Photonic Quantum Information Processing
OPTI 647: Lecture 22

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Recap: QFI and CRB for multiple parameters

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Let us have a multiple unknown parameters to be estimated simultaneously:

$$\vec{\theta} = (\theta_1, \ldots, \theta_N)$$

For multiple parameters the generalization is straightforward.

via Fidelity: $H_{ij} = -2 \frac{\partial^2 F(\vec{\epsilon})}{\partial \epsilon_i \partial \epsilon_j}\bigg|_{\vec{\epsilon}=\vec{0}}$ $\quad \vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)$

via SLD's: $H_{ij} = \frac{1}{2} \mathrm{tr}\left[\hat{\rho}_{\vec{\theta}} \left(\hat{L}_i \hat{L}_j + \hat{L}_j \hat{L}_i\right)\right]$

One SLD for each parameter:

$$\frac{\partial \hat{\rho}_{\vec{\theta}}}{\partial \theta_i} = \frac{1}{2}\left(\hat{L}_i \hat{\rho}_{\vec{\theta}} + \hat{\rho}_{\vec{\theta}} \hat{L}_i\right)$$

From numbers (single parameter), we go to matrices (multiple parameters). The CRB now is a matrix inequality in the positive semidefinite sense:

$$\mathrm{Cov}(\vec{\theta}) \geq I^{-1} \geq H^{-1}$$

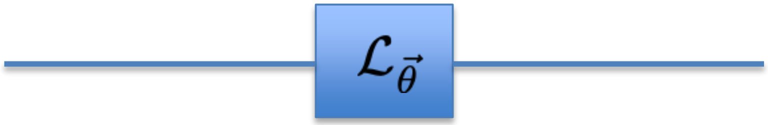
Where all Fisher matrices are positive semidefinite (eigenvalues non-negative) and symmetric, just like the covariance matrices. The off-diagonal elements represent correlated error.

The QFI is attainable when: $[\hat{L}_i, \hat{L}_j] = 0$ (one shot), and $\mathrm{tr}\left(\hat{\rho}_{\vec{\theta}}[\hat{L}_i, \hat{L}_j]\right) = 0$ (asymptotic limit of many measurements). In that case **the measurement (POVM)** is given by **the eigenvectors of the SLD's**.

For single parameter problems: $[\hat{L}, \hat{L}] = 0$. The single parameter QFI is always attainable.

Recap: QFI using SLD's

Initial state (probe state) $\hat{\rho}_0$



Quantum channel

Final state

$$\hat{\rho}_{\vec{\theta}} = \sum_k E_k |e_k\rangle\langle e_k|$$

$$H_{ij} = \mathrm{Re}\left[\sum_n \frac{\partial_i(E_n)\partial_j(E_n)}{E_n} + 4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2} \langle e_n|\partial_i e_m\rangle\langle\partial_j e_m|e_n\rangle\right]$$

Special cases

$$H_{ij} = 4\mathrm{Re}\left[\langle\partial_i e_0|\partial_j e_0\rangle - \langle e_0|\partial_i e_0\rangle\langle\partial_j e_0|e_0\rangle\right]$$

Pure final state

$$H_{ij} = 4\mathrm{Re}\left[\langle\phi_0|(\partial_i\hat{U}^\dagger_{\vec{\theta}})(\partial_j\hat{U}_{\vec{\theta}})|\phi_0\rangle - \langle e_0|(\partial_i\hat{U}_{\vec{\theta}})|\phi_0\rangle\langle\phi_0|(\partial_j\hat{U}^\dagger_{\vec{\theta}})|e_0\rangle\right]$$

Pure states under unitary evolution

$$H_{ij} = 4\mathrm{Re}\left[\langle\phi_0|\hat{H}_i\hat{H}_j|\phi_0\rangle - \langle\phi_0|\hat{H}_i|\phi_0\rangle\langle\phi_0|\hat{H}_j|\phi_0\rangle\right]$$

Pure states under unitary evolution where Hamiltonians commute

$$H_{ij} = 4\,\mathrm{Re}\left[\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2}\langle e_n|\partial_i\hat{U}_{\vec{\theta}}|\phi_m\rangle\langle\phi_m|\partial_j\hat{U}^\dagger_{\vec{\theta}}|e_n\rangle\right]$$

