Forbes Q Polynomials

\[ 2(r) = \frac{r^2/R}{1 + \sqrt{1 - \frac{r^2}{R^2}}} + \frac{1}{\sqrt{1 - \frac{r^2}{R^2}}} \left(\frac{r^2}{R^2}\right) \left(1 - \frac{r^2}{r_{\text{max}}^2}\right) \sum_{n=0}^{M} a_n Q_n^{bfs} (p^2) \]

- \[ p = \frac{r}{r_{\text{max}}} \]
- \[ R \] is the radius of the actual object
- \[ r_{\text{max}} \] is the radius of the best-fit sphere
- \[ Q_n^{bfs} (p^2) \] are set of basis polynomials which will be defined below
- \[ a_n \] are expansion coefficients

\[ \left(\frac{r^2}{R^2}\right) \left(1 - \frac{r^2}{r_{\text{max}}^2}\right) \] ensures sum = 0 when \( r=0 \) and \( r = r_{\text{max}} \)

\( bfs \): "Best Fit Sphere"  
- sphere that matches our cut edge

\[ \Delta = \text{sag difference between asphere and sphere} \]

\[ \Delta = \frac{1}{\sqrt{1 - \frac{r^2}{R^2}}} \left(\frac{r}{R}\right) (1 - p^2) \sum a_n Q_n^{bfs} (p^2) \]

\( \Delta \cos \theta \) = difference between asphere and sphere along normal to sphere

- slope of sphere: \[ \frac{\partial}{\partial \eta} \left(\frac{r^2}{1 + \sqrt{1 - \frac{r^2}{R^2}}}\right) = \frac{\partial}{\partial r} \left(r - \sqrt{r^2 - r^2}\right) \]

\[ = -\frac{1}{2} \frac{-2r}{\sqrt{R^2 - r^2}} = \frac{r}{R^2 - r^2} = \tan \theta \]

\[ \Delta \cos \theta = \frac{1}{\sqrt{R^2 - r^2}} \]

\[ \cos \theta = \frac{r}{R} \]
\[ \Delta \cos \theta = \sum p^2 (1 - p^2) \sum q_{\nu} Q_{\nu}^{\text{bfs}} (p^2) \]

so the \( Q_{\nu}^{\text{bfs}} \) describes the difference between the sphere and the normal to the sphere.

**Definition of \( Q_{\nu}^{\text{bfs}} (x) \)**

**Iterative technique used**

**Constants**

\[ f_0 = 2, \quad f_1 = \frac{\sqrt{19}}{2}, \quad g_0 = -\frac{1}{2} \]

\[ \begin{align*}
  f_m & \quad m \geq 2 \\
  h_{m-1} & = \frac{-m (m-1)}{2 f_{m-2}} \\
  g_{m-1} & = \frac{-(1 + g_{m-2} h_{m-2})}{f_{m-1}} \\
  f_m & = \sqrt{m (m+1) + 3 - g_{m-1}^2 - h_{m-2}^2}
\end{align*} \]

**Functions**

\[ \begin{align*}
  P_0 (x) & = 2 \\
  P_1 (x) & = 6 - 8x \\
  P_{n+1} (x) & = (2 - 4x) P_n (x) - P_{n-1} (x)
\end{align*} \]

\[ \begin{align*}
  Q_{n+1}^{\text{bfs}} (x) & = \left[ P_{n+1} (x) - g_m Q_{n}^{\text{bfs}} (x) - h_{n-1} Q_{n-1}^{\text{bfs}} (x) \right] \\
  & \quad f_{n+1}
\end{align*} \]
Example: Calculate $Q_2(x)$

For $Q_2(x) \Rightarrow m+1 = 2 \Rightarrow m = 1$

Need to know $p_2(x), g_1, Q_1, Q_0(x)$

\[
p_2(x) = (2-4x)p_0(x) - p_0(x)
\]

\[
= (2-4x)(6-8x) - 2
\]

\[
= 12 - 40x + \frac{32}{x^2} - 2
\]

\[
p_2(x) = \frac{32}{x^2} - 40x + 10
\]

\[
g_1 = -\left(1 + g_0 h_0 \right) \frac{h_0}{p_0}
\]

\[
h_0 = \frac{-2(2-1)}{2f_0} = -\frac{1}{2}
\]

\[
h_0 = \frac{-2}{2f_0} = -\frac{1}{2}
\]

\[
f_0 = \sqrt{2(3) + 3 - \left(-\frac{5}{2\sqrt{19}}\right)^2 - \left(-\frac{1}{2}\right)^2}
\]

\[
f_0 = \sqrt{9 - \frac{25}{4\cdot19} - \frac{1}{4}}
\]

\[
f_1 = \sqrt{9 \cdot 4\cdot19 - \frac{25}{4\cdot19} - \frac{1}{4}}
\]

\[
f_1 = \sqrt{640 - \frac{25}{4\cdot19} - \frac{19}{4\cdot19}}
\]

\[
f_2 = \sqrt{\frac{640}{76}} = \sqrt{160}
\]
\[ Q_{x}^{bf} (x) = \sqrt{ \frac{19}{160} \left[ \frac{315}{119} x^2 - 90 x + 10 + \frac{5}{119} \frac{1}{119} (13-16x) + \frac{1}{2} \right] } \]

\[ Q_{z}^{bf} (x) = \sqrt{ \frac{38}{5} x^2 - 20 \sqrt{ \frac{10}{19} x + 29 \frac{1}{119} } } \]

**Slope Difference Along Normal**

\[ \frac{d}{dp} [\Delta \cos \Theta] = \frac{d}{dp} \left[ (1 - p^2) \sum a_n Q_{2n}^{bf} (p^2) \right] = S_n (p) \]

\( S_n (p) \) is the difference in surface slope between the "best-fit" sphere and the asphere along the normal to the sphere.

