Photonic Quantum Information Processing
OPTI 647: Lecture 21

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Annoucements

1. Thanks for handing in your assignments on time.

2. I will correct everything I have by the end of the week.

3. If you haven’t received as many graded problem sets as you expected, do not worry. I just need to catch up grading.

Recap: QFI and CRB for multiple parameters

Let us have a multiple unknown parameters to be estimated simultaneously:

$$\vec{\theta} = (\theta_1, \ldots, \theta_N)$$

For multiple parameters the generalization is straightforward.

via Fidelity: $H_{ij} = -2 \frac{\partial^2 F(\vec{\epsilon})}{\partial \epsilon_i \partial \epsilon_j}\Big|_{\vec{\epsilon}=\vec{0}}$ $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_N)$

via SLD's: $H_{ij} = \frac{1}{2}\mathrm{tr}\left[\hat{\rho}_{\vec{\theta}}\left(\hat{L}_i\hat{L}_j + \hat{L}_j\hat{L}_i\right)\right]$

One SLD for each parameter:

$$\frac{\partial \hat{\rho}_{\vec{\theta}}}{\partial \theta_i} = \frac{1}{2}\left(\hat{L}_i\hat{\rho}_{\vec{\theta}} + \hat{\rho}_{\vec{\theta}}\hat{L}_i\right)$$

From numbers (single parameter), we go to matrices (multiple parameters). The CRB now is a matrix inequality in the positive semidefinite sense:

$$\mathrm{Cov}(\vec{\theta}) \geq I^{-1} \geq H^{-1}$$

Where all Fisher matrices are positive semidefinite (eigenvalues non-negative) and symmetric, just like the covariance matrices. The off-diagonal elements represent correlated error.

The QFI is attainable when: $[\hat{L}_i, \hat{L}_j] = 0$ (one shot), and $\mathrm{tr}\left(\hat{\rho}_{\vec{\theta}}[\hat{L}_i, \hat{L}_j]\right) = 0$ (asymptotic limit of many measurements). In that case **the measurement (POVM)** is given by **the eigenvectors of the SLD's**.

For single parameter problems: $[\hat{L}, \hat{L}] = 0$. The single parameter QFI is always attainable.

Today's plan: QFI using the SLD's

1. Formal solution to the Lyapunov equation.

2. The unitary evolution case.

3. The pure state case.

4. Example.

Lyapunov equation

$$\frac{\partial \hat{\rho}_{\vec{\theta}}}{\partial \theta_i} = \frac{1}{2}\left(\hat{L}_i \hat{\rho}_{\vec{\theta}} + \hat{\rho}_{\vec{\theta}} \hat{L}_i\right)$$

$$\vec{\theta} = (\theta_1, \ldots, \theta_N)$$

The SLD operator $\hat{L}_i$ is operator version of the derivate with respect to the estimated parameter θ_i .

The solution of the **Lyapunov equation** (general form: $\hat{A}\,\hat{X} + \hat{X}\hat{A}^{\dagger} + \widehat{D} = 0$) gives the SLD $\hat{L}_i$ associated with each unknown parameter θ_i. In our case the SLD's are Hermitian operators and the formal solution is:

$$\hat{L}_i = 2\int_0^{\infty} dt e^{-\hat{\rho}_{\vec{\theta}} t} \partial_i \hat{\rho}_{\vec{\theta}} e^{-\hat{\rho}_{\vec{\theta}} t}$$

Where $\partial_i \equiv \frac{\partial}{\partial_{\theta_i}}$ is a just a shorthand to denote derivatives.

In general it is not an easy task to find the SLD: It involves exponentials and derivatives of operators. Surely, we need to express the density operator (the state) on a basis.

Lyapunov equation (continued)

$$\hat{L}_i = 2\int_0^{\infty} dt e^{-\hat{\rho}_{\vec{\theta}} t} \partial_i \hat{\rho}_{\vec{\theta}} e^{-\hat{\rho}_{\vec{\theta}} t}$$

The state admits a diagonal form: $\hat{\rho}_{\vec{\theta}} = \sum_k E_k |e_k\rangle\langle e_k|$

Note that the eigenvalues E_k and eigenvectors $|e_k\rangle$ might depend on the parameters $\vec{\theta}$.