Mixed state under unitary evolution

$$H_{ij} = 4\,\mathrm{Re}\left[\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2}\langle e_n|\hat{H}_i|e_m\rangle\langle e_m|\hat{H}^\dagger_j|e_n\rangle\right]$$

Mixed state with unitary evolution, where the Hamiltonians commute

Today's plan: QFI using the SLD's

1. Ways to calculate the CFI.

2. Advanced problems and discussion.

Classical Fisher information (CFI)

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Remember: The task is to find the CFI, denoted as I, and compare it to the QFI. The CFI can be a standalone calculation.

$$I_{ij} = \mathbb{E}\left[-\frac{\partial^2 \ln P(\vec{x};\vec{\theta})}{\partial\theta_i \partial\theta_j}\right] = \mathbb{E}\left[\frac{\partial \ln P(\vec{x};\vec{\theta})}{\partial\theta_i}\frac{\partial \ln P(\vec{x};\vec{\theta})}{\partial\theta_j}\right] = \int d^N x P(\vec{x};\vec{\theta})^{-1}\frac{\partial P(\vec{x};\vec{\theta})}{\partial\theta_i}\frac{\partial P(\vec{x};\vec{\theta})}{\partial\theta_i}$$

Problem 87: Prove the expression after last =

$\vec{\theta}$: Unknown parameters.

$\vec{x}$: Estimated unknown parameter.

$P(\vec{x};\vec{\theta})$: Probability that given the unknown parameters to be $\vec{\theta}$, the measurement will return values $\vec{x}$

Let a POVM set $\{\widehat{M}_i\}, \sum_i \widehat{M}_i = I$. $P_i = \mathrm{tr}(\hat{\rho}\, \widehat{M}_i)$ is the probability of the i-th outcome to occur.

$\int dq\, |q\rangle\langle q| = I$, the $\{|q\rangle\langle q|\}$ is the POVM for measurement on position basis, i.e., homodyne measurement on the position quadrature. (Continuous index q.)

$\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = I$, the $\{\frac{1}{\pi}|\alpha\rangle\langle\alpha|\}$ is the POVM for measurement on coherent basis, i.e., ideal heterodyne. (Continuous index α.)

$\sum_n |n\rangle\langle n| = I$ where $|n\rangle$ is a Fock vector, is the POVM for photon counting.

For example: $P\left(\vec{q};\vec{\theta}\right) = \mathrm{tr}(\hat{\rho}_{\vec{\theta}}\, |\vec{q}\rangle\langle\vec{q}|\,)$, $P\left(\vec{n};\vec{\theta}\right) = \mathrm{tr}((\hat{\rho}_{\vec{\theta}}\, |\vec{n}\rangle\langle\vec{n}|)$, etc…

Homodyne detection

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Let the final state $\hat{\rho}_{\vec{\theta}}$. From that you calculate the corresponding Wigner function $W_{\vec{\theta}}(\vec{q},\vec{p})$. The marginal probability

$$P\left(\vec{q};\vec{\theta}\,\right) = \int d^N p\, W_{\vec{\theta}}(\vec{q},\vec{p}),$$

is the one which should be used in the CFI formula.

See Lecture 7, slides 17, 18 (a bit different notation there). There's an easier way to do that **for Gaussian states**:

Calculate the Wigner Characteristic function $X_W(\vec{\eta}) = \mathrm{tr}\left[\hat{\rho}_{\vec{\theta}}\, D(\vec{\eta})\right], \vec{\eta}$ are complex numbers. For position measurement, find $X_W\left(i\frac{\vec{p}_\eta}{\sqrt{2}}\right), \vec{p}_\eta$ is a real number. Read out the covariance matrix and displacements of $X_W\left(i\frac{\vec{p}_\eta}{\sqrt{2}}\right)$, that will be the covariance matrix and displacements of $P\left(\vec{q};\vec{\theta}\,\right)$.

Problem 88: How would calculate $P(q\cos\omega + p\sin;\omega;\theta\,)$, (single parameter case) for a Gaussian state?

