What I will assume you know!

• Multi-mode Gaussian states (pure and mixed) and Gaussian transformations both in phase space (Wigner, Q, P) and in Heisenberg picture (symplectic transformation on mode operators)
• Homodyne and Heterodyne detection
• CV teleportation; definition of Fidelity
• Universal bosonic n-mode unitary operation, and the role of non-Gaussian operations
• Creation of non-Gaussian states by photon number subtraction on Gaussian entangled states
• Quantum sensing: Heisenberg vs. standard shot-noise limit, Quantum vs. Classical Fisher Information
Plan for today

- Quantum state discrimination
- Application to quantum radar
Trace norm

- Trace norm (lets assume Hermitian $M$)

$$\|M\|_1 \equiv \text{Tr}\left\{\sqrt{M^\dagger M}\right\} = \text{Tr}(\|M\|)$$

- $M$ Hermitian,
  $$M = \sum_i \mu_i |i\rangle \langle i|; \quad \|M\|_1 = \sum_i |\mu_i|$$

- Satisfies properties for being a “norm”
  - Positive semidefinite $\|M\|_1 \geq 0 \quad \|M\|_1 = 0 \iff M = 0$
  - Homogeneity $\|cM\|_1 = |c|\|M\|_1$
  - Triangle inequality $\|M + N\|_1 \leq \|M\|_1 + \|N\|_1$
Trace distance

• Trace distance as the twice the largest probability difference that two states \( \rho \) and \( \sigma \) could give to the same measurement outcome \( \Lambda \)

\[
\|\rho - \sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} \text{Tr}\{\Lambda (\rho - \sigma)\}
\]

– Maximization is over all positive operators \( \Lambda \) with eigenvalues bounded from above by 1.
– Optimal \( \Lambda \) is the projector onto the positive eigenspace of \( (\rho - \sigma) \)

State discrimination

- Choose between $\rho_0$ and $\rho_1$ with minimum error

\[
p_e = p_{Y|X}(0|1)p_X(1) + p_{Y|X}(1|0) p_X(0)
= \text{Tr}\{\Lambda_0 \rho_1\} \frac{1}{2} + \text{Tr}\{\Lambda_1 \rho_0\} \frac{1}{2}
= \frac{1}{2} \left( \text{Tr}\{\Lambda_0 \rho_1\} + \text{Tr}\{(I - \Lambda_0) \rho_0\} \right)
= \frac{1}{2} \left( \text{Tr}\{\Lambda_0 \rho_1\} + \text{Tr}\{\rho_0\} - \text{Tr}\{\Lambda_0 \rho_0\} \right)
= \frac{1}{2} \left( \text{Tr}\{\Lambda_0 \rho_1\} + 1 - \text{Tr}\{\Lambda_0 \rho_0\} \right)
= \frac{1}{4} \left( 2 - 2\text{Tr}\{\Lambda_0 (\rho_0 - \rho_1)\} \right).
\]
Minimum probability of error

- Minimizing the average probability of error,

\[ P_{e,\text{min}} = \min_{\Lambda_0, \Lambda_1} \frac{1}{4} (2 - 2 \text{Tr}\{\Lambda_0 (\rho_0 - \rho_1)\}) \]

\[ = \frac{1}{4} \left( 2 - 2 \max_{\Lambda_0, \Lambda_1} \text{Tr}\{\Lambda_0 (\rho_0 - \rho_1)\} \right) \]

\[ = \frac{1}{2} \left( 1 - \frac{1}{2} \|\rho_0 - \rho_1\|_1 \right) \]

- For unequal priors, \( P_{e,\text{min}} = \frac{1}{2} \left( 1 - \|p_0 \rho_0 - p_1 \rho_1\|_1 \right) \)
Discriminating pure states, $|\psi_0\rangle$, $|\psi_1\rangle$

- Inner product, $\sigma \equiv \langle \psi_0 | \psi_1 \rangle$

- Recall, we had shown earlier,

$$P_{e,\text{min}} = \frac{1}{2} \left[ 1 - \sqrt{1 - |\sigma|^2} \right]$$

- Re-derive the above expression using the general trace-distance formula in the previous slide

- Relationship between Fidelity and trace distance

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2} \| \rho - \sigma \|_1 \leq \sqrt{1 - F(\rho, \sigma)}$$
Multi-copy state discrimination

• Consider the following problem:
  – We are given \( M \) copies of one of two states:
    
    \[
    \rho_0^\otimes n \quad \text{versus} \quad \rho_1^\otimes n
    \]
  – Minimum error probability (exact)
    
    \[
    P_{e,\min,n} := \left( 1 - \| \pi_1 \rho_1^\otimes n - \pi_0 \rho_0^\otimes n \|_1 \right)/2
    \]
    
    \[
    \sim e^{-\xi n},
    \]
    