Expressing the exponential of the density operator with Taylor series: $e^{-\hat{\rho}_{\vec{\theta}} t} = \sum_{k=0}^{\infty} \frac{\left(-\hat{\rho}_{\vec{\theta}} t\right)^k}{k!}$

We get the SLD:

$$\hat{L}_i = 2\sum_{n,m} \frac{\langle e_m | \partial_i \hat{\rho}_{\vec{\theta}} | e_n \rangle}{E_n + E_m} |e_m\rangle\langle e_n|$$

Problem 82: Prove the last equation.

The QFI from SLS's

$$\hat{L}_i = 2\sum_{n,m} \frac{\left\langle e_m \middle| \partial_i \hat{\rho}_{\vec{\theta}} \middle| e_n \right\rangle}{E_n + E_m} |e_m\rangle\langle e_n|$$

$$\hat{L}_i \hat{L}_j = 4\sum_{n,m,n'} \frac{\left\langle e_m \middle| \partial_i \hat{\rho}_{\vec{\theta}} \middle| e_n \right\rangle}{E_n + E_m} \frac{\left\langle e_n \middle| \partial_j \hat{\rho}_{\vec{\theta}} \middle| e_{n'} \right\rangle}{E_{n'} + E_n} |e_m\rangle\langle e_{n'}|$$

$$\hat{\rho}_{\vec{\theta}} \hat{L}_i \hat{L}_j = 4\sum_{n,m,n'} E_m \frac{\left\langle e_m \middle| \partial_i \hat{\rho}_{\vec{\theta}} \middle| e_n \right\rangle}{E_n + E_m} \frac{\left\langle e_n \middle| \partial_j \hat{\rho}_{\vec{\theta}} \middle| e_m \right\rangle}{E_{n'} + E_n} |e_m\rangle\langle e_n|$$

Problem 83: Prove the equation on the left .

Final aim is to find an expression of the QFI, which is general is given by:

$$H_{ij} = \frac{1}{2}\text{tr}\left[\hat{\rho}_{\vec{\theta}}\left(\hat{L}_i\hat{L}_j + \hat{L}_j\hat{L}_i\right)\right]$$

We need to use a few more tricks….

The QFI from SLD's

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"Trick 1": Care is required as the eigenvalues and eigenvector might be parameter-dependent.

$$\partial_i \hat{\rho}_{\vec{\theta}} = \sum_k (\partial_i E_k)|e_k\rangle\langle e_k| + E_k|\partial_i e_k\rangle\langle e_k| + E_k|e_k\rangle\langle\partial_i e_k|$$

"Trick 2": The derivative of the overlap below is 0. (Why?)

$$\partial_i(\langle e_n|e_m\rangle) = 0 \Rightarrow \langle\partial_i e_n|e_m\rangle + \langle e_n|\partial_i e_m\rangle = 0$$

From these relations, and $\hat{\rho}_{\vec{\theta}}\hat{L}_i\hat{L}_j = 4\sum_{n,m,n'} E_m \frac{\langle e_m|\partial_i\hat{\rho}_{\vec{\theta}}|e_n\rangle}{E_n + E_m}\frac{\langle e_n|\partial_j\hat{\rho}_{\vec{\theta}}|e_m\rangle}{E_{n'} + E_n}|e_m\rangle\langle e_n|$

we find the QFI:

Problem 84: Prove the equation in the box below:

$$H_{ij} = \mathrm{Re}\left[\sum_n \frac{\partial_i(E_n)\partial_j(E_n)}{E_n} + 4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2}\langle e_n|\partial_i e_m\rangle\langle\partial_j e_m|e_n\rangle\right]$$

Were the Real Part (Re) accounts for the anticommutor in $H_{ij} = \frac{1}{2}\mathrm{tr}\left[\hat{\rho}_{\vec{\theta}}\left(\hat{L}_i\hat{L}_j + \hat{L}_j\hat{L}_i\right)\right]$

The pure state (a.k.a. rank 1 matrices) case

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Rank **1** matrices: They have only **one** non-zero eigenvalue. Since we work with density matrices, the non-zero eigenvalue will be $E_{m=0} \equiv E_0 = 1$, and $E_{m\geq 1} = 0$.

