Photonic Quantum Information Processing
OPTI 647: Lecture 23

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What I will assume you know!

- Multi-mode Gaussian states (pure and mixed) and Gaussian transformations both in phase space (Wigner, Q, P) and in Heisenberg picture (symplectic transformation on mode operators)
- Homodyne and Heterodyne detection
- CV teleportation; definition of Fidelity
- Universal bosonic n-mode unitary operation, and the role of non-Gaussian operations
- Creation of non-Gaussian states by photon number subtraction on Gaussian entangled states
- Quantum sensing: Heisenberg vs. standard shot-noise limit, Quantum vs. Classical Fisher Information

Plan for today

- Quantum state discrimination
- Application to quantum radar

Trace norm

- Trace norm (lets assume Hermitian M)

$$\|M\|_1 \equiv \mathrm{Tr}\left\{\sqrt{M^\dagger M}\right\} = \mathrm{Tr}(|M|)$$

  – M Hermitian, $M = \sum_i \mu_i |i\rangle\langle i|; \quad ||M||_1 = \sum_i |\mu_i|$

- Satisfies properties for being a "norm"
  – Positive semidefinite $\|M\|_1 \geq 0 \qquad \|M\|_1 = 0 \quad \Leftrightarrow \quad M = 0$
  – Homogeneity $\|cM\|_1 = |c|\|M\|_1$
  – Triangle inequality $\|M + N\|_1 \leq \|M\|_1 + \|N\|_1$

Trace distance

- Trace distance as the twice the largest probability difference that two states ρ and σ could give to the same measurement outcome Λ

$$\|\rho - \sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} \mathrm{Tr}\{\Lambda(\rho - \sigma)\}$$

  - Maximization is over all positive operators Λ with eigenvalues bounded from above by 1.
  - Optimal Λ is the projector onto the positive eigenspace of (ρ - σ)

[Helstrom] Read proof in Book by Mark Wilde, https://arxiv.org/abs/1106.1445

State discrimination

- Choose between ρ_0 and ρ_1 with minimum error

$$
\begin{aligned}
p_e &= p_{Y|X}(0|1)p_X(1) + p_{Y|X}(1|0)p_X(0) \\
&= \mathrm{Tr}\{\Lambda_0\rho_1\}\frac{1}{2} + \mathrm{Tr}\{\Lambda_1\rho_0\}\frac{1}{2}. \\
&= \frac{1}{2}(\mathrm{Tr}\{\Lambda_0\rho_1\} + \mathrm{Tr}\{(I-\Lambda_0)\rho_0\}) \\
&= \frac{1}{2}(\mathrm{Tr}\{\Lambda_0\rho_1\} + \mathrm{Tr}\{\rho_0\} - \mathrm{Tr}\{\Lambda_0\rho_0\}) \\
&= \frac{1}{2}(\mathrm{Tr}\{\Lambda_0\rho_1\} + 1 - \mathrm{Tr}\{\Lambda_0\rho_0\}) \\
&= \frac{1}{4}(2 - 2\mathrm{Tr}\{\Lambda_0(\rho_0-\rho_1)\}).
\end{aligned}
$$

Minimum probability of error

- Minimizing the average probability of error,

$$
\begin{aligned}
P_{e,\min} &= \min_{\Lambda_0,\Lambda_1} \frac{1}{4}(2 - 2\mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}) \\
&= \frac{1}{4}\left(2 - 2\max_{\Lambda_0,\Lambda_1} \mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2}\|\rho_0 - \rho_1\|_1\right)
\end{aligned}
$$

  – For unequal priors, $P_{e,\min} = \frac{1}{2}\left(1 - \|p_0\rho_0 - p_1\rho_1\|_1\right)$

Discriminating pure states, $|\psi_0\rangle, |\psi_1\rangle$

- Inner product, $\sigma \equiv \langle \psi_0 | \psi_1 \rangle$
- Recall, we had shown earlier,

$$P_{e,\min} = \frac{1}{2}\left[1 - \sqrt{1 - |\sigma|^2}\right]$$

- Re-derive the above expression using the general trace-distance formula in the previous slide

Problem 89

- Relationship between Fidelity and trace distance

$$1 - \sqrt{F(\rho,\sigma)} \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho,\sigma)}$$

Multi-copy state discrimination

- Consider the following problem:
  - We are given M copies of one of two states:

$$\rho_0^{\otimes n} \text{ versus } \rho_1^{\otimes n}$$

  - Minimum error probability (exact)

$$P_{e,\min,n} := (1 - ||\pi_1 \rho_1^{\otimes n} - \pi_0 \rho_0^{\otimes n}||_1)/2$$

$$\sim e^{-\xi n},$$

Audenaert et al., Phys. Rev. Lett. 98, 160501 (2007)

- Quantum Chernoff exponent, $\xi = -\log\left(\min_{0 \le s \le 1} \mathrm{Tr}(\rho_0^s \rho_1^{1-s})\right)$
- When the states are Gaussian, ξ can be calculated from the symplectic eigenvalues of the density operators

Pirandola and Lloyd, Phys. Rev. A 78, 012331 (2008)

Quantum Chernoff bound

- QCB: Minimum probability of error of n-copy state discrimination, $\rho_0^{\otimes n}$ versus $\rho_1^{\otimes n}$

$$P_{e,\min}^{(n)} \leq \frac{1}{2} e^{-\xi n} \qquad \xi = -\log\left(\min_{0\leq s\leq 1} \mathrm{Tr}(\rho_0^s \rho_1^{1-s})\right)$$

  - Bhattacharyya bound: s = ½ (looser upper bound)
- When the states are simultaneously diagonal, this reduces to the classical problem of telling apart two distributions p_0 and p_1 with n samples

$$P_{e,\min}^{(n)} \leq \frac{1}{2} e^{-\xi n} \qquad \xi = -\log\left(\min_{0\leq s\leq 1} \sum_i p_0(i)^s p_1(i)^{1-s}\right)$$

Optimal measurements for one copy discrimination versus multi-copy

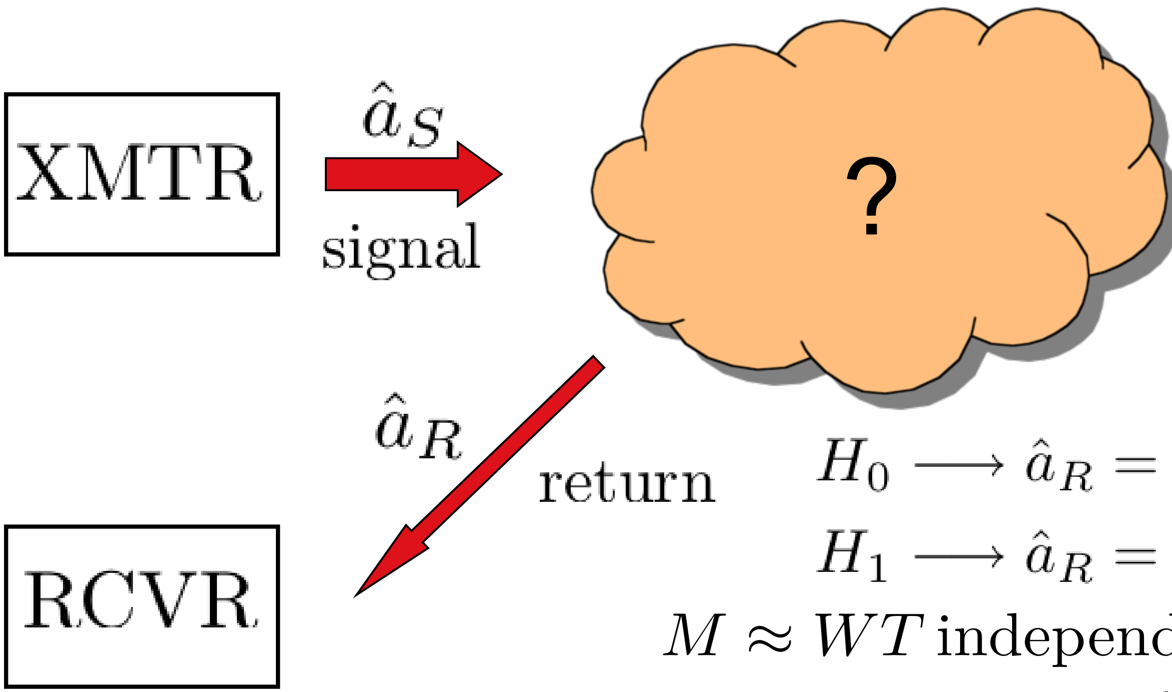
- Consider the following problem:
  - We are given n copies of one of two coherent states:

$$|\alpha\rangle^{\otimes n} \text{ versus } |\beta\rangle^{\otimes n}$$

  - Assume equally likely hypotheses
  - Inner product, $\sigma = \langle\alpha|\beta\rangle$, $|\sigma|^2 = e^{-|\alpha-\beta|^2}$
- Prove that: **Problem 90**
  - Optimal measurement $P_{e,\min} \sim e^{-\xi_{\text{opt}} n}, \xi_{\text{opt}} = -2\log|\sigma|$
  - Optimal single-mode measurement followed by majority vote, $P_e \sim e^{-\frac{\xi_{\text{opt}}}{2} n}$
  - Kennedy receiver, $P_e \sim e^{-\xi_{\text{opt}} n}$

Target detection (radar)

- Binary hypothesis test



$H_0 =$ target absent

$H_1 =$ target present

equally likely

$$H_0 \longrightarrow \hat{a}_R = \hat{a}_B$$

$$H_1 \longrightarrow \hat{a}_R = \sqrt{\kappa}\,\hat{a}_S + \sqrt{1-\kappa}\,\hat{a}_B$$

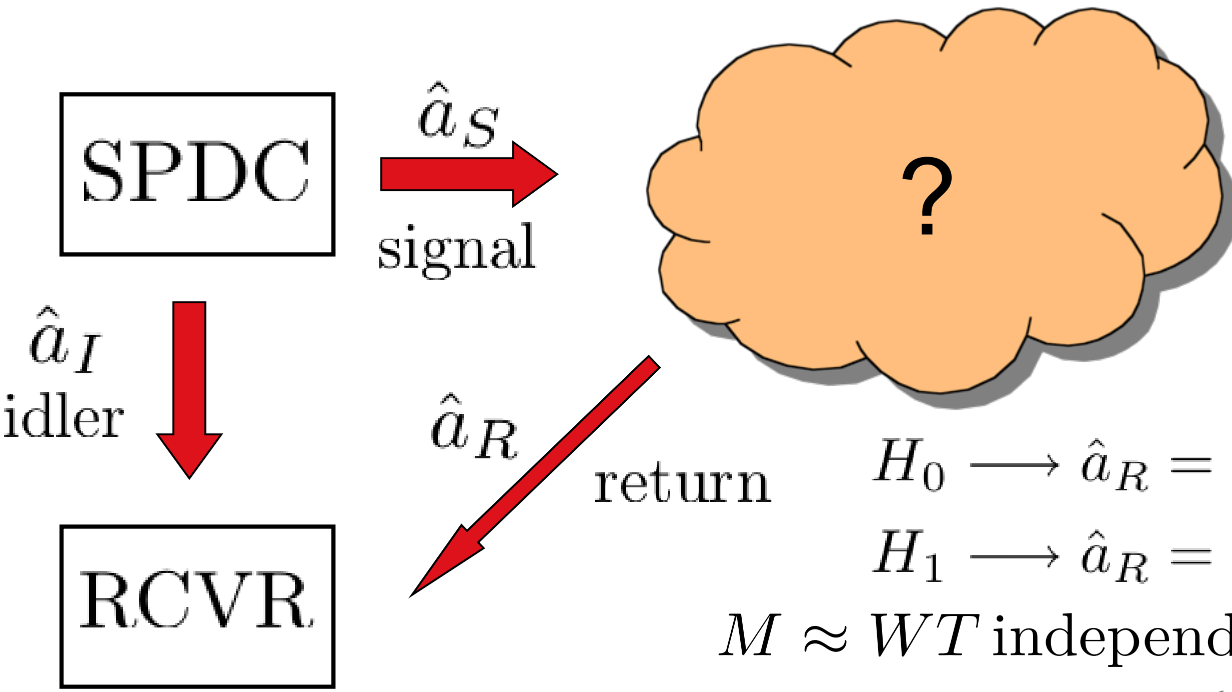
$M \approx WT$ independent temporal modes

$\kappa \ll 1$ and $N_B \gg 1$

- signal is coherent state: $\langle \hat{a}_S^\dagger \hat{a}_S \rangle = N_S$
- background state is thermal: $\langle \hat{a}_B^\dagger \hat{a}_B \rangle = \begin{cases} N_B & \text{under } H_0 \\ \dfrac{N_B}{1-\kappa} & \text{under } H_1 \end{cases}$

Quantum illumination (entangled probe)

- Two-mode squeezed vacuum as transmitter



$H_0 =$ target absent

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$$H_0 \longrightarrow \hat{a}_R = \hat{a}_B$$

$$H_1 \longrightarrow \hat{a}_R = \sqrt{\kappa}\,\hat{a}_S + \sqrt{1-\kappa}\,\hat{a}_B$$

$M \approx WT$ independent temporal modes

$\kappa \ll 1$ and $N_B \gg 1$

- signal state is thermal: $\langle \hat{a}_S^\dagger \hat{a}_S \rangle = N_S$
- background state is thermal: $\langle \hat{a}_B^\dagger \hat{a}_B \rangle = \begin{cases} N_B & \text{under } H_0 \\ \dfrac{N_B}{1-\kappa} & \text{under } H_1 \end{cases}$
- receiver uses return + idler to decide

The state discrimination problem

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- Both hypotheses produce *M* zero-mean two-mode Gaussian states, with covariance matrices given by
  - Operating regime: Highly lossy, highly noisy, low-brightness transmitter $\kappa \ll 1, N_s \ll 1 \text{ and } N_B \gg 1$

$$\mathbf{\Lambda}_{SI} = \frac{1}{4}\begin{bmatrix} S & 0 & C_q & 0 \\ 0 & S & 0 & -C_q \\ C_q & 0 & S & 0 \\ 0 & -C_q & 0 & S \end{bmatrix}$$

$$S \equiv 2N_S + 1$$
$$C_q \equiv 2\sqrt{N_S(N_S+1)}$$

H_0 $\qquad$ H_1

$$\mathbf{\Lambda}_{RI}^{(0)} = \frac{1}{4}\begin{bmatrix} B & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & S \end{bmatrix}$$

$$\mathbf{\Lambda}_{RI}^{(1)} = \frac{1}{4}\begin{bmatrix} A & 0 & \sqrt{\kappa}C_q & 0 \\ 0 & A & 0 & -\sqrt{\kappa}C_q \\ \sqrt{\kappa}C_q & 0 & S & 0 \\ 0 & -\sqrt{\kappa}C_q & 0 & S \end{bmatrix}$$

