Zernike Polynomials

- Application of Zernike polynomials has been used to represent both wavefront shape and corneal topography in the eye.
- Would like to recover basic shape information such as radius of curvature, astigmatism and asphericity based on Zernike coefficients.
- For wavefronts, radius of curvature and astigmatism is related to refractive error, and asphericity is related to spherical aberration.
- For corneal topography, radius of curvature and astigmatism is related to keratometry and asphericity is related to corneal eccentricity.

Radii and Astigmatic Axis

Consider the first six terms of a Zernike expansion of a surface, with $\rho = r / r_{max}$

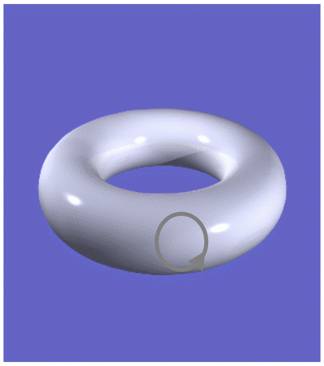
$$f(\rho,\theta) = a_{00} + 2a_{11}\rho\cos\theta + 2a_{1-1}\rho\sin\theta + \sqrt{6}a_{2-2}\rho^2\sin 2\theta + \sqrt{3}a_{20}\left(2\rho^2 - 1\right) + \sqrt{6}a_{22}\rho^2\cos 2\theta$$

If the z axis is perpendicular to the surface at the origin, then a_{11} and a_{1-1} are both zero. We can also switch from normalized coordinates to regular coordinates such that

$$f(r,\theta) = a_{00} + \sqrt{6}a_{2-2}\frac{r^2}{r_{max}^2}\sin 2\theta + \sqrt{3}a_{20}\left(2\frac{r^2}{r_{max}^2} - 1\right) + \sqrt{6}a_{22}\frac{r^2}{r_{max}^2}\cos 2\theta$$

Axis of Astigmatism

The height of an astigmatic surface will oscillate up and down as it is circumnavigated. The extrema will be along the principal meridia.



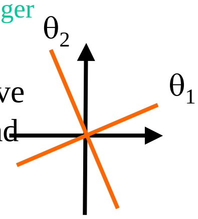
$$\frac{df}{d\theta}(r,\theta) = 2\sqrt{6}a_{2-2}\frac{r^2}{r_{max}^2}\cos 2\theta - 2\sqrt{6}a_{22}\frac{r^2}{r_{max}^2}\sin 2\theta = 0$$

$$a_{2-2}\cos 2\theta = a_{22}\sin 2\theta$$

$$\tan 2\theta = \frac{a_{2-2}}{a_{22}}$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{a_{2-2}}{a_{22}}\right) + \frac{m\pi}{2} \qquad m \text{ integer}$$

Two values of m will give θs that lie between 0° and 180°



Radii of Curvature

Along θ_1

$$f(r,\theta_1) = a_{00} + \sqrt{6}a_{2-2}\frac{r^2}{r_{max}^2}\sin 2\theta_1 + \sqrt{3}a_{20}\left(2\frac{r^2}{r_{max}^2} - 1\right) + \sqrt{6}a_{22}\frac{r^2}{r_{max}^2}\cos 2\theta_1$$

$$f(r,\theta_1) = \underbrace{a_{00} - \sqrt{3}a_{20}}_{\text{Constant Offset}} + \underbrace{r^2\left[\frac{\sqrt{6}}{r_{max}^2}\left(a_{2-2}\sin 2\theta_1 + a_{22}\cos 2\theta_1\right) + \frac{2\sqrt{3}}{r_{max}^2}a_{20}\right]}_{\text{Parabola } \frac{r^2}{2R_1}}$$

$$R_1 = \frac{r_{max}^2}{2}\left[\sqrt{6}\left(a_{2-2}\sin 2\theta_1 + a_{22}\cos 2\theta_1\right) + 2\sqrt{3}a_{20}\right]^{-1}$$

A similar expression holds for θ_2

Average Conic Constant

Equation of a conic

$$z = \frac{1}{K+1}\left[R - \sqrt{R^2 - (K+1)r^2}\right] \cong \left[\frac{r^2}{2R} + \frac{(K+1)r^4}{8R^3} + \ldots\right]$$

Find expansion terms that go as ρ^4

$$K = \frac{8R^3}{r_{max}^4}\left[6\sqrt{5}a_{40} - 30\sqrt{7}a_{60} + 270a_{80} + \cdots\right] - 1$$

Spherical Aberration

$$W(\rho, \theta) = a_{40}\sqrt{5}\left(6\rho^4 - 6\rho^2 + 1\right)$$

$$d\phi = \frac{-1000}{r}\frac{dW(r,\theta)}{dr} \quad \text{(in Diopters for W in mm)}$$