The \( \{ S_n (p) \} \) satisfy

\[ \int_0^1 S_n (p) S_m (p) \frac{1}{\sqrt{1-p^2}} dp = \frac{\pi}{2} \delta_{nm} \]

The \( \{ S_n (p) \} \) are orthogonal. The \( \{ Q_{2n}^{bf} (p^2) \} \) were specifically chosen so that the \( \{ S_n (p) \} \) would satisfy the above orthogonality requirement.

Recall for Zernike Radial Polynomials, we had

\[ \int_0^1 k_n (p) R_n^m (p) p dp = \frac{1}{2n+2} \delta_{mn} \]

This \( p \) is called a weighting function.

For the Zernike Radial polynomials, the weighting function ensures equal areas of the pupil contribute equally to the function shape.
For the Forbes $Q$ polynomials the weighting function is $\frac{1}{1 - p^2}$.

This choice of weighting function tends to minimize the maximum slope that appears in $\{S_n(p)\}$.

**Mean Square Slope**

$$\left( \int_0^1 \left[ \frac{1}{r_{\text{max}}} \sum_{m=0}^M a_m S_m(p) \right]^2 \frac{1}{\sqrt{1 - p^2}} \, dp \right)^{1/2}$$

$$\frac{1}{r_{\text{max}}} \sum_{m=0}^M a_m^2 \quad \text{METRIC OF SLOPE DEPARTURE FROM A SPHERE}$$

This will be important when we talk about optical testing.

**Fitting $Q$ Polynomials**

Suppose we have a spherical surface $\Omega$ with $\partial B R^2$.

First, find radius $R$ of “best-fit” sphere which is sphere with same origin and endpoints as $Z(\Omega)$.

$$Z(R_{\text{max}}) = R - \sqrt{R^2 - r_{\text{max}}^2} \quad \Rightarrow \quad R = \sqrt{r_{\text{max}}^2 + 2Z(r_{\text{max}})}$$

Second, arrange seg equation

$$Z(R) = \frac{r^2 / R}{1 + \sqrt{1 - r^2 / R^2}} + \frac{1}{\sqrt{1 - r^2 / R^2}} \left( \frac{r^2}{R^2} \right) \left( 1 - \frac{r^2}{R^2} \right) \sum_{m=0}^M a_m Q_m^{\text{best}}(p^2)$$

Replace $r \rightarrow r_{\text{max}}$ to have everything in normalized coordinates:

$$Z(p) : \frac{r_{\text{max}}^2 e^2 / R^2}{1 + \sqrt{1 - r_{\text{max}}^2 e^2 / R^2}} + \frac{1}{\sqrt{1 - r_{\text{max}}^2 e^2 / R^2}} \rho^2 (1 - p^2) \sum_{m=0}^M a_m Q_m^{\text{best}}(p^2)$$
\[ F(p_i) = \sqrt{1 - \frac{\max P_i}{R^2}} \left[ \sqrt{\frac{\nu_i}{P_i}} - \frac{\nu_i P_i^2/R}{1 + \sqrt{1 - \frac{\max P_i}{R^2}}} \right] \]

Third, write as a matrix equation assuming a set of points \( x_i, i = 1, N \):

\[
\begin{bmatrix}
    P_1(x_1) Q_0(x_1) & P_1(x_1) Q_1(x_1) & \cdots & P_1(x_1) Q_M(x_1)
    \\
    \vdots & \vdots & \ddots & \vdots
    \\
    P_N(x_N) Q_0(x_N) & P_N(x_N) Q_1(x_N) & \cdots & P_N(x_N) Q_M(x_N)
\end{bmatrix}
\begin{bmatrix}
    a_0 \\
    a_1 \\
    \vdots \\
    a_M
\end{bmatrix} =
\begin{bmatrix}
    F(x_1) \\
    F(x_2) \\
    \vdots \\
    F(x_N)
\end{bmatrix}
\]

where
\[ F(x_i) = \sqrt{1 - \frac{\max P_i}{R^2}} \left[ \sqrt{\frac{\nu_i}{P_i}} - \frac{\nu_i P_i^2/R}{1 + \sqrt{1 - \frac{\max P_i}{R^2}}} \right] \]

Exclude points where \( P = 0 \) and \( p_i = 0 \) as these will make the matrix singular!

\[ Q_{\text{cone}}(p_i) = \text{Similar to } Q_{\text{sph}}(p_i) \text{ but base surface is a cone instead of a sphere} \]

Can also define over an annular region.