$$B \equiv 2N_B + 1$$

$$A \equiv 2\kappa N_S + B$$

6 dB improvement in the exponent; Chernoff exponent can be seen as SNR

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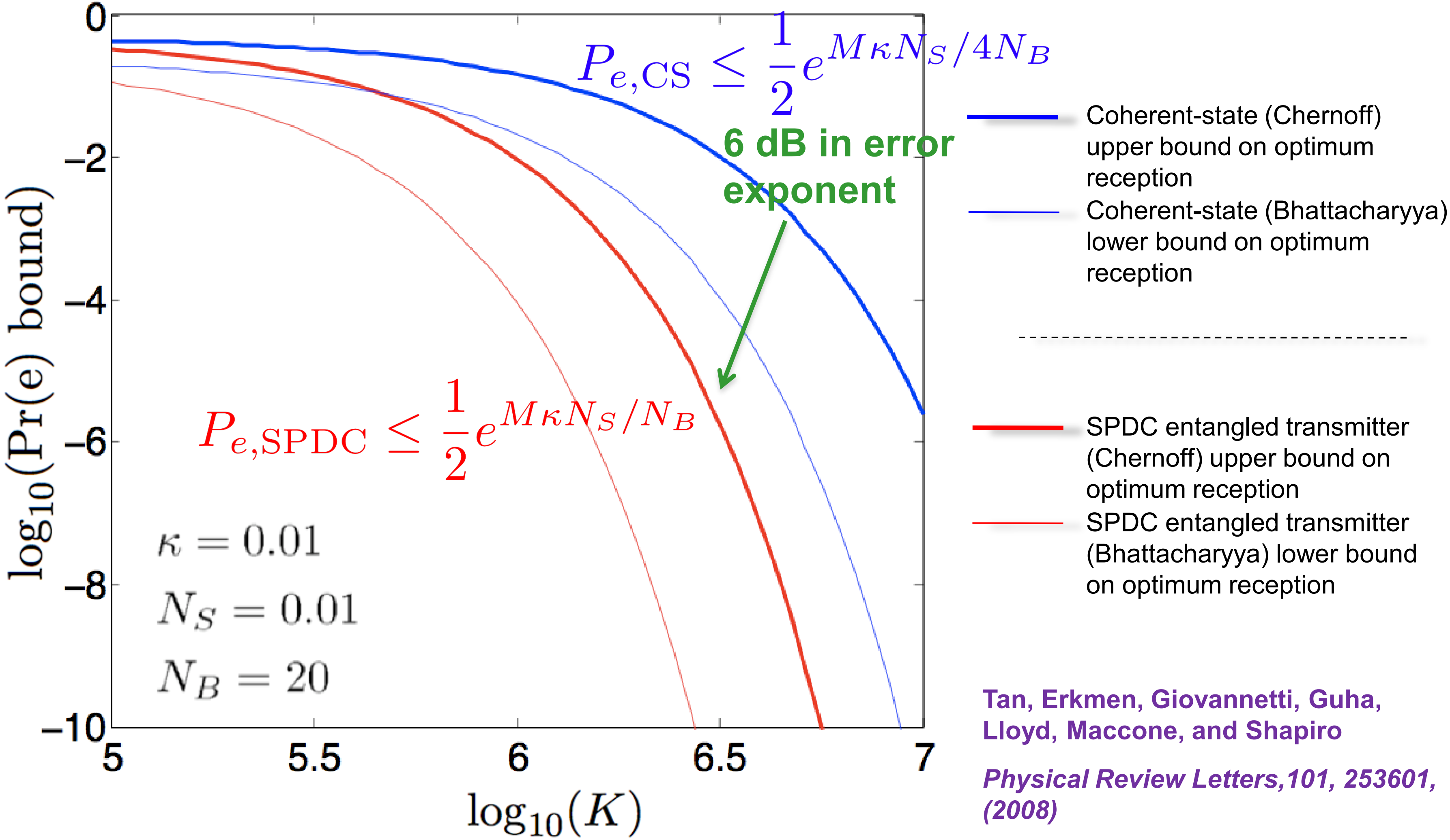
$\rho_0^{\otimes M}$ vs. $\rho_1^{\otimes M}$ $\quad P_{e,\min} \leq \frac{1}{2} e^{-M\xi}$ $\quad \xi = -\min_{s\in(0,1)} \log(\mathrm{Tr}[\rho_0^s \rho_1^{1-s}])$



Tan, Erkmen, Giovannetti, Guha, Lloyd, Maccone, and Shapiro

***Physical Review Letters*, 101, 253601, (2008)**

OPA receiver for quantum illumination

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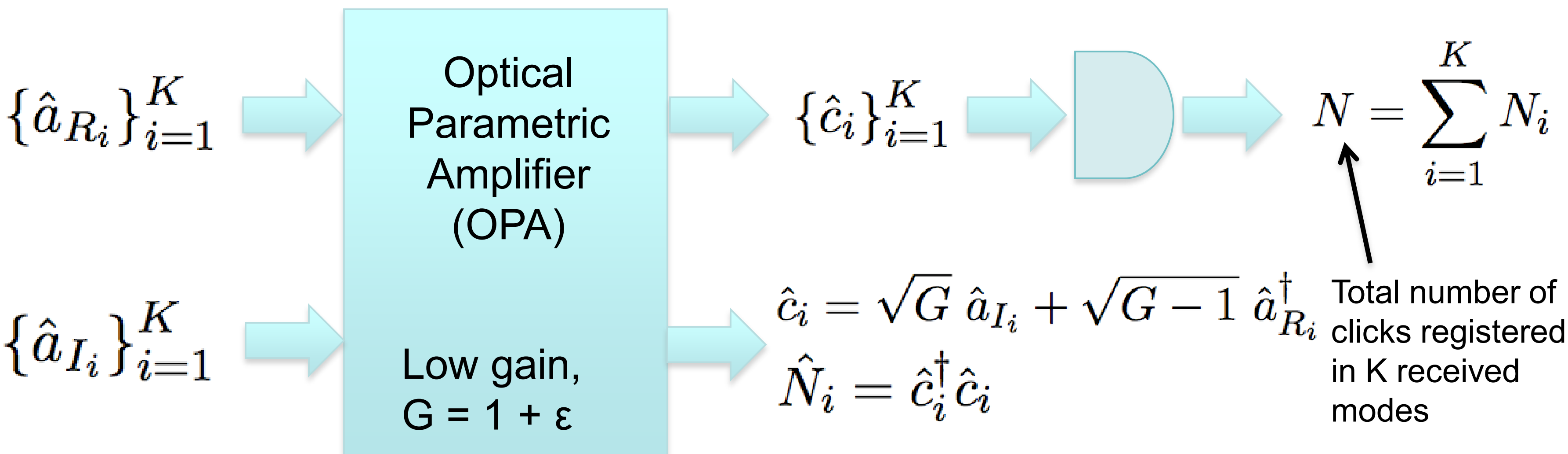
- How do we build a receiver that harnesses the promise of the 6 dB improvement?
- Consider a receiver which mixes the return and idler beams on an optical parametric amplifier (OPA) and detects the output by photon counting measurement
- $K = M$, the number of temporal modes integrated over



Guha, arXiv: quant-ph/0902.2932

ISIT 2009

Performance analysis of OPA receiver

- Output $\hat{c}_i$ of the OPA is in a zero-mean thermal state with mean photon number given by
  - Hypothesis H$_0$: $\langle \hat{c}_i^\dagger \hat{c}_i \rangle = GN_S + (G-1)(1+N_B) \triangleq N_0$
  - Hypothesis H$_1$:

$$\langle \hat{c}_i^\dagger \hat{c}_i \rangle = GN_S + (G-1)(1+N_B+\kappa N_S) + 2\sqrt{G(G-1)}\underbrace{\sqrt{\kappa N_S(N_S+1)}}$$
$$\triangleq N_1$$

Signature of remnant phase-sensitive cross-correlation between return and idler modes, $\Re(\hat{a}_{R_i}\hat{a}_{I_i})$

- Optimum measurement to distinguish between two zero-mean thermal states of $\hat{c}_i$ is photon counting on all received modes. Under H$_0$ & H$_1$:

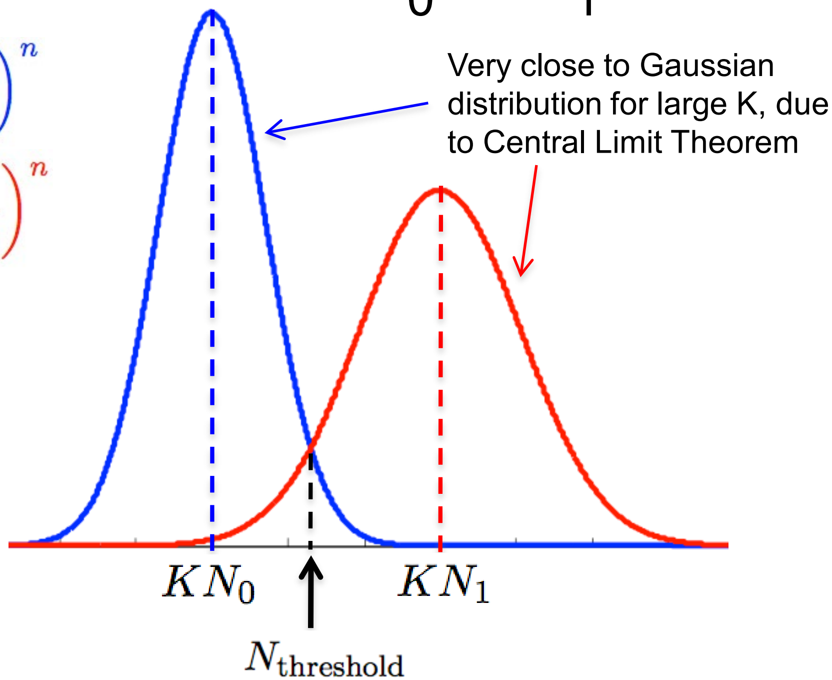
$$\hat{\rho}_{c_i}^{(k)} = \sum_{n=0}^{\infty} \frac{{N_k}^n}{(1+N_k)^{n+1}} |n\rangle\langle n|, \quad k \in \{0,1\}, 1 \le i \le K$$

Performance analysis of OPA receiver

- Detection problem: Based on the observed value of the total clicks N, decide between H_0 & H_1

$$p_{N|H_0}(n|H_0) = \frac{1}{(1+N_0)^K}\binom{n+K+1}{n}\left(\frac{N_0}{1+N_0}\right)^n$$

$$p_{N|H_1}(n|H_1) = \frac{1}{(1+N_1)^K}\binom{n+K+1}{n}\left(\frac{N_1}{1+N_1}\right)^n$$



- Decision rule
  - Say "H_0" if $N \le N_{\text{threshold}}$
  - Say "H_1" if $N > N_{\text{threshold}}$

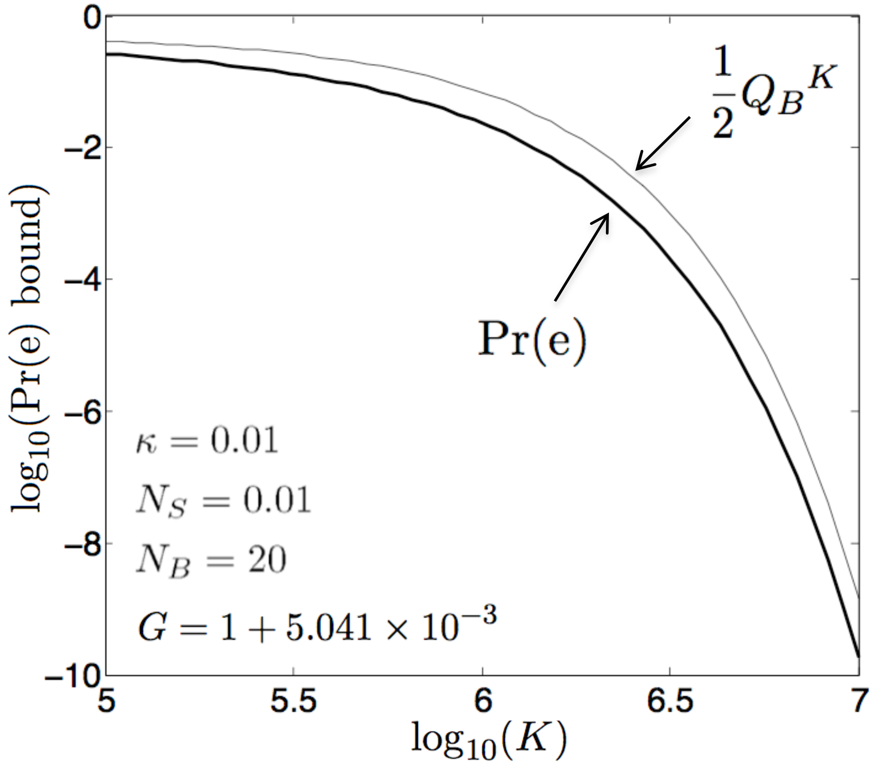
- $$\Pr(e) = \frac{1}{2}\sum_{n=N_{\text{threshold}}+1}^{\infty} p_{N|H_0}(n|H_0) + \frac{1}{2}\sum_{n=0}^{N_{\text{threshold}}} p_{N|H_1}(n|H_1) \approx \frac{1}{2}\text{erfc}(C\sqrt{K}) \approx \frac{1}{4}e^{-(C^2)K}$$

$$(C = (N_1 - N_0)/(\sqrt{2}(\sigma_0+\sigma_1)), \quad \sigma_k = \sqrt{N_k(N_k+1)}, k \in \{0,1\})$$

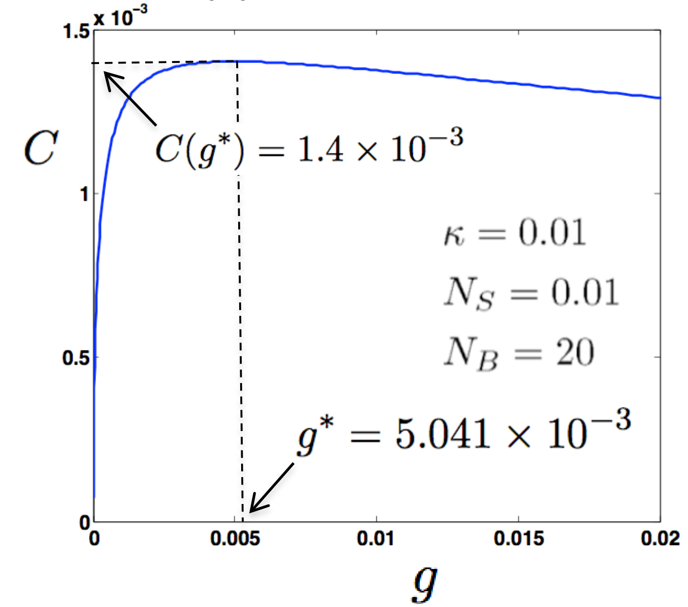
Bhattacharyya bound on performance

- Bhattacharyya (upper) bound on performance of OPA-based receiver
  - $\Pr(\mathrm{e}) \leq \frac{1}{2}{Q_B}^K$, where
  - Bound asymptotically tight as $K \to \infty$

$$Q_B = \sum_{n=0}^{\infty} \sqrt{\frac{{N_0}^n}{(1+N_0)^{1+n}} \times \frac{{N_1}^n}{(1+N_1)^{1+n}}} = \left(\sqrt{(1+N_0)(1+N_1)} - \sqrt{N_0 N_1}\right)^{-1}$$



