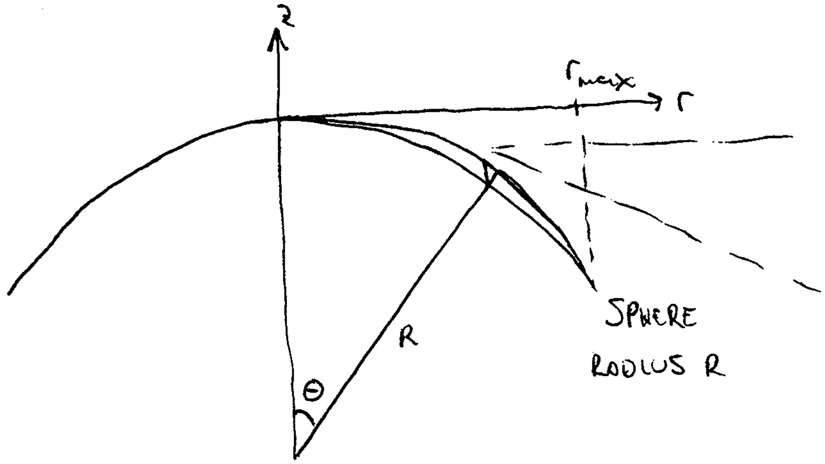
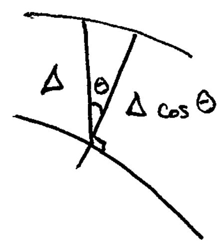
FORBES Q POLYNOMIALS





$$z(r) = \frac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r^2}{R^2}}}\left(\frac{r^2}{r_{max}^2}\right)\left(1-\frac{r^2}{r_{max}^2}\right)\sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

$\rho = \dfrac{r}{r_{max}}$ sag of a sphere additional sag of asphere

$Q_m^{bfs}(\rho^2)$ are set of ~~other~~ polynomials which will be defined below

a_m are expansion coefficients

$\left(\frac{r^2}{r_{max}^2}\right)\left(1-\frac{r^2}{r_{max}^2}\right)$ ensures sum = 0 when $r=0$ and $r = \pm r_{max}$

bfs = "Best Fit Sphere" := sphere that matches origin and edge

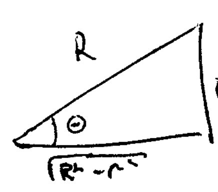
Δ = sag difference between asphere and sphere

$$\Delta = \frac{1}{\sqrt{1-\frac{r^2}{R^2}}}(\rho^2)(1-\rho^2)\sum a_m Q_m^{bfs}(\rho^2)$$

$\Delta\cos\theta$ = difference between asphere and sphere along normal to sphere

slope of sphere $\dfrac{d}{dr}\left(\dfrac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}}\right) = \dfrac{d}{dr}\left(R-\sqrt{R^2-r^2}\right)$

$$= -\frac{1}{\cancel{2}}\frac{-\cancel{2}r}{\sqrt{R^2-r^2}} = \frac{r}{\sqrt{R^2-r^2}} = \tan\theta$$



$\cos\theta = \dfrac{\sqrt{R^2-r^2}}{R}$

$\cos\theta = \sqrt{1-\frac{r^2}{R^2}}$

$$\Delta \cos\Theta = \cancel{\#\#}\ \rho^2(1-\rho^2)\sum a_m Q_m^{bfs}(\rho^2)$$

so the $\{Q_m^{bfs}\}$ describes the difference between the asphere and sphere along the normal to the sphere