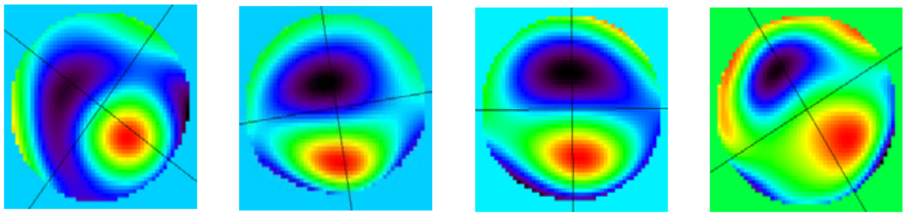
$$W(r, \theta) = a_{40}\sqrt{5}\left(6(r/r_{max})^4 - 6(r/r_{max})^2 + 1\right)$$

$$d\phi = \frac{-1000}{r}a_{40}\sqrt{5}\left(24\frac{r^3}{r_{max}^4} - 12\frac{r}{r_{max}^2}\right)$$

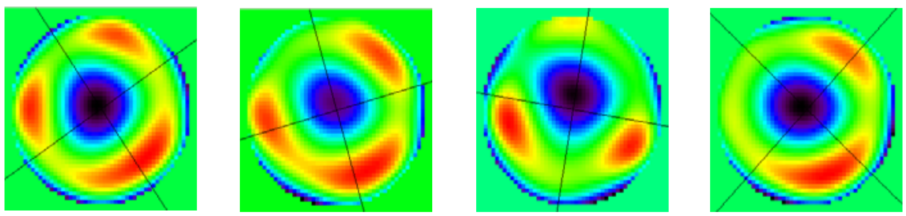
$$d\phi = \frac{-24000a_{40}\sqrt{5}}{r_{max}^4}r^2 + \frac{12000a_{40}\sqrt{5}}{r_{max}^2}$$

Keratoconus Detection

Keratoconics

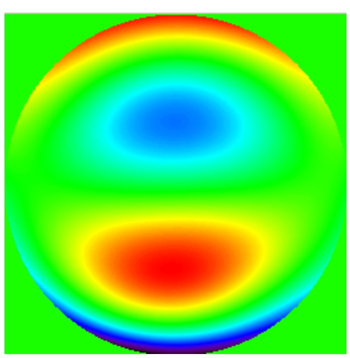


Normals

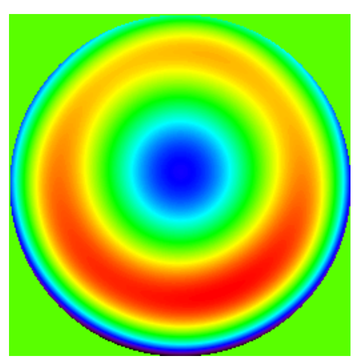


Keratoconus Detection

Keratoconus Feature



Normal Feature



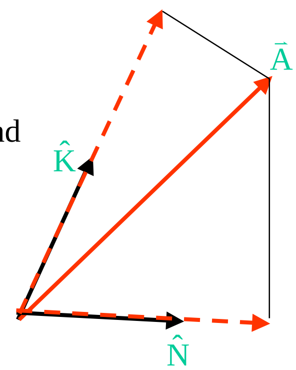
Keratoconus Detection

$\hat{K}$ = unit vector of average higher order coeffcients of Keratoconus Patients

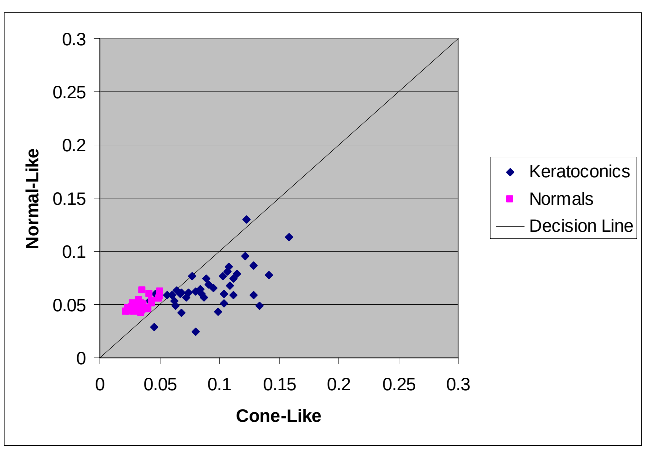
$\hat{N}$ = unit vector of average higher order coeffcients of Normal Patients

$\vec{A} = \langle a_{3,-3}, a_{3,-1}, a_{3,1}, a_{3,3}, a_{4,-4}, \ldots \rangle$

The dot product of the feature vectors and the vector under test gives a measure of how much the test vector looks like the feature vector.



Keratoconus Detection



35/40 (87%) Cones Correctly Classified

40/40 (100%) Normals Correctly Classified

Misclassified Cones