A. Variable Power Elements

Decentering coma and trefoil terms introduces defocus and astigmatism terms, and that the magnitude of the defocus and astigmatism is linearly proportional to the amount of decentration $\Delta\rho$. Consider two complementary thin elements whose phase is given by

$$\phi_1(x, y) = a_{31} Z_3^1(x, y) + a_{33} Z_3^3(x, y) \qquad (1)$$

And

$$\phi_2(x, y) = -a_{31} Z_3^1(x, y) - a_{33} Z_3^3(x, y) \qquad (2)$$

If one element is shifted along the x-axis by an amount Δx and the other element is shifted an equal distance in the opposite direction, then variable levels of defocus and astigmatism can be produced. Specifically,

$$\begin{aligned}
&\phi_1(x + \Delta x, y) \\
&\quad = a_{31} Z_3^1(x + \Delta x, y) + a_{33} Z_3^3(x + \Delta x, y) \\
&\quad = a_{31}\left[-2\sqrt{2}\left(\Delta x + 3\Delta x^3\right)Z_0^0(x, y) + 9\sqrt{2}\Delta x^2 Z_1^1(x, y) - 2\sqrt{6}\Delta x Z_2^0(x, y) - 2\sqrt{3}\Delta x Z_2^2(x, y) + Z_3^1(x, y)\right] \\
&\qquad + a_{33}\left[-2\sqrt{2}\Delta x^3 Z_0^0(x, y) + 3\sqrt{2}\Delta x^2 Z_1^1(x, y) - 2\sqrt{3}\Delta x Z_2^2(x, y) + Z_3^3(x, y)\right]
\end{aligned} \qquad (3)$$

and

$$\begin{aligned}
&\phi_2(x - \Delta x, y) \\
&\quad = -a_{31} Z_3^1(x + \Delta x, y) - a_{33} Z_3^3(x + \Delta x, y) \\
&\quad = -a_{31}\left[2\sqrt{2}\left(\Delta x + 3\Delta x^3\right)Z_0^0(x, y) + 9\sqrt{2}\Delta x^2 Z_1^1(x, y) + 2\sqrt{6}\Delta x Z_2^0(x, y) + 2\sqrt{3}\Delta x Z_2^2(x, y) + Z_3^1(x, y)\right] \\
&\qquad - a_{33}\left[2\sqrt{2}\Delta x^3 Z_0^0(x, y) + 3\sqrt{2}\Delta x^2 Z_1^1(x, y) + 2\sqrt{3}\Delta x Z_2^2(x, y) + Z_3^3(x, y)\right]
\end{aligned} \qquad (4)$$

where the decentered version of the coma and trefoil terms are simply obtained from their respective rows in Table 1. Note that in combining these two phase-elements, terms with even powers of Δx cancel, and that the piston terms $Z_0^0(x, y)$ can be ignored since they represent only a constant phase shift of the net wavefront passing through the elements. The phase of a plane wave passing through the elements is then given by

$$\phi_1(x+\Delta x, y)+\phi_2(x-\Delta x, y)=-4\sqrt{6}\Delta x a_{31} Z_2^0(x, y)-4\sqrt{3}\Delta x(a_{31}+a_{33}) Z_2^2(x, y) \quad (5)$$

There are two special cases of this result. First, when $a_{33} = -a_{31}$. In this case, only pure defocus with a magnitude that is linearly dependent upon Δx appears. The phase of the first element is given by

$$\phi_1(x, y)=a_{31}\left(Z_3^1(x, y)-Z_3^3(x, y)\right)=\frac{a_{31}\sqrt{2}}{3}\left(\frac{x^3}{3}+xy^2-\frac{x}{3}\right) \quad . (6)$$

The phase of the second element is just the negative of the first. Equation (23) describes one form of a Variable Focus Lens (VFL) first described by Alvarez. The generalized form for the VFL such that the phase of one of the elements is given by

$$\phi(x, y)=A\left(\frac{x^3}{3}+xy^2\right)+Bx^2+Cxy+Dx+E+F(y) \quad , (7)$$

where B, C, D, and E are arbitrary constants and F(y) is an arbitrary function of y. Interestingly, the specific example of the VFL given in equation (23) is unique in that it minimizes the rms phase difference between the phase plate and a plane wave. This minimization is due to the rms being related to the sum of the squares of the Zernike expansion coefficients. The function $\phi_1(x,y)$ requires only two Zernike terms ($Z_3^1(x, y)$ and $Z_3^3(x, y)$) to represent its shape. Additional Zernike terms would be necessary to represent the generalized VFL shape in equation (24). These additional terms can only increase the rms phase difference, since the squares of the coefficients are necessarily positive.

The second special case of the variable power elements is when $a_{31} = 0$. In this case, pure

90/180 astigmatism results in equation (22). The phase of the first element in this case is given by

$$\phi_1(x, y) = a_{33} Z_3^3(x, y) = a_{33} 3\sqrt{8}\left(\frac{x^3}{3} - xy^2\right) \quad (8)$$

Again, the phase of the second element is just the negative of the first. This element is exactly the Variable Astigmatic Lens (VAL) described by Humphrey, Alvarez's student and colleague. The element can be generalized as well such that the phase is given by

$$\phi(x, y) = A\left(\frac{x^3}{3} - xy^2\right) + Bx^2 + Cxy + Dy^2 + Ex + Fy + G \quad (9)$$

where B, C, D, E, F and G are arbitrary constants. As with the VFL, the phase surface described in equation (25) represents the minimum rms phase surface for the VAF.