Definition of $Q_m^{bfs}(x)$

Iterative technique used

Constants

$$f_0 = 2 \qquad f_1 = \frac{\sqrt{19}}{2} \qquad g_0 = -\frac{1}{2}$$

for $m \geq 2$

$$h_{m-2} = \frac{-m(m-1)}{2 f_{m-2}}$$

$$g_{m-1} = \frac{-(1 + g_{m-2} h_{m-2})}{f_{m-1}}$$

$$f_m = \sqrt{m(m+1) + 3 - g_{m-1}^2 - h_{m-2}^2}$$

Functions

$$\left.\begin{aligned} P_0(x) &= 2 \\ P_1(x) &= 6 - 8x \\ P_{m+1}(x) &= (2-4x)P_m(x) - P_{m-1}(x) \end{aligned}\right\} \text{Scaled Jacobi Polynomials}$$

$$Q_0^{bfs}(x) = 1$$

$$Q_1^{bfs}(x) = \frac{1}{\sqrt{19}}(13 - 16x)$$

$$Q_{m+1}^{bfs}(x) = \frac{\left[P_{m+1}(x) - g_m Q_m^{bfs}(x) - h_{m-1} Q_{m-1}^{bfs}(x)\right]}{f_{m+1}}$$

EXAMPLE: CALCULATE $Q_2^{bfs}(x)$

For $Q_2^{bfs}(x) \Rightarrow m+1 = 2 \Rightarrow m = 1$

Need to know $P_2(x)$, g_1, $Q_1^{bfs}(x)$, h_0, $Q_0(x)$

$$P_2(x) = (2-4x)\,P_1(x) - P_0(x)$$

$$= (2-4x)(6-8x) - 2$$

$$= 12 - 40x + 32x^2 - 2$$

$$\boxed{P_2(x) = 32x^2 - 40x + 10}$$

$$g_1 = \frac{-(1 + g_0 h_0)}{f_1}$$

$$\boxed{h_0 = \frac{-2(2-1)}{2f_0} = \frac{-1}{2}}$$

$$g_1 = \frac{-\left(1 + \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\right)}{\frac{\sqrt{19}}{2}}$$

$$\boxed{g_1 = -\frac{5}{4}\,\frac{2}{\sqrt{19}} = \frac{-5}{2\sqrt{19}}}$$

$$f_2 = \sqrt{2(3) + 3 - \left(\frac{-5}{2\sqrt{19}}\right)^2 - \left(\frac{-1}{2}\right)^2}$$

$$f_2 = \sqrt{9 - \frac{25}{4\cdot 19} - \frac{1}{4}}$$

$$f_2 = \sqrt{\frac{9\cdot 4\cdot 19}{4\cdot 19} - \frac{25}{4\cdot 19} - \frac{19}{4\cdot 19}}$$

$$\boxed{f_2 = \sqrt{\frac{640}{76}} = \sqrt{\frac{160}{19}}}$$

~~$Q_2^{bfs}(x)$~~

$$Q_2^{bfs}(x) = \sqrt{\frac{19}{160}} \left[32x^2 - 40x + 10 + \frac{5}{2\sqrt{19}} \frac{1}{\sqrt{19}}(13-16x) + \frac{1}{2} \right]$$

$$\boxed{Q_2^{bfs}(x) = 4\sqrt{\frac{38}{5}}\, x^2 - 20\sqrt{\frac{10}{19}}\, x + 29\sqrt{\frac{2}{95}}}$$

Slope Difference Along Normal

$$\frac{d}{d\rho}\left[\Delta \cos\Theta\right] = \frac{d}{d\rho}\left[\rho^2(1-\rho^2)\sum a_m Q_m^{bfs}(\rho^2)\right] = S_m(\rho)$$

$S_m(\rho)$ is the difference in surface slope between the "best-fit" sphere and the asphere along the normal to the sphere.

The $\{S_m(\rho)\}$ satisfy

$$\int_0^1 S_m(\rho)\, S_n(\rho) \frac{1}{\sqrt{1-\rho^2}}\, d\rho = \frac{\pi}{2}\delta_{mn}$$

The $\{S_m(\rho)\}$ are orthogonal! The $\{Q_m^{bfs}(\rho^2)\}$ were specifically chosen so that the $\{S_m(\rho)\}$ would satisfy the above orthogonality requirement.