Therefore, the QFI can be simplified substantially:

$$H_{ij} = \mathrm{Re}\left[\sum_n \frac{\partial_i(E_n)\partial_j(E_n)}{E_n} + 4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2}\langle e_n|\partial_i e_m\rangle\langle\partial_j e_m|e_n\rangle\right]$$

This is 0. If $n = 0$ it's obvious: it's two derivatives on 1 (i.e. zero), divided by 1. If $n \geq 1$: The fraction behaves as $\frac{0^2}{0} \to 0$

$$4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2}\langle e_n|\partial_i e_m\rangle\langle\partial_j e_m|e_n\rangle(1 - \delta_{n,m})\delta_{m,0}$$

Because if $E_n = E_m$ then the ratio $\left(\frac{E_n - E_m}{E_n + E_m}\right)^2 \to 0$

$$H_{ij} = 4\mathrm{Re}\left[\langle\partial_i e_0|\partial_j e_0\rangle - \langle e_0|\partial_i e_0\rangle\langle\partial_j e_0|e_0\rangle\right]$$

Problem 85: Prove the the last equation. You'll need to use $\sum_{n=0} |e_n\rangle\langle e_n| = I$, and a *small trick.*

QFI for unitary evolution of mixed states

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$$\hat{\rho}_0 = \sum_n E_n |\phi_n\rangle\langle\phi_n| \xrightarrow{\hat{U}_{\vec{\theta}} = \exp\left(i\sum_j \theta_j \hat{H}_j\right)} \hat{\rho}_{\vec{\theta}} = \sum_n E_n \hat{U}_{\vec{\theta}} |\phi_n\rangle\langle\phi_n| \hat{U}^\dagger_{\vec{\theta}} = \sum_n E_n |e_n\rangle\langle e_n|$$

Initial state $\hat{\rho}_0$ obtains a set of parameters $\vec{\theta}$ via a unitary $\widehat{U}_{\vec{\theta}}$. The final state $\hat{\rho}_{\vec{\theta}}$, will have the same eigenvalues as the initial state and the new eigenvectors will have (in general) dependence on the parameters.

$$H_{ij} = \mathrm{Re}\left[\sum_n \frac{\partial_i(E_n)\partial_j(E_n)}{E_n} + 4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2} \langle e_n|\partial_i e_m\rangle\langle\partial_j e_m|e_n\rangle\right]$$

$$|\partial_i e_n\rangle = (\partial_i \hat{U}_{\vec{\theta}})|\phi_n\rangle$$

$$H_{ij} = 4\,\mathrm{Re}\left[\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2} \langle e_n|\partial_i \hat{U}_{\vec{\theta}}|\phi_m\rangle\langle\phi_m|\partial_j \hat{U}^\dagger_{\vec{\theta}}|e_n\rangle\right]$$

If the Hamiltonian $\widehat{H}_i$ associated with each parameter θ_i, satisfy $[\widehat{U}_{\vec{\theta}}, \widehat{H}_i] = 0 \Rightarrow [\widehat{H}_i, \widehat{H}_j] = 0$.

$$H_{ij} = 4\,\mathrm{Re}\left[\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2} \langle e_n|\hat{H}_i|e_m\rangle\langle e_m|\hat{H}^\dagger_j|e_n\rangle\right]$$

Problem 86: Prove the the last equation.

QFI for unitary evolution of pure states

$$\hat{U}_{\vec{\theta}} = \exp\left(i\sum_j \theta_j \hat{H}_j\right)$$

$$|\phi_0\rangle \longrightarrow |e_0\rangle = \hat{U}_{\vec{\theta}}|\phi_0\rangle$$

$$H_{ij} = 4\mathrm{Re}\left[\langle\partial_i e_0|\partial_j e_0\rangle - \langle e_0|\partial_i e_0\rangle\langle\partial_j e_0|e_0\rangle\right]$$

QFI for pure states

$$|\partial_i e_n\rangle = (\partial_i \hat{U}_{\vec{\theta}})|\phi_n\rangle$$

$$H_{ij} = 4\mathrm{Re}\left[\langle\phi_0|(\partial_i \hat{U}^\dagger_{\vec{\theta}})(\partial_j \hat{U}_{\vec{\theta}})|\phi_0\rangle - \langle e_0|(\partial_i \hat{U}_{\vec{\theta}})|\phi_0\rangle\langle\phi_0|(\partial_j \hat{U}^\dagger_{\vec{\theta}})|e_0\rangle\right]$$

If the Hamiltonian $\hat{H}_i$ associated with each parameter θ_i, satisfy $[\hat{U}_{\vec{\theta}}, \hat{H}_i] = 0 \Rightarrow [\hat{H}_i, \hat{H}_j] = 0$.

$$H_{ij} = 4\mathrm{Re}\left[\langle\phi_0|\hat{H}_i\hat{H}_j|\phi_0\rangle - \langle\phi_0|\hat{H}_i|\phi_0\rangle\langle\phi_0|\hat{H}_j|\phi_0\rangle\right]$$

If we have only one estimated parameter.

$$H = 4\left[\langle\phi_0|\hat{H}^2|\phi_0\rangle - |\langle\phi_0|\hat{H}|\phi_0\rangle|^2\right] = 4(\Delta\hat{H})^2$$

For that case, the QFI is the variance of the Hamiltonian which corresponding to the parameter under consideration. Examples?