• Quantum Chernoff exponent, \( \xi = -\log \left( \min_{0 \leq s \leq 1} \text{Tr}(\rho_0^s \rho_1^{1-s}) \right) \)

• When the states are Gaussian, \( \xi \) can be calculated from the symplectic eigenvalues of the density operators

Quantum Chernoff bound

- QCB: Minimum probability of error of n-copy state discrimination, $\rho_0^\otimes n$ versus $\rho_1^\otimes n$

$$P_{e,\min}^{(n)} \leq \frac{1}{2} e^{-\xi n} \quad \xi = -\log \left( \min_{0 \leq s \leq 1} \text{Tr}(\rho_0^s \rho_1^{1-s}) \right)$$

- Bhattacharyya bound: $s = \frac{1}{2}$ (looser upper bound)

- When the states are simultaneously diagonal, this reduces to the classical problem of telling apart two distributions $p_0$ and $p_1$ with $n$ samples

$$P_{e,\min}^{(n)} \leq \frac{1}{2} e^{-\xi n} \quad \xi = -\log \left( \min_{0 \leq s \leq 1} \sum_i p_0(i)^s p_1(i)^{1-s} \right)$$
Consider the following problem:

- We are given \( n \) copies of one of two coherent states:

\[
|\alpha\rangle^\otimes n \text{ versus } |\beta\rangle^\otimes n
\]

- Assume equally likely hypotheses
- Inner product, \( \sigma = \langle \alpha | \beta \rangle \), \( |\sigma|^2 = e^{-|\alpha - \beta|^2} \)

Prove that:

- Optimal measurement \( P_{e,\text{min}} \sim e^{-\xi_{\text{opt}} n} \), \( \xi_{\text{opt}} = -2 \log |\sigma| \)
- Optimal single-mode measurement followed by majority vote, \( P_e \sim e^{-\frac{\xi_{\text{opt}}}{2} n} \)
- Kennedy receiver, \( P_e \sim e^{-\xi_{\text{opt}} n} \)
Target detection (radar)

- Binary hypothesis test

\[
\hat{a}_S \quad \rightarrow \quad \hat{a}_R = \hat{a}_B
\]

\[
H_0 \rightarrow \hat{a}_R = \hat{a}_B
\]

\[
H_1 \rightarrow \hat{a}_R = \sqrt{\kappa} \hat{a}_S + \sqrt{1 - \kappa} \hat{a}_B
\]

\[
M \approx WT \text{ independent temporal modes}
\]

\[
\kappa \ll 1 \text{ and } N_B \gg 1
\]

- signal is coherent state: \( \langle \hat{a}_S^\dagger \hat{a}_S \rangle = N_S \)
- background state is thermal: \( \langle \hat{a}_B^\dagger \hat{a}_B \rangle = \begin{cases} N_B & \text{under } H_0 \\ \frac{N_B}{1 - \kappa} & \text{under } H_1 \end{cases} \)
Quantum illumination (entangled probe)

- Two-mode squeezed vacuum as transmitter

$q_S$ signal

$\hat{a}_S$ signal

$H_0 = \text{target absent}$

$H_1 = \text{target present}$

equally likely

$H_0 \rightarrow \hat{a}_R = \hat{a}_B$

$H_1 \rightarrow \hat{a}_R = \sqrt{\kappa} \hat{a}_S + \sqrt{1 - \kappa} \hat{a}_B$

$M \approx WT$ independent temporal modes

$\kappa \ll 1$ and $N_B \gg 1$

- signal state is thermal: $\langle \hat{a}_S^\dagger \hat{a}_S \rangle = N_S$

- background state is thermal: $\langle \hat{a}_B^\dagger \hat{a}_B \rangle = \begin{cases} N_B & \text{under } H_0 \\ \frac{N_B}{1 - \kappa} & \text{under } H_1 \end{cases}$

- receiver uses return + idler to decide
The state discrimination problem

- Both hypotheses produce $M$ zero-mean two-mode Gaussian states, with covariance matrices given by
  - Operating regime: Highly lossy, highly noisy, low-brightness transmitter $\kappa \ll 1, N_s \ll 1$ and $N_B \gg 1$

\[
\Lambda_{SI} = \frac{1}{4} \begin{bmatrix}
S & 0 & C_q & 0 \\
0 & S & 0 & -C_q \\
C_q & 0 & S & 0 \\
0 & -C_q & 0 & S
\end{bmatrix}
\]

\[
S \equiv 2N_s + 1 \\
C_q \equiv 2\sqrt{N_S(N_S + 1)}
\]