Recall for Zernike Radial Polynomials we had

$$\int_0^1 R_N^n(\rho)\, R_{N'}^{n'}(\rho)\, \rho\, d\rho = \frac{1}{2N+2}\delta_{NN'}$$

This ρ is called a weighting function

For the Zernike Radial polynomials the weighting function ensures equal areas of the pupil contribute equally to the function shape.

For the Forbes Q polynomials the weighting function is $\frac{1}{\sqrt{1-\rho^2}}$.
This choice of weighting function tends to minimize the maximum slope that appears in $\{S_n(\rho)\}$.

Mean Square Slope

$$\left[\int_0^1 \left[\frac{1}{r_{max}} \sum_{m=0}^{M} a_m S_m(\rho) \right]^2 \frac{1}{\sqrt{1-\rho^2}} \, d\rho \right]$$

$$= \frac{1}{r_{max}^2} \sum_m a_m^2$$ METRIC OF SLOPE DEPARTURE FROM A SPHERE

This will be important when we talk about optical testing

Fitting Q Polynomials

Suppose we have aspheric surface $z(r)$ with $z(0) = 0$

First, find radius R of "Best-fit" sphere which is sphere with same origin and endpoints as $z(r)$

$$z(r_{max}) = R - \sqrt{R^2 - r_{max}^2} \Rightarrow R = \frac{r_{max}^2 + z^2(r_{max})}{2z(r_{max})}$$

Second, rearrange sag equation

$$z(r) = \frac{r^2/R}{1+\sqrt{1-\frac{r^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r^2}{R^2}}} \left(\frac{r^2}{r_{max}^2}\right)\left(1 - \frac{r^2}{r_{max}^2}\right) \sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

replace $r \Rightarrow r_{max}\rho$ to have everything in normalized coordinates

$$z(\rho) = \frac{r_{max}^2 \rho^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}} + \frac{1}{\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}} \rho^2(1-\rho^2) \sum_{m=0}^{M} a_m Q_m^{bfs}(\rho^2)$$

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$$\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}\left[\cancel{z}(\rho)-\frac{r_{max}^2\rho^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho^2}{R^2}}}\right]=\sum_{m=0}^{M} a_m\,\rho^2(1-\rho^2)\,Q_m^{bfs}(\rho^2)$$

Third, write as a matrix equation assuming a set of points $\{\rho_i\}$ $i=1,N$

$$\begin{pmatrix} \rho_1^2(1-\rho_1^2)Q_0^{bfs}(\rho_1^2) & \rho_1^2(1-\rho_1^2)Q_1^{bfs}(\rho_1^2) & \cdots & \rho_1^2(1-\rho_1^2)Q_M^{bfs}(\rho_1^2) \\ \vdots & & & \vdots \\ \rho_N^2(1-\rho_N^2)Q_0^{bfs}(\rho_N^2) & \cdots & \cdots & \rho_N^2(1-\rho_N^2)Q_M^{bfs}(\rho_N^2) \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{pmatrix} = \begin{pmatrix} F(\rho_1) \\ F(\rho_2) \\ \vdots \\ F(\rho_N) \end{pmatrix}$$

where $F(\rho_i)=\sqrt{1-\frac{r_{max}^2\rho_i^2}{R^2}}\left[\cancel{z}(\rho_i)-\frac{r_{max}^2\rho_i^2/R}{1+\sqrt{1-\frac{r_{max}^2\rho_i^2}{R^2}}}\right]$

Exclude points where $\rho=0$ and $\rho=1$ as these will make matrices singular!

$\underline{Q}\,\underline{a}=\underline{F}$ Matrix equation

$$\boxed{a=\left[Q^T Q\right]^{-1}Q^T F}$$ Least Squares fit

Related Sets

$Q_m^{con}(\rho^2)$ - Similar to $Q_m^{bfs}(\rho^2)$ but base surface is a conic instead of a sphere

Can also define over ~~a~~ an annular region