- OPA Gain $G = 1 + g$ is optimized for min $\Pr(\mathrm{e})$, i.e. max C



OPA receiver yields 3 dB improvement

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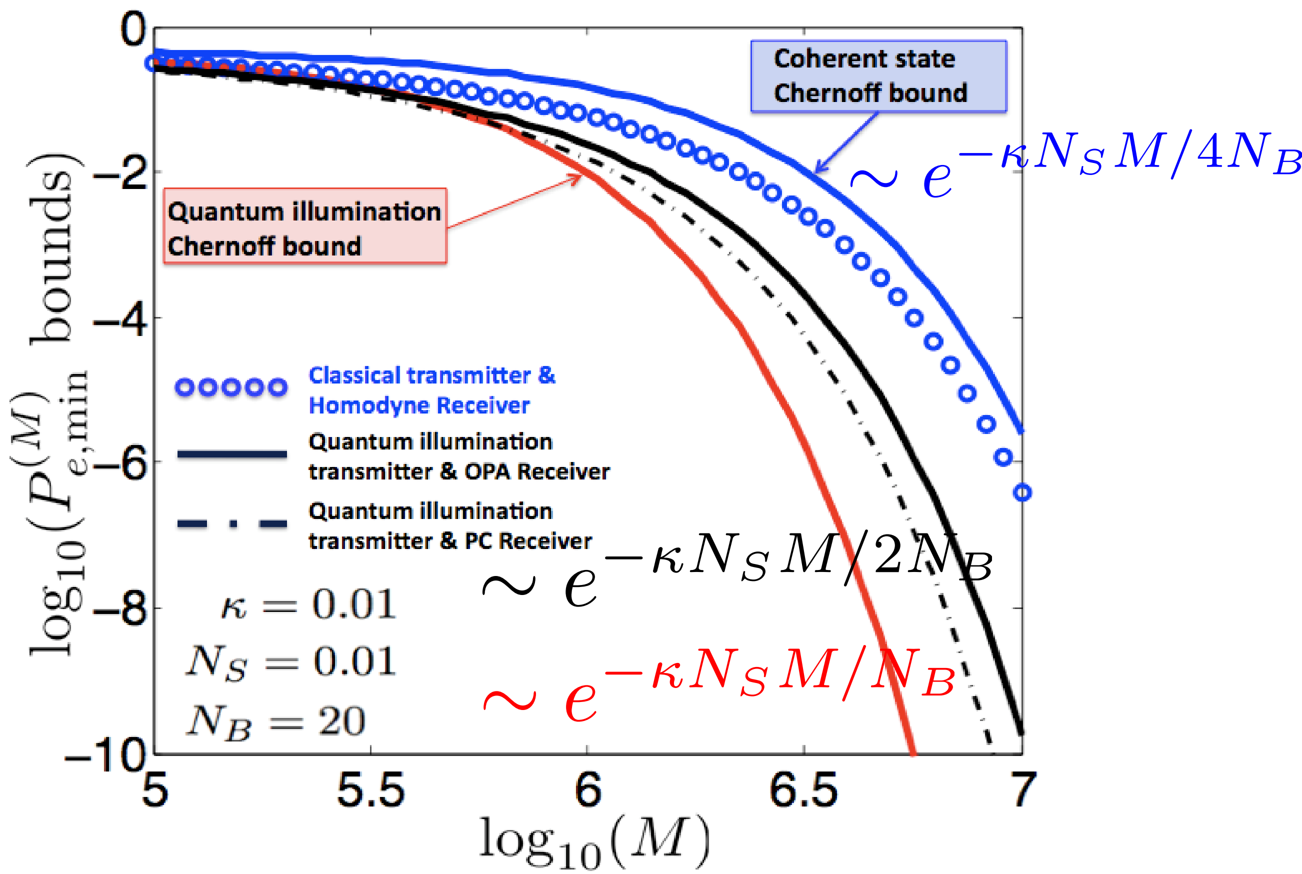
Quantum Radar

$$\mathrm{SNR}_Q \propto \frac{\kappa N_S}{N_B}$$

Classical Radar

$$\mathrm{SNR}_C \propto \frac{\kappa N_S}{4N_B}$$



Quantum Radar with OPA receiver

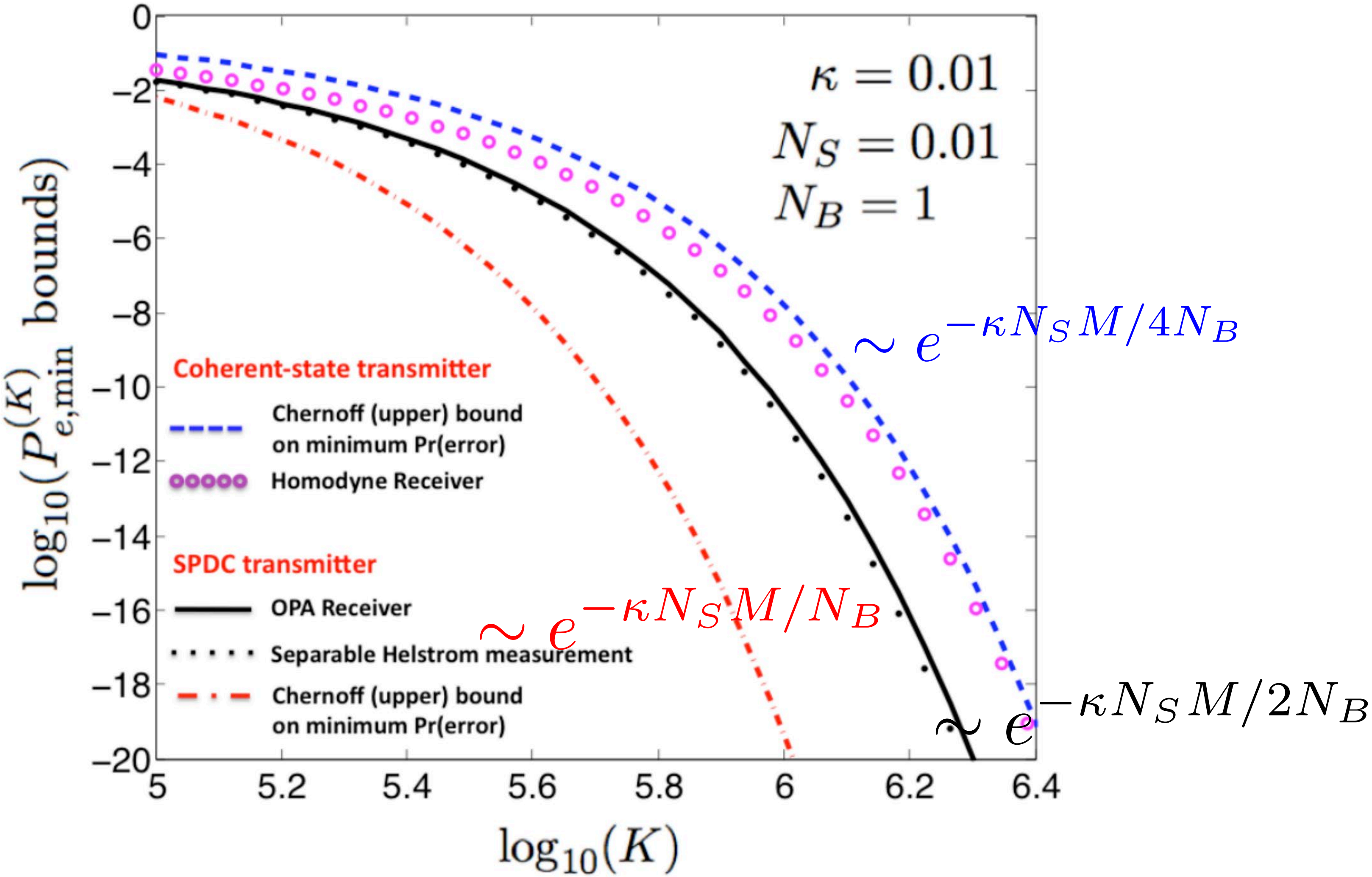
$$\mathrm{SNR}_{Q,\mathrm{OPA}} = \frac{\kappa N_S}{2N_B}$$