Heterodyne Detection

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Let the final state $\hat{\rho}_{\vec{\theta}}$. From that you calculate the corresponding Husimi Q function $Q_{\vec{\theta}}(\vec{\alpha})$,

$$Q_{\vec{\theta}}(\vec{\alpha}) = \frac{1}{\pi^N} \langle \vec{\alpha} | \hat{\rho}_{\vec{\theta}} | \vec{\alpha} \rangle$$

which is a proper probability distribution and the one which should be used in the CFI formula.

Advanced Problem 13

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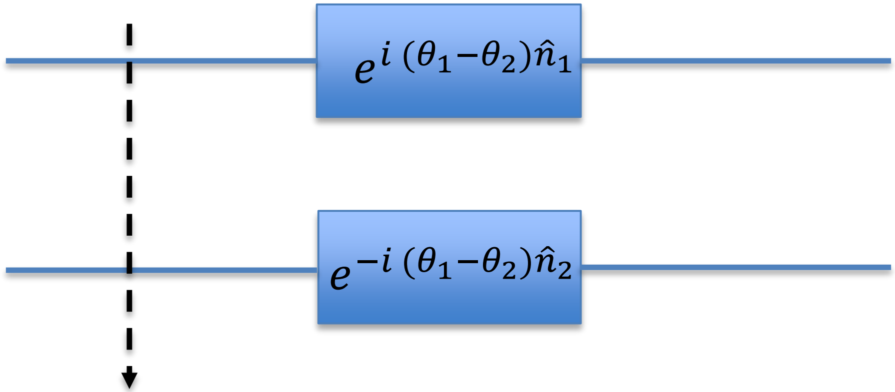
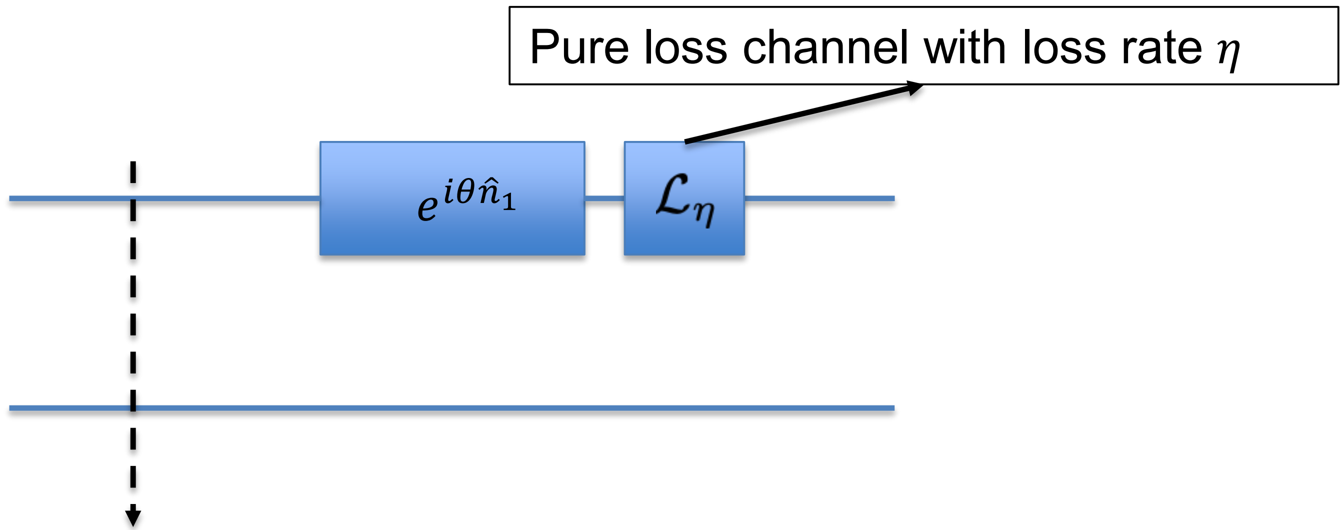


Two mode squeezed state.

