 & 8 & 0 \end{pmatrix}$$

NOT EVEN SAME DIMENSIONS, LET ALONE EQUAL

CONSIDER ANOTHER EXAMPLE

$$\vec{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \vec{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\vec{A}\vec{B} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \vec{0}$$

ZERO MATRIX

THIS RESULTS IN ZERO MATRIX EVEN THOUGH $\vec{A} \neq \vec{0}$ AND $\vec{B} \neq \vec{0}$

IDENTITY MATRIX

$$\vec{I}_n = \begin{array}{cc} & \text{n columns} \\ \text{n rows} & \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \end{array}$$

ALL ONES ALONG DIAGONAL AND ZEROES OTHERWISE

SUBSCRIPT OFTEN DROPPED

SATISFIES $\vec{I}\vec{A} = \vec{A}\vec{I} = \vec{A}$

$\vec{I}$ is the identity element for matrix multiplication.

Continuing the previous ~~[illegible]~~ example

$$\vec{B}\vec{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \vec{A}$$

so for this example $\vec{B}\vec{A} = \vec{A}$, but $\vec{B} \neq \vec{I}$

SUMMARY:
unlike arithmetic
- $\vec{A}\vec{B} = \vec{0}$ doesn't mean one of them is $\vec{0}$
- $\vec{A}\vec{B} = \vec{A}$ doesn't mean $\vec{B}$ is $\vec{I}$

SOLVING LINEAR EQUATIONS - often we run into cases where we have N equations and N unknowns. Linear algebra provides a systematic method for determining if solutions exist, and if they do, what the solutions are.

EXAMPLE 1

$$x + 2y = 3 \quad (1)$$
$$4x + 5y = 6 \quad (2)$$

Two equations, two unknowns: x, y

The way you probably learned to solve this is something like

STEP 1: multiply both sides of (1) and subtract it from (2)

4 times (1) gives $4x + 8y = 12$

subtracting this from (2) gives $-3y = -6$

STEP 2: Solve for $y \Rightarrow y = 2$

STEP 3: Plug result into (1) and solve for x

$$x + 2(2) = 3 \Rightarrow x = -1$$

STEP 4: For good measure, plug x and y in (2) $\quad 4(-1) + 5(2) = 6$