$M \approx WT$: number of temporal modes

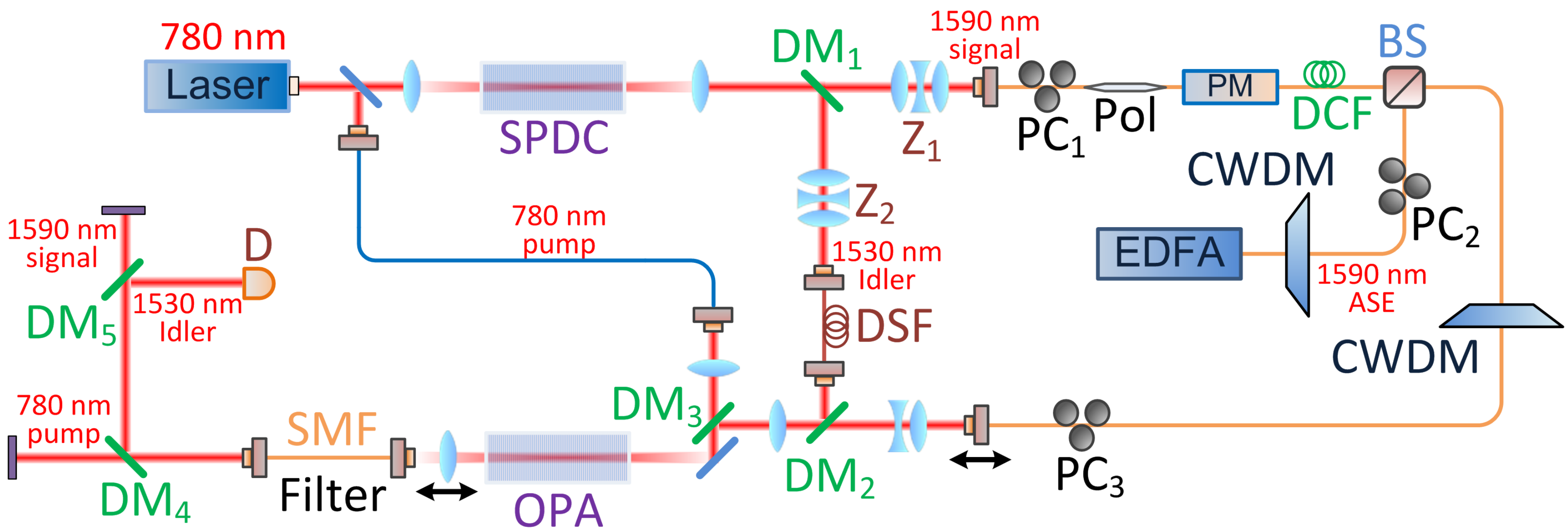
Optimal copy-by-copy measurement is worse by 3 dB from optimal

We saw this curious 3 dB difference in the pure-state case. Is this a general feature in binary state discrimination?

Advanced Problem 17 (open)



Quantum illumination radar experiment

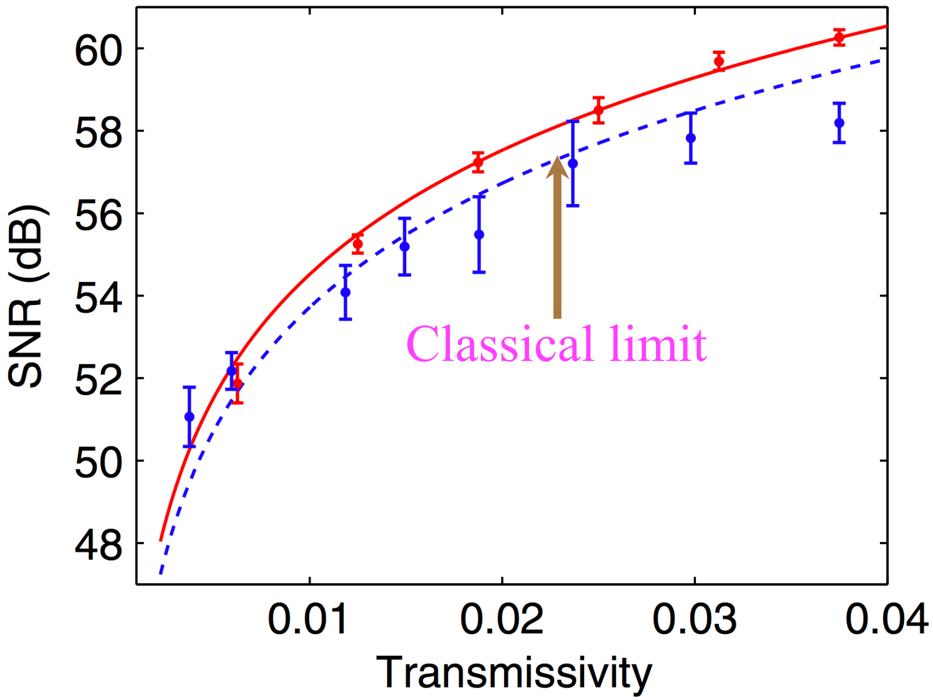


ASE: amplified spontaneous emission; BS: beam splitter;
CWDM: corse wavelength-division multiplexer; D: detector;
DCF: dispersion-compensating fiber; DM: dichroic mirror; DSF: dispersion-shifted LEAF fiber;
EDFA: erbium-doped fiber amplifier; OPA: optical parametric amplifier; PC: polarization controller;
PM: phase modulator; Pol: polarizer; SMF: single-mode fiber;
SPDC: spontaneous parametric down conversion; Z: zoom-lens systems

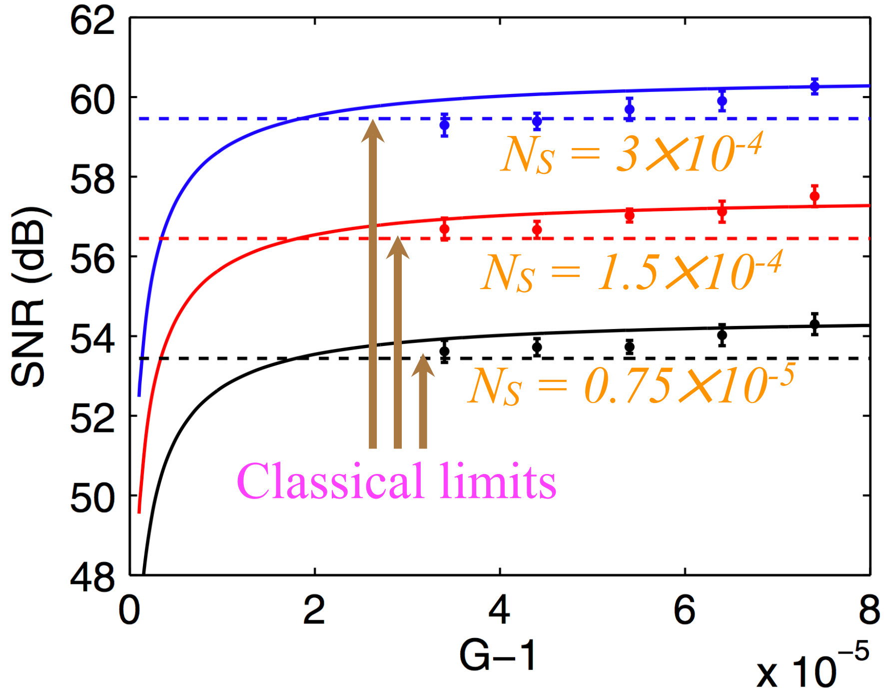
Z. Zhang *et al.*, Phys. Rev. Lett. 114, 110506 (2015)