1. Calculate the QFI matrix for the unknown parameters θ_1, θ_2
2. Calculate the CFI matrix for homodyne on position.
3. Calculate the CFI matrix for homodyne by keeping one quadrature to be position and rotating the other quadrature by ω with respect to the position. Optimize the trace of the CFI matrix over ω.
4. Compare the trace of the CFIs and QFI matrices.

$$\mathrm{tr}[\mathrm{Cov}(\vec{\theta})] \geq \mathrm{tr}[I^{-1}] \geq \mathrm{tr}[H^{-1}]$$

Advanced Problem 14



Fixed photon number state:

$$|\psi_0\rangle = \sum_{k=0}^{N} a_k |k, N-k\rangle$$

for some a_k such that $\sum_{k=0}^{N} |a_k|^2 = 1$

1. Calculate the QFI matrix for the unknown parameters θ, η.
2. Is the simultaneous QFI matrix (from question 1) attainable by a measurement?
3. What is the QFI for just θ? (If η is known). What is the optimal a_k in that case?
4. What is the QFI for just η? (If θ is known). What is the optimal a_k in that case?

Advanced Problem 15

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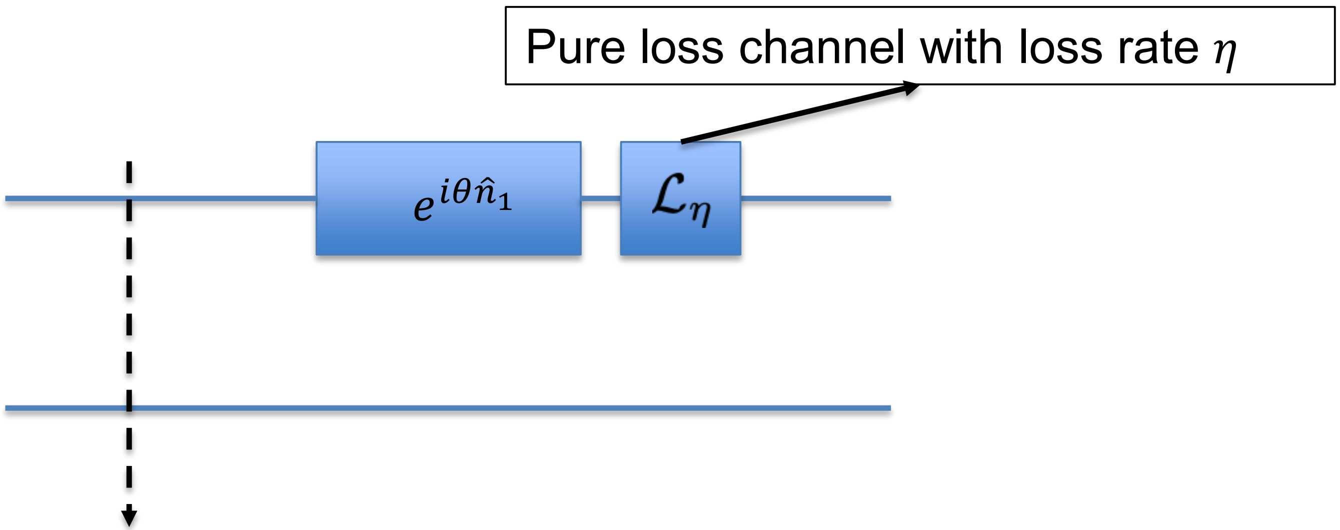
For Gaussian single parameter estimation (pure or mixed Gaussian states, parameters are imprinted through Gaussian operators), argue, as generally as you can, and provide examples:

1. That in order to get Heisenberg limit QFI ($\sim E^2$), the parameter must be imprinted in the covariance matrix.
2. Classical limit QFI ($\sim E$), is found if the parameter is picked up by the first moments only.

Advanced Problem 16 (hard)

What is the optimal state with fixed **mean** photon number for simultaneous estimation of θ and η?



$|\psi_0\rangle$, such that $\langle\psi_0|(\hat{n}_1 + \hat{n}_2)|\psi_0\rangle = N$ (known)

Next lectures:

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1. ~~Quantum sensing (3-4 lectures).~~

2. Soft introduction to Optical Quantum Computers (2-3 lectures).

3. Quantum discrimination (1-2 lectures).

4. Quantum Communications (2-3 lectures).