SLD's for unitary evolution of pure states

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We found a general formula for the SLD's: $\hat{L}_i = 2\sum_{n,m} \frac{\langle e_m | \partial_i \hat{\rho}_{\vec{\theta}} | e_n \rangle}{E_n + E_m} |e_m\rangle\langle e_n|$

From that, we found a general formula for QFI:

$$H_{ij} = \mathrm{Re}\left[\sum_n \frac{\partial_i(E_n)\partial_j(E_n)}{E_n} + 4\sum_{n,m} E_m \frac{(E_n - E_m)^2}{(E_n + E_m)^2} \langle e_n | \partial_i e_m \rangle \langle \partial_j e_m | e_n \rangle\right]$$

Then we focused on special cases (unitary evolution, pure states, unitary evolution for pure states). We could have done the same thing on the SLD's level and *then* calculate the QFI.

The SLD for unitary evolution of pure states is given by the commutator of the Hamiltonian (responsible for imprinting the parameter on the state) and the final state:

$$\hat{L} = 2i[\widehat{H}, |e_0\rangle\langle e_0|]$$

Example

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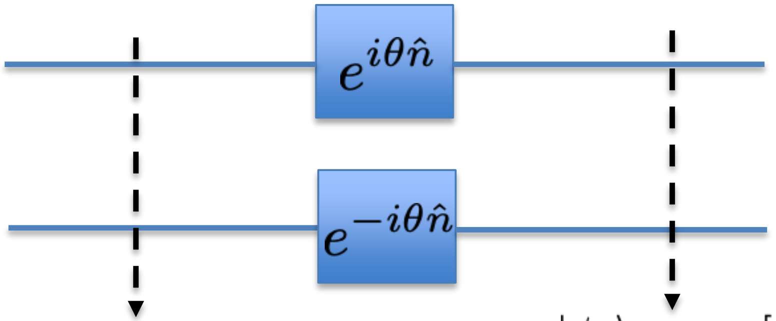

$$|\psi_0\rangle = \sum_{k=0}^{N} \sqrt{\binom{N}{k}\eta^k(1-\eta)^{N-k}}|k, N-k\rangle$$

$$|\psi_\theta\rangle = \exp\left[i\theta(\hat{n}_1 - \hat{n}_2]\right) \sum_{k=0}^{N} \sqrt{\binom{N}{k}\eta^k(1-\eta)^{N-k}}|k, N-k\rangle$$

Fixed photon number state $\quad 0 \leq \eta \leq 1$

$$H = 4\left[\langle\phi_0|\hat{H}^2|\phi_0\rangle - |\langle\phi_0|\hat{H}|\phi_0\rangle|^2\right] = 4(\Delta\hat{H})^2$$

For $|\phi_0\rangle \equiv |\psi_0\rangle$ and $\hat{H} = \hat{n}_1 - \hat{n}_2$

$$\langle\psi_0|\hat{n}_1|\psi_0\rangle = \eta M$$

$$\langle\psi_0|\hat{n}_2|\psi_0\rangle = (1-\eta)M$$

Total mean photon number: $\langle\psi_0|(\hat{n}_1 + \hat{n}_2)|\psi_0\rangle = M$

$$\langle\psi_0|(\hat{n}_1^2 - 2\hat{n}_1\hat{n}_2 + \hat{n}_2^2)|\psi_0\rangle = M^2(1-2\eta)^2 + 4M(1-\eta)\eta$$

$$\langle\psi_0|(\hat{n}_1 - \hat{n}_2)|\psi_0\rangle = -M(1-2\eta)$$

$$H = 16M(1-\eta)\eta \sim E$$

Classical limit

Next lecture: Further topics on quantum estimation theory

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Final lecture on quantum sensing and the QFI approach:

1. You've got all tools for quantum sensing!

2. No more theory.

3. Applications involving QFI and CFI calculations.

4. Advanced problems (probably two).