SNR measurements

Quantum vs Classical SNR



SNR vs Receiver gain



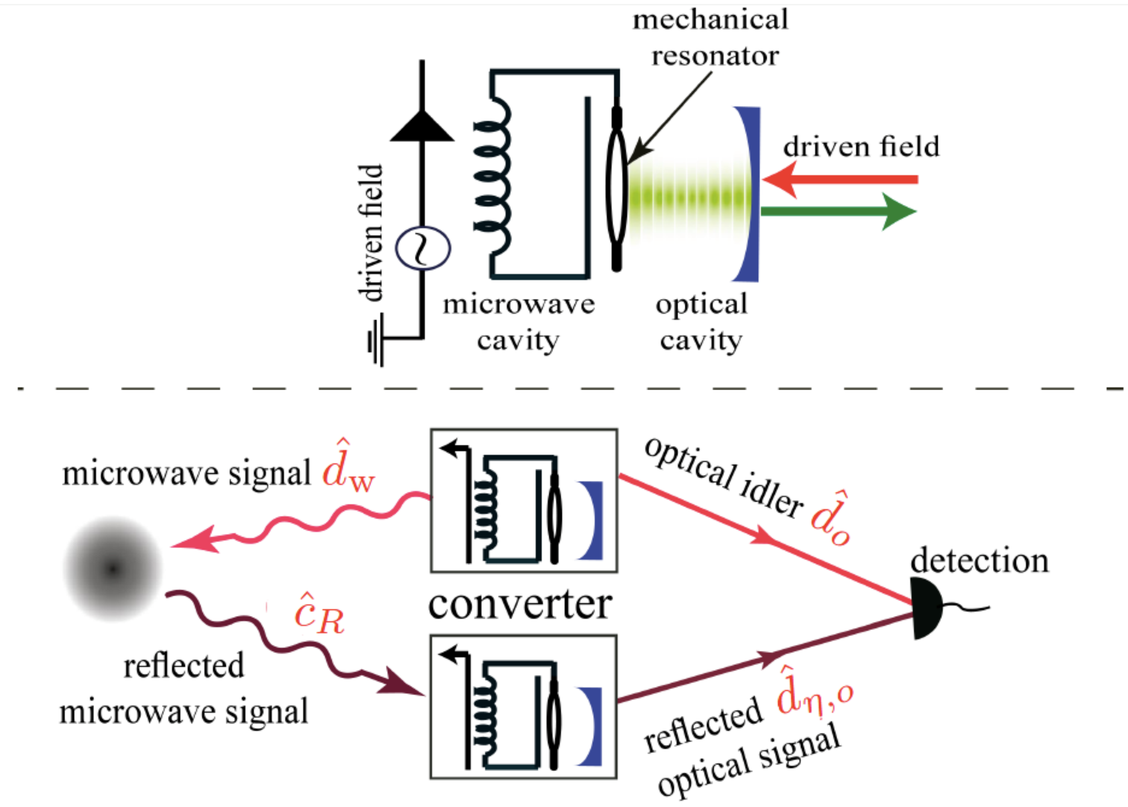
Environmental loss: 14 dB; Noise background: 75 dB

Quantum sensing outperforms optimum classical sensing

Z. Zhang *et al.*, Phys. Rev. Lett. 114, 110506 (2015)

Microwave quantum radar

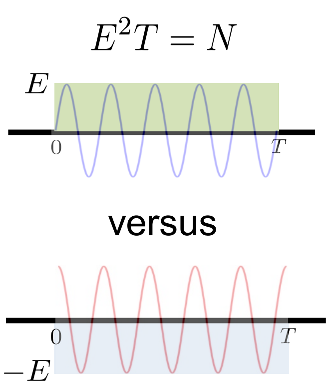
- The high-noise requirement makes microwave-wavelength operation a naturally-suited regime for quantum illumination
- Quantum advantage most pronounced at a high enough M. To get to a given M ~ WT, since W is lower (than optical), higher integration time T is needed. Applications with long dwell time?

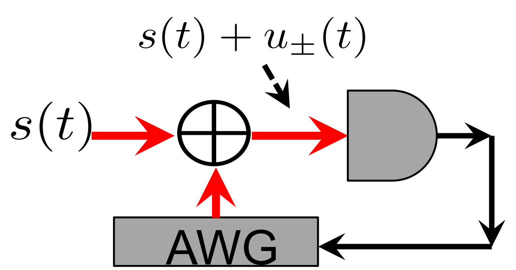


Barzanjeh, Guha, Weedbrook, Vitali, Shapiro, Pirandola,
Phys. Rev. Lett. 114, 080503 (2015)

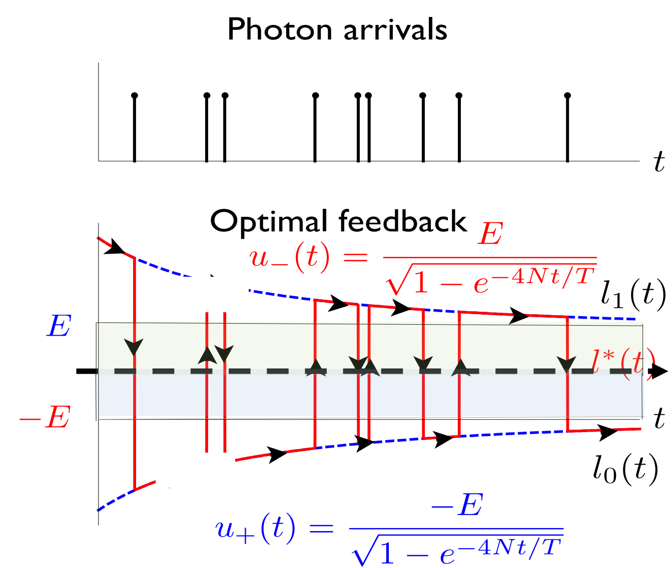
Optimal quantum receiver design

- Calculating minimum probability of error for discriminating states in $\{p_i, |\psi_i\rangle\}$ is conceptually simple
- But structured receiver designs that achieve optical state discrimination at the quantum minimum error, are far and few
- Binary coherent states, $\{|-\alpha\rangle, |\alpha\rangle\}$, $|\alpha|^2 \equiv N$
  - Minimum error probability (equal priors), $P_{e,\min} = \frac{1}{2}\left[1 - \sqrt{1 - e^{-4N}}\right]$
  - Dolinar's receiver [1973]: **OPTI 595B**





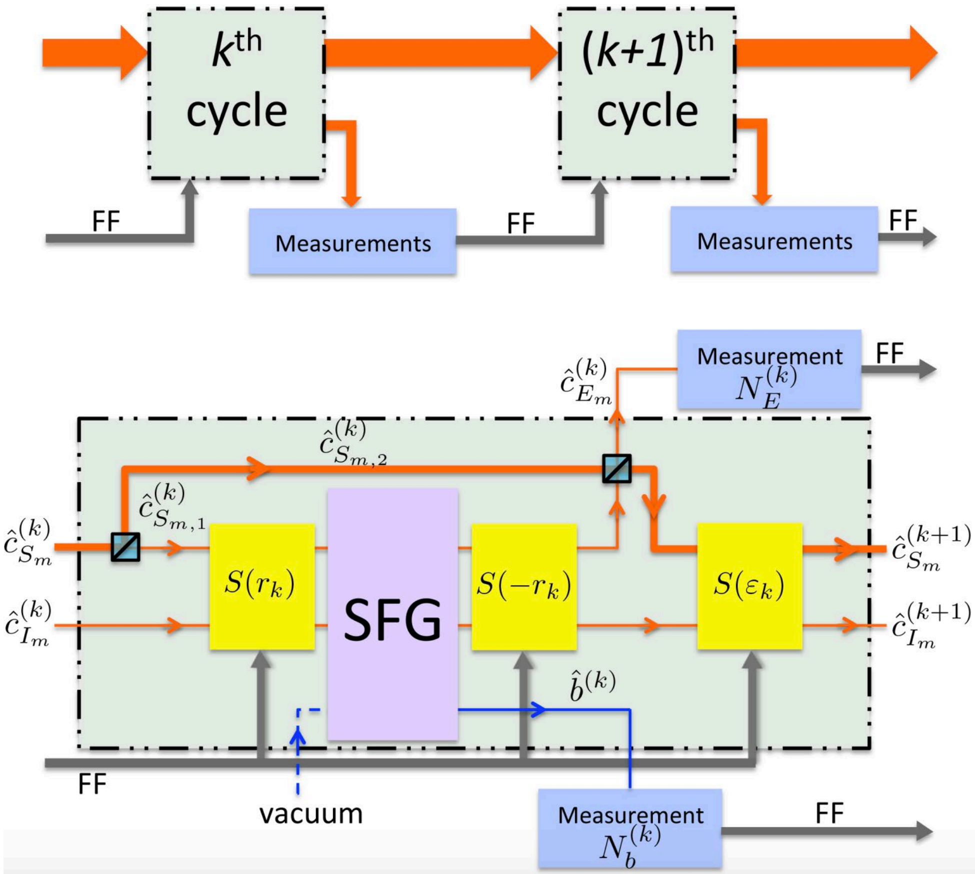
Detector PPP rate:
$\lambda(t) = |s(t) + u_\pm(t)|^2$