\[
\Lambda_{RI}^{(0)} = \frac{1}{4} \begin{bmatrix}
B & 0 & 0 & 0 \\
0 & B & 0 & 0 \\
0 & 0 & S & 0 \\
0 & 0 & 0 & S
\end{bmatrix}
\]

\[
B \equiv 2N_B + 1
\]

\[
\Lambda_{RI}^{(1)} = \frac{1}{4} \begin{bmatrix}
A & 0 & \sqrt{\kappa}C_q & 0 \\
0 & A & 0 & -\sqrt{\kappa}C_q \\
\sqrt{\kappa}C_q & 0 & S & 0 \\
0 & -\sqrt{\kappa}C_q & 0 & S
\end{bmatrix}
\]

\[
A \equiv 2\kappa N_S + B
\]
6 dB improvement in the exponent; Chernoff exponent can be seen as SNR

\[ P_{e,\text{min}} \leq \frac{1}{2} e^{-M\xi} \quad \xi = -\min_{s \in (0,1)} \log(\text{Tr}[\rho_0^s\rho_1^{1-s}]) \]

\[ P_{e,\text{CS}} \leq \frac{1}{2} e^{M\kappa N_S/4N_B} \]

\[ P_{e,\text{SPDC}} \leq \frac{1}{2} e^{M\kappa N_S/N_B} \]

\[ \kappa = 0.01 \]

\[ N_S = 0.01 \]

\[ N_B = 20 \]

Tan, Erkmen, Giovannetti, Guha, Lloyd, Maccone, and Shapiro

OPA receiver for quantum illumination

- How do we build a receiver that harnesses the promise of the 6 dB improvement?
- Consider a receiver which mixes the return and idler beams on an optical parametric amplifier (OPA) and detects the output by photon counting measurement
- \( K = M \), the number of temporal modes integrated over

\[
\begin{align*}
\{ \hat{a}_{R_i} \}_{i=1}^{K} & \quad \text{ Optical Parametric Amplifier (OPA)} \\
\{ \hat{a}_{I_i} \}_{i=1}^{K} & \quad \text{ Low gain, } G = 1 + \epsilon
\end{align*}
\]

\[
\hat{c}_i = \sqrt{G} \hat{a}_{I_i} + \sqrt{G - 1} \hat{a}_{R_i}^\dagger
\]

\[
\hat{N}_i = \hat{c}_i^\dagger \hat{c}_i
\]

\[
N = \sum_{i=1}^{K} N_i
\]

Total number of clicks registered in K received modes

ISIT 2009
Performance analysis of OPA receiver

- Output $\hat{c}_i$ of the OPA is in a zero-mean thermal state with mean photon number given by
  
  - Hypothesis $H_0$: $\langle \hat{c}_i^\dagger \hat{c}_i \rangle = GN_S + (G - 1)(1 + N_B) \overset{\Delta}{=} N_0$
  
  - Hypothesis $H_1$:
    
    $$
    \langle \hat{c}_i^\dagger \hat{c}_i \rangle = GN_S + (G - 1)(1 + N_B + \kappa N_S) + 2\sqrt{G(G - 1)}\sqrt{\kappa N_S(N_S + 1)}
    \overset{\Delta}{=} N_1
    $$

    Signature of remnant phase-sensitive cross-correlation between return and idler modes, $\mathcal{R}(\hat{a}_{Ri}, \hat{a}_{Ii})$

- Optimum measurement to distinguish between two zero-mean thermal states of $\hat{c}_i$ is photon counting on all received modes. Under $H_0$ & $H_1$:

  $$
  \hat{\rho}^{(k)}_{c_i} = \sum_{n=0}^{\infty} \frac{N_k^n}{(1 + N_k)^{n+1}} |n\rangle\langle n|, \quad k \in \{0, 1\}, 1 \leq i \leq K
  $$
Performance analysis of OPA receiver

• Detection problem: Based on the observed value of the total clicks N, decide between H₀ & H₁

\[
p_{N|H_0}(n|H_0) = \frac{1}{(1 + N_0)^K} \binom{n + K + 1}{n} \left( \frac{N_0}{1 + N_0} \right)^n
\]

\[
p_{N|H_1}(n|H_1) = \frac{1}{(1 + N_1)^K} \binom{n + K + 1}{n} \left( \frac{N_1}{1 + N_1} \right)^n
\]

• Decision rule
  – Say “H₀” if \( N \leq N_{\text{threshold}} \)
  – Say “H₁” if \( N > N_{\text{threshold}} \)

