Chebyshev Polynomials - Good for square or rectangular pupils

In one dimension

$$T_N(\rho_x) = \cos(N\cos^{-1}(\rho_x)) \qquad N = 0..\infty \qquad -1 \le \rho_x \le 1$$

They satisfy the following recurrence relationship

$$T_0(\rho_x) = 1$$

Liu2011-3.pdf

$$T_1(\rho_x) = \rho_x$$

$$T_{N+1}(\rho_x) = 2\rho_x T_N(\rho_x) - T_{N-1}(\rho_x)$$

0	1
1	ρx
2	$2\,\rho x^2 - 1$
3	$4\,\rho x^3 - 3\,\rho x$
4	$8\,\rho x^4 - 8\,\rho x^2 + 1$
5	$16\,\rho x^5 - 20\,\rho x^3 + 5\,\rho x$
6	$32\,\rho x^6 - 48\,\rho x^4 + 18\,\rho x^2 - 1$
7	$64\,\rho x^7 - 112\,\rho x^5 + 56\,\rho x^3 - 7\,\rho x$
8	$128\,\rho x^8 - 256\,\rho x^6 + 160\,\rho x^4 - 32\,\rho x^2 + 1$
9	$256\,\rho x^9 - 576\,\rho x^7 + 432\,\rho x^5 - 120\,\rho x^3 + 9\,\rho x$

IN 2D

$$T_{NM}(\rho_x, \rho_y) = T_N(\rho_x)\,T_M(\rho_y)$$

JUST PRODUCT OF 1D FUNCTIONS

Orthogonality

$$\int_{-1}^{1}\int_{-1}^{1} T_{NM}(\rho_x,\rho_y)\,T_{N'M'}(\rho_x,\rho_y)\,\frac{d\rho_x\,d\rho_y}{\sqrt{1-\rho_x^2}\,\sqrt{1-\rho_y^2}} = \begin{cases} (1+\delta_{N0})(1+\delta_{M0})\dfrac{\pi^2}{4} & N=N',\ M=M' \\ 0 & \text{otherwise} \end{cases}$$

Table of 2D CHEBYSHEV POLYNOMIALS

j	N	m	$T_{Nm}(\rho_x, \rho_y)$
0	0	0	1
1	1	0	ρx
2	0	1	ρy
3	2	0	$2\rho x^2 - 1$
4	1	1	$\rho x \rho y$
5	0	2	$2\rho y^2 - 1$
6	3	0	$4\rho x^3 - 3\rho x$
7	2	1	$2\rho x^2 \rho y - \rho y$
8	1	2	$2\rho x \rho y^2 - \rho x$
9	0	3	$4\rho y^3 - 3\rho y$
10	4	0	$8\rho x^4 - 8\rho x^2 + 1$
11	3	1	$4\rho x^3 \rho y - 3\rho x \rho y$
12	2	2	$4\rho x^2 \rho y^2 - 2\rho x^2 - 2\rho y^2 + 1$
13	1	3	$4\rho x \rho y^3 - 3\rho x \rho y$
14	0	4	$8\rho y^4 - 8\rho y^2 + 1$
15	5	0	$16\rho x^5 - 20\rho x^3 + 5\rho x$
16	4	1	$8\rho x^4 \rho y - 8\rho x^2 \rho y + \rho y$
17	3	2	$8\rho x^3 \rho y^2 - 4\rho x^3 - 6\rho x \rho y^2 + 3\rho x$
18	2	3	$8\rho x^2 \rho y^3 - 6\rho x^2 \rho y - 4\rho y^3 + 3\rho y$
19	1	4	$8\rho x \rho y^4 - 8\rho x \rho y^2 + \rho x$
20	0	5	$16\rho y^5 - 20\rho y^3 + 5\rho y$
21	6	0	$32\rho x^6 - 48\rho x^4 + 18\rho x^2 - 1$
22	5	1	$16\rho x^5 \rho y - 20\rho x^3 \rho y + 5\rho x \rho y$
23	4	2	$16\rho x^4 \rho y^2 - 8\rho x^4 - 16\rho x^2 \rho y^2 + 8\rho x^2 + 2\rho y^2 - 1$
24	3	3	$16\rho x^3 \rho y^3 - 12\rho x^3 \rho y - 12\rho x \rho y^3 + 9\rho x \rho y$
25	2	4	$16\rho x^2 \rho y^4 - 16\rho x^2 \rho y^2 + 2\rho x^2 - 8\rho y^4 + 8\rho y^2 - 1$
26	1	5	$16\rho x \rho y^5 - 20\rho x \rho y^3 + 5\rho x \rho y$
27	0	6	$32\rho y^6 - 48\rho y^4 + 18\rho y^2 - 1$
28	7	0	$64\rho x^7 - 112\rho x^5 + 56\rho x^3 - 7\rho x$
29	6	1	$32\rho x^6 \rho y - 48\rho x^4 \rho y + 18\rho x^2 \rho y - \rho y$
30	5	2	$32\rho x^5 \rho y^2 - 16\rho x^5 - 40\rho x^3 \rho y^2 + 20\rho x^3 + 10\rho x \rho y^2 - 5\rho x$
31	4	3	$32\rho x^4 \rho y^3 - 24\rho x^4 \rho y - 32\rho x^2 \rho y^3 + 24\rho x^2 \rho y + 4\rho y^3 - 3\rho y$
32	3	4	$32\rho x^3 \rho y^4 - 32\rho x^3 \rho y^2 + 4\rho x^3 - 24\rho x \rho y^4 + 24\rho x \rho y^2 - 3\rho x$
33	2	5	$32\rho x^2 \rho y^5 - 40\rho x^2 \rho y^3 + 10\rho x^2 \rho y - 16\rho y^5 + 20\rho y^3 - 5\rho y$
34	1	6	$32\rho x \rho y^6 - 48\rho x \rho y^4 + 18\rho x \rho y^2 - \rho x$
35	0	7	$64\rho y^7 - 112\rho y^5 + 56\rho y^3 - 7\rho y$