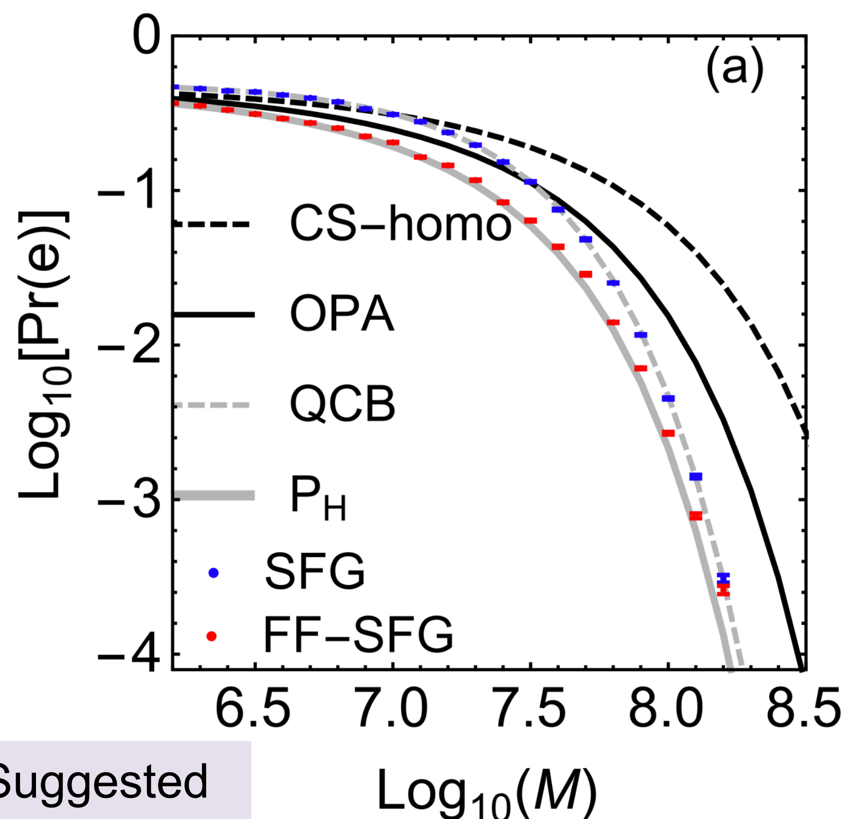


Optimum receiver for quantum illumination

Sum frequency generation (SFG): $\hat{H}_I = \hbar g \sum_{m=1}^{M} (\hat{b}^\dagger \hat{a}_{S_m} \hat{a}_{I_m} + \hat{b} \hat{a}_{S_m}^\dagger \hat{a}_{I_m}^\dagger)$

Inspired by Dolinar receiver: feedback using squeezing instead of displacement





Suggested reading

Zhuang, Zhang, Shapiro, Phys. Rev. Lett. 118, 040801 (2017)

One-versus-Two Target Detection

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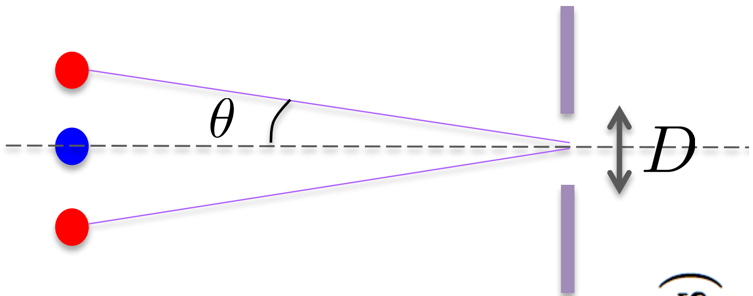
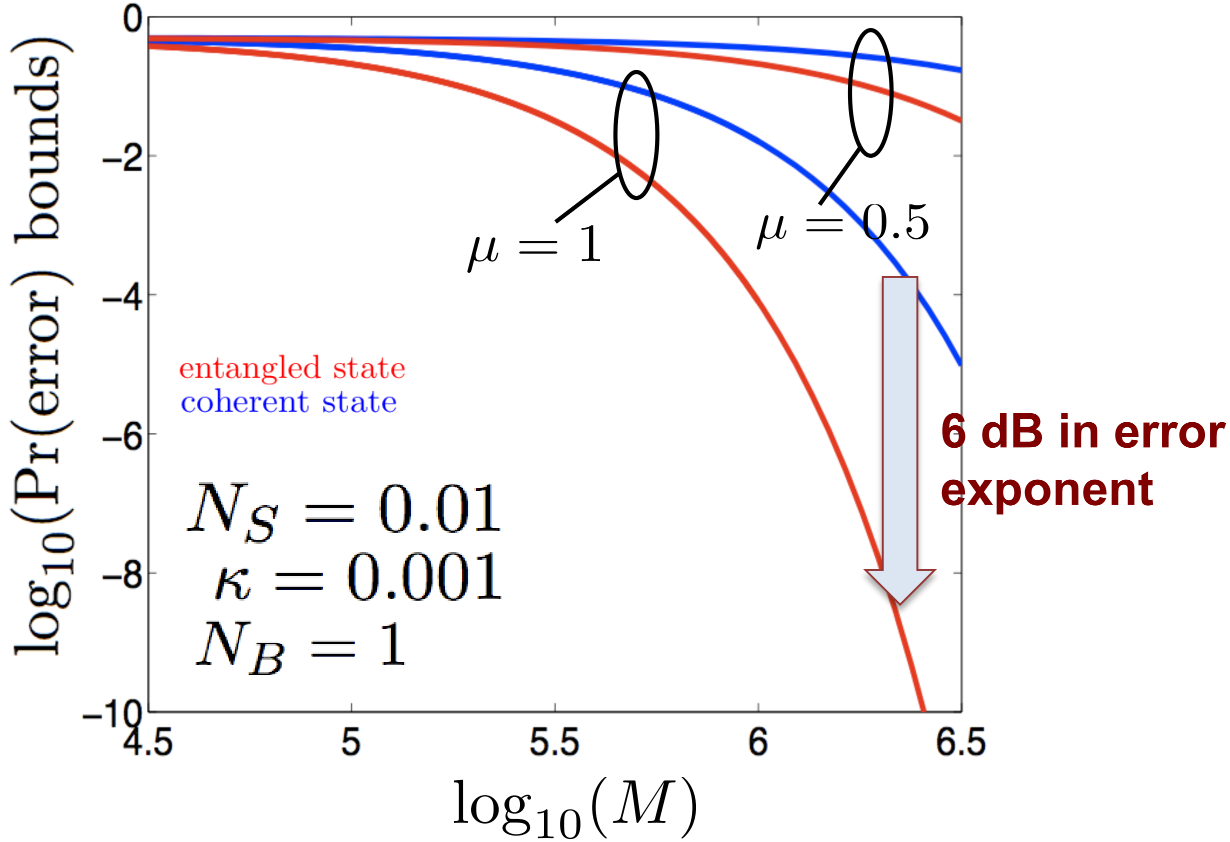


$$\theta_{\mathrm{Ray}} \equiv \frac{\lambda}{D}\,, \theta = \mu\theta_{\mathrm{Ray}}$$



Does this quantum illumination improvement prevail in imaging problems? More complex optical sensing / discrimination tasks?

S. Guha and J. H. Shapiro, QCMC 2010, arXiv:1012.2548v1

Upcoming topics

- Quantum limits of optical communications