• Probability of error

\[
Pr(e) = \frac{1}{2} \sum_{n=N_{\text{threshold}}+1}^{\infty} p_{N|H_0}(n|H_0) + \frac{1}{2} \sum_{n=0}^{N_{\text{threshold}}} p_{N|H_1}(n|H_1) \approx \frac{1}{2} \text{erfc}(C \sqrt{K}) \approx \frac{1}{4} e^{-(C^2)K}
\]

\[
(C = (N_1 - N_0)/(\sqrt{2}(\sigma_0 + \sigma_1)), \quad \sigma_k = \sqrt{N_k(N_k + 1)}, k \in \{0, 1\})
\]
Bhattacharyya bound on performance

- Bhattacharyya (upper) bound on performance of OPA-based receiver
  - \[ \Pr(e) \leq \frac{1}{2} Q_B^K \], where
  - Bound asymptotically tight as \( K \to \infty \)

\[
Q_B = \sum_{n=0}^{\infty} \sqrt{\frac{N_0^n}{(1 + N_0)^{1+n}}} \times \frac{N_1^n}{(1 + N_1)^{1+n}} = \left( \sqrt{(1 + N_0)(1 + N_1)} - \sqrt{N_0 N_1} \right)^{-1}
\]

- OPA Gain \( G = 1 + g \) is optimized for min \( \Pr(e) \), i.e. max C

![Graph showing \( \log_{10}(\Pr(e) \text{ bound}) \) vs \( \log_{10}(K) \), with \( \kappa = 0.01 \), \( N_S = 0.01 \), \( N_B = 20 \), and \( G = 1 + 5.041 \times 10^{-3} \).]

![Graph showing \( C \) vs \( g \), with \( \kappa = 0.01 \), \( N_S = 0.01 \), \( N_B = 20 \), and \( g^* = 5.041 \times 10^{-3} \).]
OPA receiver yields 3 dB improvement

**Quantum Radar**

\[ SNR_Q \propto \frac{\kappa N_S}{N_B} \]

**Classical Radar**

\[ SNR_C \propto \frac{\kappa N_S}{4N_B} \]

**Quantum Radar with OPA receiver**

\[ SNR_{Q, OPA} = \frac{\kappa N_S}{2N_B} \]

\[ M \approx WT : \text{number of temporal modes} \]
Optimal copy-by-copy measurement is worse by 3 dB from optimal

We saw this curious 3 dB difference in the pure-state case. Is this a general feature in binary state discrimination?

Advanced Problem 17 (open)
Quantum illumination radar experiment

ASE: amplified spontaneous emission; BS: beam splitter; CWDM: coarse wavelength-division multiplexer; D: detector; DCF: dispersion-compensating fiber; DM: dichroic mirror; DSF: dispersion-shifted LEAF fiber; EDFA: erbium-doped fiber amplifier; OPA: optical parametric amplifier; PC: polarization controller; PM: phase modulator; Pol: polarizer; SMF: single-mode fiber; SPDC: spontaneous parametric down conversion; Z: zoom-lens systems

Environmental loss: 14 dB; Noise background: 75 dB
Quantum sensing outperforms optimum classical sensing

Microwave quantum radar

- The high-noise requirement makes microwave-wavelength operation a naturally-suited regime for quantum illumination
- Quantum advantage most pronounced at a high enough $M$. To get to a given $M \sim WT$, since $W$ is lower (than optical), higher integration time $T$ is needed. Applications with long dwell time?

Optimal quantum receiver design

- Calculating minimum probability of error for discriminating states in \( \{ p_i, |\psi_i\rangle \} \) is conceptually simple
- But structured receiver designs that achieve optical state discrimination at the quantum minimum error, are far and few
- Binary coherent states, \( \{|-\alpha\rangle, |\alpha\rangle\} \) \( |\alpha|^2 \equiv N \)
  - Minimum error probability (equal priors), \( P_{e,\text{min}} = \frac{1}{2} \left[ 1 - \sqrt{1 - e^{-4N}} \right] \)
  - Dolinar’s receiver [1973]: **OPTI 595B**
Optimum receiver for quantum illumination

Sum frequency generation (SFG):
\[
\hat{H}_I = \hbar g \sum_{m=1}^{M} (\hat{b}^{\dagger} \hat{a}_{S_m} \hat{a}_{I_m} + \hat{b} \hat{a}_{S_m}^{\dagger} \hat{a}_{I_m}^{\dagger})
\]

Inspired by Dolinar receiver: feedback using squeezing instead of displacement

Suggested reading

One-versus-Two Target Detection

\[ \theta_{\text{Ray}} = \frac{\lambda}{D}, \theta = \mu \theta_{\text{Ray}} \]

Does this quantum illumination improvement prevail in imaging problems? More complex optical sensing / discrimination tasks?

\[ N_S = 0.01 \]
\[ \kappa = 0.001 \]
\[ N_B = 1 \]

6 dB in error exponent

S. Guha and J. H. Shapiro, QCMC 2010, arXiv:1012.2548v1
Upcoming topics

• Quantum limits of optical communications