Order of polynomial $= N + m$

Single Index j given N and m

$$j = \frac{(N+m)(N+m+1)}{2} + m$$

Double indices given j

$$\text{order} = \text{ROUND}\left[\sqrt{1+2j} - 1\right]$$

$$m = j - \frac{\text{order}\,(\text{order}+1)}{2}$$

$$N = \text{order} - m$$

For continuous functions $W(\rho_x, \rho_y)$, the Chebyshev expansion coefficients are just given by

$$a_{nm} = \frac{4}{(1+\delta_{n0})(1+\delta_{m0})\pi^2} \int_{-1}^{1}\int_{-1}^{1} W(\rho_x, \rho_y)\, T_{nm}(\rho_x, \rho_y) \frac{d\rho_x\, d\rho_y}{\sqrt{1-\rho_x^2}\,\sqrt{1-\rho_y^2}}$$

The square roots tend to make these integrals nasty. For simple polynomial functions, we can usually just compare like terms

e.g.

$$W(\rho_x, \rho_y) = \rho_x^2 + \rho_y^2 \qquad \text{defocus}$$

From the table

$$T_{20}(\rho_x, \rho_y) = 2\rho_x^2 - 1 \Rightarrow \rho_x^2 = \frac{T_{20}(\rho_x, \rho_y) + T_{00}(\rho_x, \rho_y)}{2}$$

$$T_{02}(\rho_x, \rho_y) = 2\rho_y^2 - 1 \Rightarrow \rho_y^2 = \frac{T_{02}(\rho_x, \rho_y) + T_{00}(\rho_x, \rho_y)}{2}$$

So

$$W(\rho_x, \rho_y) = T_{00}(\rho_x, \rho_y) + \frac{1}{2} T_{20}(\rho_x, \rho_y) + \frac{1}{2} T_{02}(\rho_x, \rho_y)$$

Fit Chebyshev's in a similar manner as we did with Zernikes for discretely sampled data

Suppose you have a set of measurements $\{ w(\rho_{xi}, \rho_{yi}) \}$ at points (ρ_{xi}, ρ_{yi}) where $i = 1..N$. We can write a system of equations in matrix form

$$\overleftrightarrow{A} \vec{x} = \vec{b}$$

where

$$\overleftrightarrow{A} = \begin{pmatrix} T_{00}(\rho_{x1}, \rho_{y1}) & T_{10}(\rho_{x1}, \rho_{y1}) & \cdots & T_{nmax, mmax}(\rho_{x1}, \rho_{y1}) \\ T_{00}(\rho_{x2}, \rho_{y2}) & T_{10}(\rho_{x2}, \rho_{y2}) & \cdots & \vdots \\ T_{00}(\rho_{x3}, \rho_{y3}) & T_{10}(\rho_{x3}, \rho_{y2}) & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ T_{00}(\rho_{xN}, \rho_{yN}) & T_{10}(\rho_{xN}, \rho_{yN} & \cdots & T_{nmax, mmax}(\rho_{xN, yN}) \end{pmatrix}$$

and

$$\vec{b} = \begin{pmatrix} w(\rho_{x1}, \rho_{y1}) \\ w(\rho_{x2}, \rho_{y2}) \\ \vdots \\ \vdots \\ w(\rho_{xN}, \rho_{yN} \end{pmatrix}$$

Solve for the expansion coefficients with

$$\vec{x} = \left[\overleftrightarrow{A}^T \overleftrightarrow{A} \right]^{-1} \overleftrightarrow{A}^T \vec{b}$$

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